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CONVOLUTIONS OF DISTRIBUTIONS ATTRACTED TO STABLE LAWS[1]

By Howard G. Tucker[2]

University of California, Riverside

0. Summary. This paper deals with the domains of attraction of the stable distributions and the normalizing coefficients associated with distributions in those domains of attraction. Using the notation $F \varepsilon \mathfrak{D}(\alpha)$ and $F \varepsilon \mathfrak{D}_{\mathfrak{N}}(\alpha)$ to mean that the distribution function F is in the domain of attraction and the domain of normal attraction respectively of a stable law of characteristic exponent α, the following result is obtained: if $F \varepsilon \mathfrak{D}(\alpha)$ and $G \varepsilon \mathfrak{D}(\beta)$, where $0 < \alpha \leqq \beta \leqq 2$, and if $\{B_n\}$ and $\{C_n\}$ are normalizing coefficients respectively of F and G, then $F * G \varepsilon \mathfrak{D}(\alpha)$ and its normalizing coefficients are $\{(B_n^\alpha + C_n^\alpha)^{1/\alpha}\}$. Two more specialized results are obtained on convolutions of distribution functions in $\mathfrak{D}(2)$, namely: (i) if $F \varepsilon \mathfrak{D}_{\mathfrak{N}}(2)$ and $G \varepsilon \mathfrak{D}(2)\backslash\mathfrak{D}_{\mathfrak{N}}(2)$, then $F * G \varepsilon \mathfrak{D}(2)\backslash \mathfrak{D}_{\mathfrak{N}}(2)$, and (ii) if F and G are distribution functions, and if the four tail probabilities vary regularly with exponent -2 and involve possibly four different slowly varying functions, then F, G and $F * G$ are in $\mathfrak{D}(2)$. These latter two results hold *only* for $\mathfrak{D}(2)$ and not for $\mathfrak{D}(\alpha)$ for $0 < \alpha < 2$, thus adding two *exceptional properties* to the normal law within the family of stable laws.

1. Introduction and lemmas. The probabilistic terminology used here is fairly standard and can be found, for example, in [3]. However, for the sake of completeness, certain definitions and known results should be included here along with the lemmas subsequently needed. Since much of what will be done concerns slowly varying functions, a discussion of these is needed first.

Definition. A real-valued function $L(\cdot)$ defined over some terminal interval (a, ∞) is said to be *slowly varying* if (i) $L(x) > 0$ for all $x \varepsilon (a, \infty)$, (ii) $L(xy)/L(y) \to 1$ as $y \to \infty$ for every $x > 0$, and (iii) L is bounded over every bounded subinterval of some terminal subinterval of (a, ∞). A function φ defined over some terminal interval (a, ∞) is said to *vary regularly with exponent* ρ if there is a slowly varying function L such that $\varphi(x) \sim x^\rho L(x)$ (as $x \to \infty$).

If L is a slowly varying function, then a result frequently used states that $xL(x) \to \infty$ as $x \to \infty$ ([4], page 59); this result is not necessarily true if (iii) in the above definition were omitted. If now requirement (iii) in the definition is replaced by (iii′) which states that L is measurable, then it is known [2] that (iii) also holds (but that (iii) and (iii′) are not equivalent). Under requirements (i), (ii), (iii′), i.e., that L be a *measurable slowly varying function*, Karamata's representation theorem ([4], page 59) holds; namely,

$$L(x) = c(x) \exp \int_a^x (\theta(t)/t)\, dt, \tag{1}$$

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[2] Now at University of California, Irvine.

where $c(x)$ is a measurable function satisfying $c(x) \to$ (some constant)$c > 0$ as $x \to \infty$, and where $\theta(t)$ is Lebesgue-integrable over (a, x) for all $x > a$ and $\theta(t) \to 0$ as $t \to \infty$. For a complete development of this theory, see Karamata's basic paper [4] and Feller's recent book [1].

In view of the definition given above, if L_1 and L_2 are two slowly varying functions, then their product is also. If, in addition, L_1 and L_2 are *measurable* slowly varying functions (i.e., if (iii) is replaced by the stronger requirement (iii$'$)), then the quotient is also slowly varying; this is not necessarily true if (iii) holds, but (iii$'$) does not hold [2]. When a proof requires the use of Karamata's representation theorem or requires somewhere that a quotient of slowly varying functions be slowly varying, then the slowly varying functions involved must be and will be assumed to be measurable if they are not so already.

LEMMA 1. *If L_1 and L_2 are measurable slowly varying functions, then $L_1 + L_2$ is also a measurable slowly varying function.*

(I do not know whether this lemma is true without the word "measurable".)

Proof of this lemma follows from Karamata's representation (1) and routine computation.

LEMMA 2. *If L_1 and L_2 are measurable slowly varying functions, and if $\delta > 0$, then*

$$x^{-\delta}L_1(x) + L_2(x) \sim L_2(x).$$

PROOF. Since quotients of measurable slowly varying functions are also slowly varying, we need only observe that by Karamata's result quoted immediately after the definition that $x^{-\delta}L_1(x)/L_2(x) \to 0$ as $x \to \infty$, and thus

$$x^{-\delta}L_1(x) + L_2(x) = L_2(x)\{x^{-\delta}L_1(x)/L_2(x) + 1\} \sim L_2(x). \quad \text{Q.E.D.}$$

LEMMA 3. Let $F_1, \cdots, F_m$ *be $m \geqq 2$ distribution functions such that*

$$1 - F_i(x) \sim L_i(x)/x^{\rho_i}, \qquad 1 \leqq i \leqq m,$$

where, for some fixed $k \in \{1, \cdots, m\}$,

$$0 < \rho_1 = \cdots = \rho_k < \rho_{k+1} \leqq \cdots \leqq \rho_m,$$

and where $L_1, \cdots, L_m$ are measurable slowly varying functions. Then $1 - F_1 * \cdots * F_m$ varies regularly with exponent $-\rho_1$ and

$$1 - F_1 * \cdots * F_m(x) \sim x^{-\rho_1} \textstyle\sum_{j=1}^{k} L_j(x).$$

PROOF. This lemma is an extension of a result due to Feller ([1], page 271). Its proof is similar to Feller's and needs only Lemmas 1 and 2 in addition to the arguments given.

LEMMA 4. *If φ is a measurable slowly varying function, then there is a differentiable slowly varying function φ_1 (over some terminal interval) such that $\varphi(x) \sim \varphi_1(x)$ and $x\varphi_1(x)$ is strictly increasing.*

This lemma follows from Karamata's representation (1).

It should be recalled that F being in the domain of attraction of a stable distribution of characteristic exponent α (denoted here by $F \varepsilon \mathfrak{D}(\alpha)$) means that if $\{X_n\}$ is a sequence of independent identically distributed random variables with common distribution function F, then there is a sequence of positive constants $\{B_n\}$, called *normalizing coefficients,* and a sequence of real numbers $\{A_n\}$, called *centering constants* such that the limiting distribution of

$$B_n^{-1}(X_1 + \cdots + X_n) - A_n$$

is a stable distribution with characteristic exponent α. Now necessary and sufficient conditions are obtained for a sequence of positive numbers to be normalizing coefficients for some distribution function $F \varepsilon \mathfrak{D}(\alpha)$, $0 < \alpha \leqq 2$. The direct statement is used in Section 2. An interesting by-product of the converse is discussed after the proof is given.

LEMMA 5. *If $F \varepsilon \mathfrak{D}(\alpha)$, $0 < \alpha \leqq 2$, and if $\{B_n\}$ is a sequence of normalizing coefficients for F, then there is a measurable slowly varying function φ defined over $(0, \infty)$, which must be asymptotic to a nondecreasing function when $\alpha = 2$, such that $B_n \sim n^{1/\alpha}\varphi(n)$. Conversely, if φ is a measurable slowly varying function over $(0, \infty)$, and if $0 < \alpha < 2$, or if $\alpha = 2$ and φ is a measurable slowly varying function asymptotic to a nondecreasing function, then there is an $F \varepsilon \mathfrak{D}(\alpha)$ such that $\{n^{1/\alpha}\varphi(n)\}$ is a sequence of normalizing coefficients for F.*

PROOF. The proof of the direct statement of this lemma follows from a theorem and an example due to J. Lamperti in [5]. From Theorem 2 on page 64 of [5], if $\{X_n\}$ are independent, identically distributed random variables with common distribution function $F \varepsilon \mathfrak{D}(\alpha)$, $0 < \alpha \leqq 2$, and if we define $X_t = \sum_{1 \leqq k \leqq t} X_k$, then there is a measurable slowly varying function $\varphi(t)$, a constant $\beta > 0$ and a function $\omega(t)$ such that the limit distribution of the process $\{t^{-\beta}X_t/\varphi(t) + \omega(t)\}$ is that of a stable process of characteristic exponent α. The value of β is computed to be $1/\alpha$ by Example 1 on page 65 of [5]. We need only prove the converse. We first do so for $\alpha < 2$. Accordingly, let φ be any measurable slowly varying function over $(0, \infty)$, let $0 < \alpha < 2$, and define $B(t) = t^{1/\alpha}\varphi(t)$. Now by Lemma 4 there is a slowly varying function φ^* such that $\varphi(x) \sim \varphi^*(x)$ and $t^{1/\alpha}\varphi^*(t)$ is strictly increasing and continuous. We need only find an $F \varepsilon \mathfrak{D}(\alpha)$ such that $\{n^{1/\alpha}\varphi^*(n)\}$ are normalizing coefficients for F. Define F by

$$1 - F(t^{1/\alpha}\varphi^*(t)) = 1/t \tag{2}$$

for sufficiently large t, and $F(t) = 0$ otherwise. Clearly $\{n^{1/\alpha}\varphi^*(n)\}$ are normalizing coefficients for F *if* we can show $F \varepsilon \mathfrak{D}(\alpha)$. In order to do this we first observe that for large t there exists a strictly increasing continuous function $n(t)$ such that

$$(n(t))^{1/\alpha}\varphi^*(n(t)) = t. \tag{3}$$

Thus for any $\kappa > 0$,

$$1 - F(\kappa t) = 1 - F(\kappa(n(t))^{1/\alpha}\varphi^*(n(t))).$$

Since φ^* is slowly varying, then for fixed ϵ, $0 < \epsilon < \kappa$, and all large t,

$$\varphi^*(n(t))[\varphi^*((\kappa - \epsilon)^\alpha n(t))]^{-1}\kappa(\kappa - \epsilon)^{-1} > 1.$$

Hence

$$\begin{aligned} 1 - F(\kappa t) &= 1 - F(\kappa(\kappa - \epsilon)^{-1}((\kappa - \epsilon)^\alpha n(t))^{1/\alpha}\varphi^*((\kappa - \epsilon)^\alpha n(t)) \\ &\quad \cdot \varphi^*(n(t))[\varphi^*((\kappa - \epsilon)^\alpha n(t))]^{-1}) \\ &\leqq 1 - F(((\kappa - \epsilon)^\alpha n(t))^{1/\alpha}\varphi^*((\kappa - \epsilon)^\alpha n(t))) \\ &\sim 1/(\kappa - \epsilon)^\alpha n(t). \end{aligned}$$

By a similar argument one can replace $\kappa - \epsilon$ by $\kappa + \epsilon$ to obtain the inequality in the other direction, and then by an argument involving lim sup and lim inf, one obtains

$$1 - F(\kappa t) \sim 1/\kappa^\alpha n(t). \tag{4}$$

By (2) and (3) one obtains

$$1 - F(t) \sim 1/n(t). \tag{5}$$

Thus, by (4) and (5), $1 - F$ is a function which varies regularly with exponent $-\alpha$, i.e., $F \varepsilon \mathfrak{D}(\alpha)$. In the case where $\alpha = 2$, $B(t) = t^{\frac{1}{2}}\varphi(t)$, where $\varphi(t)$ is a nondecreasing slowly varying function. In order that $\{B(n)\}$ be normalizing coefficients for some $F \varepsilon \mathfrak{D}(2)$, it is known ([1], pp. 304–305) that they must satisfy $nU(B_n)/B_n^2 \to$ (some) $C > 0$ as $n \to \infty$, where $U(x) = \int_{|t| \leqq x} t^2\, dF(t)$. Hence it is sufficient to find an F such that $U(x)/\varphi^2(x) \to C > 0$ as $x \to \infty$. Clearly, if such an F exists, it is in $\mathfrak{D}(2)$ since this last relation implies that $U(x)$ is slowly varying. To prove that such an F exists, let us use the Karamata representation

$$\varphi(x) = c(x) \exp \{\textstyle\int_1^x \theta(t)t^{-1}\, dt\}$$

where $c(x) \to c > 0$ as $x \to \infty$ and $\theta(t) \to 0$ as $t \to \infty$. As in one proof of Lemma 4, since $\varphi(x)$ and $c(x)$ are not necessarily differentiable, we first define

$$\varphi_1(x) = c \exp \textstyle\int_1^x (\theta(t)/t)\, dt.$$

Hence $\varphi(x) \sim \varphi_1(x)$, where φ_1 is now obviously continuous. However, $\theta(t)$ is not necessarily continuous. However, as in a proof of Lemma 4, we may re-represent

$$\varphi_1(x) = c_1(x) \exp \textstyle\int_1^x (\theta_0(t)/t)\, dt,$$

where $c_1(x) \to c_1 > 0$, and now $c_1(x)$ and $\theta_0(t)$ are continuous. Now let $u(x) = \max\{\theta_0(x), 0\}$ and $\varphi_2(x) = c_1 \exp \int_1^x (u(t)/t)\, dt$. It is easily verified that φ_2 is nondecreasing and differentiable, and $\varphi_2(x) \sim \varphi_1(x) \sim \varphi(x)$. Now set

$$(d/dx)(\varphi_2(x))^2 = c_1^2\{\exp \textstyle\int_1^x (2u(t)/t)\, dt\}2u(x)/x = Cx^2 f(x),$$

where $f(x)$ is defined by the equality. Easily, $f(x)$ as defined over, say, $(1, \infty)$ is nonnegative and integrable; adjust the value of C such that $\int_1^\infty f(x)\, dx = 1$ and let $f(x) = 0$ if $x \leqq 1$. Then, if the distribution function F is defined by $F(x) = \int_{-\infty}^x f(t)\, dt$, we have

$$C \int_{|t| \leqq x} t^2\, dF(t) = \varphi_2^2(x) - \varphi_2^2(1).$$

If $\int_{|t| \leqq x} t^2 \, dF(t)$ is bounded, then F is in the domain of normal attraction of the normal distribution; if it is unbounded, then

$$\int_{|t| \leqq x} t^2 \, dF(t)/\varphi_2^2(x) \to C^{-1} \quad \text{or} \quad \int_{|t| \leqq x} t^2 \, dF(t)/\varphi^2(x) \to C.^{-1} \quad \text{Q.E.D.}$$

Of side interest here is the following remark. In [7], S. N. Nagaev proved that if $\{B_n\}$ is a sequence of normalizing coefficients for $F \in \mathfrak{D}(\alpha)$, where $0 < \alpha < 2$, then $B_n = n^{1/\alpha}\varphi(n)$, where $\varphi(n)$ is a function defined over the positive integers which satisfies

$$(6) \qquad \varphi(mn)/\varphi(n) \to 1 \quad \text{as} \quad n \to \infty, \qquad m = 1, 2, \cdots .$$

However, the conclusion of the direct assertion of Lemma 6 is stronger than the conclusion obtained by Nagaev in that it is possible to find a positive function ψ defined over the integers which is not a restriction to the positive integers of a measurable slowly varying function and yet satisfies (6). The following example of such is due to J. Sroka [8]. Let n be represented as a product of powers of primes, $n = \prod p_i^{k_i}$, and define $\psi(n) = 1/\sum k_i p_i$. Clearly ψ satisfies (6). However, if L is a slowly varying function (satisfying (i), (ii) and (iii)), we know (as was pointed out immediately after the definition) that $xL(x) \to \infty$ as $x \to \infty$. However, when n is a prime, then $n\psi(n) = 1$, thus showing that ψ is not even a restriction to the positive integers of a slowly varying function.

As remarked above, the direct statement of Lemma 5 will be applied a number of times in the next section. The following observation gives an application of the converse. We shall show that although two sequences of normalizing coefficients for the same distribution are asymptotic to each other, nevertheless, if $\{B_n\}$ and $\{C_n\}$ are normalizing coefficients for two *different* distribution functions in the same $\mathfrak{D}(\alpha)$, $0 < \alpha \leqq 2$, then there need be no asymptotic relation between the two sequences. In order to demonstrate this, let φ_1 and φ_2 be two nondecreasing slowly varying functions such that $\limsup_{x\to\infty} (\varphi_1(x)/\varphi_2(x)) = \infty$ and $\liminf_{x\to\infty} (\varphi_1(x)/\varphi_2(x)) = 0$. [These can be constructed as follows. Let $1 = t_1 < t_2 < \cdots < t_n < \cdots$ be integers which satisfy the following. Let $\theta_1(t)$ be defined by $\theta_1(t) = 1$ for $t \in [t_1, t_2)$, $\theta_1(t) = \frac{1}{3}$ for $t \in [t_3, t_4)$, and in general $\theta_1(t) = 1/(2n-1)$ for $t \in [t_{2n-1}, t_{2n})$, and $\theta_1(t) = 0$ if $t \in \bigcup_{n=1}^{\infty} [t_{2n}, t_{2n+1})$. Let

$$\begin{aligned} \theta_2(t) &= 1/2n \quad \text{for} \quad t \in [t_{2n}, t_{2n+1}) \\ &= 0 \qquad\ \text{for} \quad t \in \textstyle\bigcup_{n=1}^{\infty} [t_{2n-1}, t_{2n}). \end{aligned}$$

We can require $\{t_n\}$ to satisfy $\varphi_1(t_2) \geqq 2$, $\varphi_2(t_2) = 1$, $\varphi_1(t_3) = \varphi_1(t_2)$, $\varphi_2(t_3) \geqq 2^2\varphi_1(t_2)$, $\varphi_1(t_4) \geqq 2^3\varphi_2(t_3)$, $\varphi_2(t_4) = \varphi_2(t_3)$, etc., where $\varphi_i(x) = \exp \int_1^x (\theta_i(t)/t) \, dt$, $i = 1, 2$. Clearly $t_n \to \infty$, $\theta_i(t) \to 0$ as $t \to \infty$, so $\varphi_1(x)$ and $\varphi_2(x)$ are *nondecreasing* slowly varying functions which obviously satisfy $\limsup \varphi_1(x)/\varphi_2(x) = \infty$, and $\liminf \varphi_1(x)/\varphi_2(x) = 0$.] For any fixed $\alpha \in (0, 2]$, let $B_n = n^{1/\alpha}\varphi_1(n)$, and let $C_n = n^{1/\alpha}\varphi_2(n)$. Now it follows by the converse in Lemma 6 that there exist distribution functions F, G in $\mathfrak{D}(\alpha)$ such that $\{B_n\}$ and $\{C_n\}$ are normalizing coefficients for F and G respectively. However, B_n/C_n oscillates arbitrarily close to 0 and $+\infty$ as n gets large. This completes the demonstration.

2. Properties of convolutions and their normalizing coefficients. In this section we investigate the following problem: If $F \varepsilon \mathfrak{D}(\alpha)$ and $G \varepsilon \mathfrak{D}(\beta)$, where $0 < \alpha \leqq \beta \leqq 2$, and if $\{B_n\}$ and $\{C_n\}$ are their respective normalizing coefficients, then what can be said about the domain of attraction and the normalizing coefficients for $F * G$?

THEOREM 1. *Let $F \varepsilon \mathfrak{D}(\alpha)$ and $G \varepsilon \mathfrak{D}(\beta)$, where $0 < \alpha < \beta \leqq 2$, and let $\{B_n\}$ be normalizing coefficients for F. Then $F * G \varepsilon \mathfrak{D}(\alpha)$, and $\{B_n\}$ are normalizing coefficients for $F * G$.*

PROOF. Let $\{X_1, Y_1, X_2, Y_2, \cdots\}$ be independent random variables such that F is the distribution function for each X_n and G correspondingly for each Y_n. Let $\{B_n\}$ and $\{C_n\}$ be sequences of normalizing coefficients for F and G respectively. There exist centering constants $\{b_n\}$ and $\{c_n\}$ such that the limiting distributions of random variables U_n and V_n defined by

$$U_n = B_n^{-1}(X_1 + \cdots + X_n) - b_n \qquad \text{and}$$
$$V_n = C_n^{-1}(Y_1 + \cdots + Y_n) - c_n$$

are stable distributions with characteristic exponents α and β respectively. Further, the limiting joint distribution of $\{(U_n, V_n)\}$ is that of two independent random variables. Let $Z_n = X_n + Y_n$, $D_n = b_n + c_nC_n/B_n$. Then

$$B_n^{-1}(Z_1 + \cdots + Z_n) - D_n = U_n + (C_n/B_n)V_n.$$

By Lemma 5 there exist measurable slowly varying functions φ and ψ over $(0, \infty)$ such that $B_n \sim n^{1/\alpha}\varphi(n)$ and $C_n \sim n^{1/\beta}\psi(n)$. Hence

$$B_n/C_n \sim n^{(1/\alpha)-(1/\beta)}\varphi(n)/\psi(n).$$

From the discussion preceding Lemma 1, it follows that φ/ψ is a measurable slowly varying function over $(0, \infty)$. This fact is needed in order to be able to conclude, by a lemma due to Karamata mentioned earlier, that $B_n/C_n \to \infty$ or $C_n/B_n \to 0$ as $n \to \infty$. Then by an easy argument it follows that the limit distribution of $\{U_n + (C_n/B_n)V_n\}$ is the same as the limit distribution of $\{U_n\}$, which proves the theorem. Q.E.D.

Theorem 1 serves mostly as a lemma for the following theorem, although it does have its independent interest.

THEOREM 2. *If $F \varepsilon \mathfrak{D}(\alpha)$ and $G \varepsilon \mathfrak{D}(\beta)$, where $0 < \alpha \leqq \beta \leqq 2$, and if $\{B_n\}$ and $\{C_n\}$ are normalizing coefficients for F and G respectively, then $F * G \varepsilon \mathfrak{D}(\alpha)$ and $\{(B_n^\alpha + C_n^\alpha)^{1/\alpha}\}$ are normalizing coefficients for $F * G$.*

PROOF. Case (i) $0 < \alpha < \beta \leqq 2$. The above theorem readily follows in this case from Theorem 1, since by Lemmas 2 and 5, $B_n^\alpha + C_n^\alpha \sim B_n^\alpha$. Case (ii) $0 < \alpha = \beta \leqq 2$. In this case the characteristic function that we use for the stable distribution is

$$(6) \qquad f(u) = \exp\{iau - c|u|^\alpha\{1 + i\beta(u/|u|)\tan(\pi\alpha/2),\}\} \quad \text{if} \quad \alpha \neq 1$$
$$= \exp\{iau - c|u|\{1 + i\beta(u/|u|)2(\log|u|)/\pi\}\} \quad \text{if} \quad \alpha = 1,$$

where a is any real number, $-1 \leqq \beta \leqq 1, 0 < \alpha \leqq 2$ and $c \geqq 0$. One can easily verify that if $a = 0$, then for $p > 0$, $q > 0$ and $p + q = 1$, the function $f(u)$ satisfies

$$(7) \qquad \begin{aligned} f(p^{1/\alpha}u)f(q^{1/\alpha}u) &= f((p+q)^{1/\alpha}u) = f(u) \quad \text{if} \quad \alpha \neq 1 \\ &= f(u)e^{i\gamma u} \qquad\qquad\qquad \text{if} \quad \alpha = 1, \end{aligned}$$

where $\gamma = -2c\beta(p \log p + q \log q)/\pi$. Now by Lemma 5, there exist measurable slowly varying functions $U(x)$, $V(x)$ such that $B_n \sim n^{1/\alpha}U(n)$ and $C_n \sim n^{1/\alpha}V(n)$. We may select the centering constants $\{b_n\}$ and $\{c_n\}$ so that the limiting distributions of $\{S_n\}$ and $\{T_n\}$ are stable with characteristic exponent α with characteristic function given by (6), where $a = 0$ in (6), where

$$S_n = \textstyle\sum_{k=1}^n X_k/n^{1/\alpha}U(n) - b_n,$$
$$T_n = \textstyle\sum_{k=1}^n Y_k/n^{1/\alpha}V(n) - c_n,$$

and where $\{X_1, Y_1, X_2, Y_2, \cdots\}$ are as in the proof of Theorem 1. Let us define

$$W(x) = (U^\alpha(x) + V^\alpha(x))^{1/\alpha}.$$

A simple argument and Lemma 1 imply that $W(x)$ is a slowly varying function. Let us define

$$\begin{aligned} Z_n = \textstyle\sum_{k=1}^n (X_k + Y_k)[n^{1/\alpha}W(n)]^{-1} \\ - (b_nU(n)(W(n))^{-1} + c_nV(n)(W(n))^{-1} + h(\alpha)), \end{aligned}$$

where $h(\alpha) = -2c\beta(p \log p + q \log q)/\pi$ if $\alpha = 1$ and $= 0$ otherwise. If we denote $p_n = U(n)/W(n)$, $q_n = V(n)/W(n)$, then $Z_n = p_nS_n + q_nT_n - h(\alpha)$, where $p_n^\alpha + q_n^\alpha = 1$. Notice that for each n, S_n and T_n are independent. In what follows, let us denote the characteristic function of a random variable X evaluated at u by $f_X(u)$. Let $\{p_{n'}\}$ be any convergent subsequence of $\{p_n\}$ and $\{q_{n'}\}$ the corresponding subsequence of $\{q_n\}$. Then $p_{n'} \to$ (some) $p \geqq 0$, $q_{n'} \to q \geqq 0$, and $p^\alpha + q^\alpha = 1$. It follows that $f_{p_{n'}S_{n'}}(u) = f_{S_{n'}}(p_{n'}u) \to f(pu)$ and $f_{q_{n'}T_{n'}}(u) = f_{T_{n'}}(q_{n'}u) \to f(qu)$, where f is as defined in (6). Hence

$$(8) \qquad f_{Z_{n'}}(u) \to f(pu)f(qu) = f((p^\alpha + q^\alpha)^{1/\alpha}u) = f(u).$$

Since for every convergent sequence of $\{p_n\}$, (8) holds, it follows that $f_{Z_n}(u) \to f(u)$, i.e., $F * G \in \mathfrak{D}(\alpha)$ with normalizing coefficients $\{(B_n^\alpha + C_n^\alpha)^{1/\alpha}\}$. Q.E.D.

3. On the domain of attraction of the normal distribution. So far results have been established which determine the domain of attraction of the convolutions of two distribution functions, each of which is attracted to a stable law; in addition, the sequence of normalizing coefficients of the convolution is obtained. In this section the finer structure of the domain of attraction of the normal distribution is examined, and two theorems on convolutions of distributions in

$\mathfrak{D}(2)$ are obtained. Each of these two theorems add to the exceptional nature of the normal distribution within the family of stable laws.

In what follows the symbol $\mathfrak{D}_{\mathfrak{N}}(\alpha)$ will denote the domain of normal attraction for the stable distribution with characteristic exponent α. If both F and G are in $\mathfrak{D}_{\mathfrak{N}}(2)$, then trivially so is $F * G$. A first case beyond this is the following.

THEOREM 3. *If F and G are in $\mathfrak{D}(2)$, if $F \varepsilon \mathfrak{D}_{\mathfrak{N}}(2)$, and if $G \varepsilon \mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$, then $F * G \varepsilon \mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$.*

NOTE. This theorem is not necessarily true when $\mathfrak{D}(2)$ and $\mathfrak{D}_{\mathfrak{N}}(2)$ are replaced by $\mathfrak{D}(\alpha)$ and $\mathfrak{D}_{\mathfrak{N}}(\alpha)$ respectively, when $0 < \alpha < 2$. As an example, consider F and G defined by $1 - F(x) = x^{-\alpha}$ if $x \geqq 1$ and $= 1$ if $x < 1$, and $1 - G(x) = x^{-\alpha}/\log x$ if $x \geqq 2$ and $= 1$ if $x < 2$. Thus by Theorem 5 on page 181 of [3], $F \varepsilon \mathfrak{D}_{\mathfrak{N}}(\alpha)$ and $G \varepsilon \mathfrak{D}(\alpha) \setminus \mathfrak{D}_{\mathfrak{N}}(\alpha)$. Now by Lemma 3, $1 - F * G(x) \sim x^{-\alpha}(1 + 1/\log x)$, which by this last reference implies that $F * G \varepsilon \mathfrak{D}_{\mathfrak{N}}(\alpha)$.

PROOF OF THEOREM 3. Let $\{B_n\}$ and $\{C_n\}$ be normalizing coefficients for F and G respectively. We may take $B_n = n^{\frac{1}{2}}$. Following the proof of Theorem 1, we need only show that $C_n/B_n \to \infty$ as $n \to \infty$. Let us denote $U(x) = \int_{|t| \leqq x} t^2 \, dG(t)$. Then $\{C_n\}$ must satisfy $nU(C_n)/C_n^2 \to$ (some) $K_2 > 0$ as $n \to \infty$. In order to prove $C_n/B_n \to \infty$ we must prove that for arbitrary (large) $K > 0$, $C_n \geqq KB_n$ for for all large n. Let us suppose the contrary. Then there is a $K_0 > 0$ such that

$$C_n < K_0 B_n = K_0 n^{\frac{1}{2}}$$

for infinitely many values of n. Hence,

$$1/C_n^2 > 1/K_0^2 B_n^2 = 1/K_0^2 n, \qquad \text{or} \qquad 1 \sim nU(C_n)/K_2 C_n^2 > U(C_n)/K_2 K_0^2$$

for infinitely many values of n. But $C_n \to \infty$ so $U(C_n) \to \infty$ since $G \not\varepsilon \mathfrak{D}_{\mathfrak{N}}(2)$ (See Theorem 4 on page 181 of [3]). This involves a contradiction, and hence $C_n/B_n \to \infty$ as $n \to \infty$. Q.E.D.

We now consider distribution functions in $\mathfrak{D}(2)$ whose tail probabilities vary regularly with exponent -2. It should be noted that it is possible to have some distribution functions in $\mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$ which do not have tails which vary regularly with exponent -2 and some that do. An example of an $F \varepsilon \mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$ and such that neither $1 - F(x)$ nor $F(-x - 0)$ varies regularly is given in [1], page 279, problem 29. The following such example seems easier. Let F be discrete, taking jumps only at points $2^{n/2}$ of size $1/2^n$, $n = 1, 2, \cdots$. We now prove that $U(x) = \int_0^x t^2 \, dF(t)$ is slowly varying. Indeed, if $x > 1$, then there is an integer r such that $2^{(r-1)/2} < x \leqq 2^{r/2}$ and

$$1 = U(y)/U(y) \leqq U(xy)/U(y) \leqq (U(y) + r)/U(y) \to 1$$

as $y \to \infty$, since clearly $U(y) \to \infty$ as $y \to \infty$. A similar argument holds for $0 < x < 1$. Thus $F \varepsilon \mathfrak{D}(2) \setminus \mathfrak{D}_{\mathfrak{N}}(2)$ by Theorem 1 on page 303 of [1] and Theorem 4 on page 181 of [3]. In order to show that $1 - F$ does not vary regularly with exponent -2, we only need show that if we let

$$\varphi(x) = (1 - F(2^{1/8}x))/(1 - F(x)),$$

then $\varphi(x)$ does not converge to $2^{-\frac{1}{2}}$ as $x \to \infty$. One easily sees that for $x = 2^{(n/2)+(1/8)}$, then $\varphi(x) = 1$, while for $x = 2^{(n/2)-(1/16)}$, then $\varphi(x) = \frac{1}{2}$, which confirms what was last asserted.

An example of a distribution function with a tail that varies regularly with exponent -2 and is in $\mathfrak{D}(2)\backslash\mathfrak{D}_{\mathfrak{N}}(2)$ is not given here. All one needs is any distribution with such a tail probability and infinite variance, and it will turn out, because of Theorem 4, that it is in $\mathfrak{D}(2)\backslash\mathfrak{D}_{\mathfrak{N}}(2)$.

THEOREM 4. *Let F and G be distribution functions such that each tail probability,* i.e., *each function $1 - F(x)$, $F(-x)$, $1 - G(x)$, $G(-x)$ defined for $x > 0$ varies regularly with exponent -2, but allowing each of the four to be associated with possibly a different slowly varying function. Then F, G and $F * G$ are in $\mathfrak{D}(2)$. In particular, if F and G are in $\mathfrak{D}(2)\backslash\mathfrak{D}_{\mathfrak{N}}(2)$, then so is $F * G$.*

PROOF. By Lemma 3, $1 - F * G(x)$ and $F * G(-x)$ each vary regularly with exponent -2. If F and G are both in $\mathfrak{D}(2)\backslash\mathfrak{D}_{\mathfrak{N}}(2)$, then Theorem 4 on page 181 of [3] implies that $F * G \not\varepsilon\, \mathfrak{D}_{\mathfrak{N}}(2)$. It remains then only to prove that F, G and $F * G \,\varepsilon\, \mathfrak{D}(2)$. By Lemma 3 we know that each of the six tail probability functions of F, G and $F * G$ varies regularly with exponent -2, with each of the two tails of the same distribution being associated with possibly different slowly varying functions. We therefore need only prove if H is a distribution function, and if $1 - H(x)$ and $H(-x)$ each vary regularly with exponent -2, then $H \,\varepsilon\, \mathfrak{D}(2)$. One way of proving this is to use Theorem 1 on page 172 of [3] which states: $H \,\varepsilon\, \mathfrak{D}(2)$ if and only if

$$\Phi(x) = x^2 \int_{t>x} dH(t) \Big/ \int_{|t| \leq x} t^2\, dH(t) \to 0. \tag{9}$$

By Lemma 1 it follows that $\int_{|t|>x} dH(t) = x^{-2}L(x)$, where $L(x)$ is a measurable, slowly varying function. Integrating the denominator of (9) by parts and taking reciprocals, we get

$$1/\Phi(x) = 2 \int_0^x (L(t)/t)\, dt / L(x) - 1.$$

Applying Theorem 1 on page 273 of [1] and by taking, in that Theorem, $p = -1$, $\delta = 0$, we have $0 < L(x)/\int_0^x (L(t)/t)\, dt \to 0$ or $1/\Phi(x) \to \infty$ or $\Phi(x) \to 0$ as $x \to \infty$. Q.E.D.

An interesting by-product of this last theorem adds to the exceptional character of the normal distribution within the realm of stable distributions. If $F \,\varepsilon\, \mathfrak{D}(\alpha)$, $0 < \alpha < 2$, if the ratio $(1 - F(x))/F(-x)$ does not converge to either 0 or ∞, and if both $1 - F(x)$ and $F(-x)$ vary regularly with exponent $-\alpha$, then for some constants $c_1 > 0$, $c_2 > 0$, both $c_1x^\alpha(1 - F(x))$ and $c_2x^\alpha F(-x)$ are asymptotically the same slowly varying function, i.e.,

$$c_1x^\alpha(1 - F(x)) \sim c_2x^\alpha F(-x) \sim L(x),$$

where $L(x)$ is a measurable slowly varying function. However, if L_1 and L_2 are any two measurable slowly varying functions, and if H is a distribution function satisfying $1 - H(x) \sim x^{-2}L_1(x)$, $H(-x) \sim x^{-2}L_2(x)$, then $H \,\varepsilon\, \mathfrak{D}(2)$.

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