The Annals of Mathematical Statistics
1968, Vol. 39, No. 3, 905–922

ASYMPTOTIC NORMALITY IN NONPARAMETRIC METHODS[1]

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0. Summary. Let $U_1, U_2, \cdots, U_N$ be a random sample from a population with a continuous distribution function and R_i, $i = 1, \cdots, N$, be the rank of U_i among the N observations. Asymptotic normality is studied for the statistics of the type

(0.1) $$\sum_{i=1}^N \sum_{j=1}^N c_{ij} a_N(R_i/N, R_j/N),$$

where constants c_{ij} satisfy certain negligibility conditions and the score function $a_N(\cdot, \cdot)$ is derived from a function $a(\cdot, \cdot)$ satisfying certain monotonicity and integrability conditions. It is shown that the statistic (0.1) is *asymptotically equivalent* to

(0.2) $$\sum_{i=1}^N \sum_{j=1}^N c_{ij} a(U_i, U_j),$$

so that the problem is reduced to a simpler one, viz. studying the asymptotic distribution of (0.2).

Similar results are obtained for the two sample analog of (0.1) viz.

(0.3) $$\sum_{i=1}^N \sum_{j=1}^M c_{ij} a_{NM}(R_i/N, S_j/M)$$

where S_j, $j = 1, \cdots, M$, are the ranks corresponding to another independent random sample of size M from some other population. Few more variants of the above and applications of these statistics are given.

The present study is a generalization of a paper by Hájek (1961).

1. Introduction. In the present day literature on nonparametric methods one finds three basic methods to study asymptotic distributions. The first one, known as the U-statistic method, was suggested by Hoeffding (1948). Although this method established asymptotic normality of many useful statistics, the class of rank score statistics was still outside its framework. A conjecture of Hodges and Lehmann regarding the superiority of the normal scores test over the t-test (à la Pitman efficiency) inspired Chernoff and Savage (1958) to study the asymptotic normality for the rank score statistics employed in the two sample location problem.

A third approach initiated by Wald and Wolfowitz (1944) was studied by several authors. Hájek (1961) led it to completion by giving useful necessary and sufficient conditions for the basic theorem regarding the asymptotic normality. This study was exploited further by Hájek (1962) to study the regression

Received 13 October 1964; revised 8 January 1968.

[1] Part of this paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under a Ford Foundation Grant for statistics.

problem. By using the concept of contiguity due to Le Cam (1960), Hájek (1962) obtained asymptotic relative efficiencies for various pairs of statistics.

The last two methods mentioned above have become basic in studying the asymptotic distribution of a nonparametric statistic. The Chernoff-Savage method has an advantage of yielding results for the so-called *fixed alternative.* However, the necessary computations become too involved for certain generalizations. If the aim of the study is to find Pitman efficiency then Hájek's method could be considered to be simpler and at the same time the class of statistics is wider.

In the present paper we have adopted the Hájek approach. The class of statistics is broad enough to embrace rank score functions having more than one argument; a typical need while testing independence in a bivariate population or testing serial independence of the observations in a sample. Wald-Wolfowitz (1943), Noether (1950) and Ghosh (1954) studied tests for serial dependence based on permutations of observations. The present method has a wider basis of applications and also has the following advantage. Typically, a nonparametric test statistic is shown to be asymptotically equivalent to a statistic of a simpler form which may be handled easily under the null hypothesis. This equivalence may be further exploited by appealing to the contiguity techniques of Le Cam and Hájek (see [6]).

Although not a prerequisite, some familiarity with the paper by Hájek (1961) would facilitate the reading of this paper.

2. Applications.

(a) *Testing the hypothesis of independence in bivariate populations against linear alternatives.* Let (X_i, Y_i) be a random sample of N paired observations from a continuous bivariate distribution function. Let R_i and S_i be the ranks of X_i and Y_i among X and Y observations respectively. It is desired to test the independence of X and Y.

Various linear alternatives may be considered. Bhuchongkul (1964) studied the following model,

$$X = \theta Z_1 + (1 - \theta)Z_2, \qquad Y = \theta Z_1 + (1 - \theta)Z_3, \tag{2.1}$$

where $0 < \theta < 1$ and Z_1, Z_2, Z_3 are independent random variables. It was shown that the normal score test statistic

$$N^{-\frac{1}{2}} \sum_{i=1}^{N} E_{R_i} E_{S_i}, \tag{2.2}$$

leads to an asymptotically efficient test for normal alternatives and has the same asymptotic properties as those of the normal score test statistic for the two sample problem. Here E_i is the expected value of the ith largest observation in the random sample of size N from a standard normal population.

The statistic (2.2) is clearly a special case of (0.1) and it will be seen later that the function

$$a_N(i/N, j/N) = E_i E_j \tag{2.3}$$

satisfies the conditions for being asymptotically equivalent to a statistic of the form

$$N^{-\frac{1}{2}} \textstyle\sum_{i=1}^{N} \phi(U_i)\phi(V_i), \tag{2.4}$$

where the function ϕ is related to the normal distribution function and $\{U_i\}$, $\{V_i\}$ are two independent random samples from the uniform distribution on $(0,1)$.

Asymptotic normality of (2.4) and hence that of (2.2) follows from well-known central limit theorems. In the case of regression alternatives,

$$Y = \alpha + \beta X + \sigma Z, \tag{2.5}$$

where X and Z are independent, the author (1962) has shown that the test statistic

$$N^{-\frac{1}{2}} \textstyle\sum_{i=1}^{N} X_i E_{S_i} \tag{2.6}$$

has some attractive asymptotic properties. Again (2.6) is another variant studied in the present paper and considerations of the asymptotic equivalence lead to a direct application of the standard central limit theorems.

(b) *Spearman's "foot-rule."* With the same notation as above, a measure of dependence based on

$$N^{-\frac{1}{2}}(N + 1)^{-1} \textstyle\sum_{i=1}^{N} |R_i - S_i|, \tag{2.7}$$

is known in the literature as *Spearman's foot-rule*. Although this is thought to be a crude measure, it can be shown that the test based on (2.7) is asymptotically efficient for testing independence of X and Y against contiguous nonlinear alternatives given by the bivariate distribution function

$$H(x, y) = F(x)G(y)[1 + \alpha\, |F(x) - G(y)|], \tag{2.8}$$

where α is a small positive number.

The results of this paper can be applied to prove that (2.7) is asymptotically equivalent to the statistic

$$N^{-\frac{1}{2}} \textstyle\sum_{i=1}^{N} |F(X_i) - G(Y_i)|, \tag{2.9}$$

where X and Y are independent. The asymptotic normality of (2.9), when properly normalized, can be handled very easily.

(c) *Tests for serial dependence.* Suppose $X_1, \cdots, X_N$ are N observations on a process taken at N successive times. It is desired to test the null hypothesis that the N observations constitute a random sample, against the alternative hypothesis of serial dependence of the first order.

Under the assumption of joint normality, it was shown by Anderson (1948) that the test based on the statistic

$$N^{-\frac{1}{2}} \textstyle\sum_{i=1}^{N-1} (X_i - X_{i+1})^2 \tag{2.10}$$

is UMP unbiased. However if the underlying distribution is not normal then the rank analog of (2.10) becomes

$$N^{-\frac{1}{2}}(N+1)^{-2}\sum_{i=1}^{N-1} R_i R_{i+1}, \tag{2.11}$$

where R_i is the rank of X_i. The statistic (2.11) is a special case of (0.1). (The author has been informed that the results of the present paper have been used by Aiyer (1968) to study the nonparametric tests for serial dependence.) It should be noted that under the assumption of normality, the distribution of the statistic (2.10) does not have a closed form even in the case of null hypothesis. Thus the rank analog has the added advantage of having a null distribution which can be computed exactly.

3. Inequalities. Let $U_1, \cdots, U_N$ be independent uniform random variables on $[0, 1]$ and R_i be the ranks of U_i. Let $Z_1 < Z_2 < \cdots < Z_N$ be the corresponding order statistic, so that

$$U_i = Z_{R_i}. \tag{3.1}$$

Let $\{a_{ij}\}$ be a set of N^2 real numbers and $a_{..}$ be their average.

Definition 3.1. A collection of N^2 numbers a_{ij} is said to possess Δ-monotonicity if

$$\Delta_{ij} = a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1} + a_{ij} \geqq 0 \quad \text{for all} \quad i, j, \tag{3.2}$$

or

$$\Delta_{ij} \leqq 0 \quad \text{for all} \quad (i, j).$$

The condition of Δ monotonicity is satisfied in most of the practical applications. For example the rank scores used in the statistics (2.2), (2.7) and (2.11) do satisfy this condition.

Theorem 3.1. *If the set $\{a_{ij}\}$ is Δ-monotone then*

$$E[a_N(U_1, U_2) - a_N(R_1/N, R_2/N)]^2 \leqq C\,(N+1)^{-\frac{1}{2}}\max\,(a_{ij} - a_{..})^2, \tag{3.3}$$

where C is a positive constant and a_N is defined by

$$a_N(\lambda, \theta) = a_{ij}; \quad (i-1)/N < \lambda \leqq i/N, \quad (j-1)/N < \lambda \leqq j/N. \tag{3.4}$$

The proof will be based on the following lemmas.

Lemma 3.1. *For the special case,*

$$\begin{aligned} \epsilon(\lambda, \theta) &= 1 \quad \textit{if} \quad \lambda > 0 \quad \textit{and} \quad \theta > 0. \\ &= 0 \quad \textit{otherwise}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} E[\epsilon(U_1 - k/N, U_2 - l/N) - \epsilon(R_1 - k)/N, (R_2 - l)/N)]^2 \\ \leqq 3(N-1)^{-1}N^{-\frac{1}{2}}(N-k)^{\frac{1}{2}}(N-l)^{\frac{1}{2}}, \end{aligned} \tag{3.6}$$

where k and l are fixed positive integers.

Proof. For $Z_1 < \cdots < Z_N$ fixed, let K and L denote the number of Z less than k/N and less l/N respectively. If $K \leqq k$ and $L \leqq l$ then it is obvious that

$$\begin{aligned}(3.7)\quad & \epsilon[Z_i - k/N,\ Z_j - l/N] - \epsilon[(i-k)/N,\ (j-l)/N] \\ & \quad = 1 \quad \text{if either } K < i \leqq k, L < j, \\ & \qquad\qquad \text{or} \quad K < i, L < j \leqq l, \\ & \quad = 0 \quad \text{otherwise.}\end{aligned}$$

By obvious geometric considerations it can be seen that for the case $K \leqq k$, $L \leqq l$, the number of pairs (i, j) such that the difference (3.7) is unity is $(k - K)(l - L) + (N - k)(l - L) + (N - l)(k - K)$.

In general, for any values of K and L the upper bound on the number of pairs (i, j) with difference (3.7) equal to ± 1 is $|K - k||L - l| + (N - k)|L - l| + (N - l)|K - k|$. Hence for $Z_1 < Z_2 < \cdots < Z_N$ fixed, the left side of (3.6) is

$$\begin{aligned}(3.8)\quad & (N(N-1))^{-1} \sum \sum_{i \neq j} \{\epsilon[Z_i - k/N, Z_j - l/N] \\ & - \epsilon[(i-k)/N, (j-l)/N)\}^2 \leqq (N(N-1))^{-1}\{|K - k||L - l| \\ & + (N-k)|L-l| + (N-l)|K-k|\}.\end{aligned}$$

(Throughout the paper the suppression of the limits would mean that the summation is taken over all possible values.)

When the statistic $Z_1 < \cdots < Z_N$ is not fixed, K and L are binomial random variables and the inequality (3.6) follows by taking the expected value on both sides of (3.8) and applying the Schwarz inequality. The proof of Lemma 3.1 is completed.

REMARK. When the elementary function ϵ has one argument the right side of (3.8) becomes $(1/N)|K - k|$. In the present case, in addition to an analogous term there are terms of higher order in $(1/N)$. This makes the upper bound (see (3.6)) tend to zero at a slower rate when compared with the one argument case (see (3.9) below). Consequently, this results in a higher moment condition on the function $a(\lambda, \theta)$ (see Theorem 4.2).

For the sake of future reference and to facilitate the comparison between the present case and the one argument case the following upper bound obtained by Hájek (1961) is stated.

$$\begin{aligned}(3.9)\quad & E[a_N(U_1) - a_N(R_1/N)]^2 \\ & \leqq CN^{-1} \max_{1 \leqq i \leqq N} |a_i - a.|[\textstyle\sum_{i=1}^N (a_i - a.)^2]^{\frac{1}{2}},\end{aligned}$$

where C is a positive constant and the other notation is obvious. (Throughout the paper the letter C with or without subscripts will be used as a generic notation for a positive constant.)

The elementary function ϵ is used in the following construction. Let

$$\begin{aligned}(3.10)\quad & b_{ij} = a_{ij} - a_{i1} - a_{1j} + a_{11} \\ & b(\lambda, \theta) = a_N(\lambda, \theta) - a_N(\lambda, 1/N) - a_N(1/N, \theta) + a_N(1/N, 1/N).\end{aligned}$$

Recalling the definition of Δ_{ij} in (3.2) it readily follows that

$$
\begin{aligned}
(3.11)\qquad b_{NN} &= a_{NN} - a_{N1} - a_{1N} + a_{11} = \textstyle\sum_k \sum_l \Delta_{kl} \\
b_{NN}^2 &= \textstyle\sum_k \sum_l \sum_m \sum_n \Delta_{kl}\Delta_{mn}\,; \\
b_{ij} &= \textstyle\sum_k \sum_l \Delta_{kl}\epsilon((i-k)/N,\,(j-l)/N), \\
(3.12)\qquad b(\lambda,\theta) &= \textstyle\sum_k \sum_l \Delta_{kl}\epsilon(\lambda - k/N,\,\theta - l/N), \\
&\qquad 0 < \lambda < 1,\, 0 < \theta < 1.
\end{aligned}
$$

Using the expression (3.12) it follows that

$$
\begin{aligned}
(3.13)\qquad \textstyle\sum_i \sum_j b_{ij}^2 &= \textstyle\sum_k \sum_l \sum_m \sum_n \Delta_{kl}\Delta_{mn} \\
&\cdot \textstyle\sum_i \sum_j \epsilon((i-k)/N,\,(j-l)/N)\epsilon((i-m)/N,\,(j-n))/N).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\epsilon((i-k)/N,\,(j-l)/N)\epsilon((i-m)/N,\,(j-n)/N) \\
(3.14)\qquad &= 1 \quad \text{for } i > \max(k, m) \text{ and } j > \max(l, n) \\
&= 0 \quad \text{otherwise,}
\end{aligned}
$$

for a fixed (k, l) and (m, n), the number of pairs (i, j) such that the left side of (3.14) is unity, equals $[N - \max(k, m)]\ [N - \max(l, n)]$. Hence

$$
\begin{aligned}
(3.15)\qquad \textstyle\sum_i \sum_j b_{ij}^2 &= \textstyle\sum_k \sum_l \sum_m \sum_n \Delta_{kl}\Delta_{mn} \\
&\cdot [N - \max(k, m)][N - \max(l, n)].
\end{aligned}
$$

LEMMA 3.2.

$$
\begin{aligned}
&\{\epsilon(Z_i - k/N,\, Z_j - l/N) - \epsilon((i-k)/N,\,(j-l)/N)\} \\
(3.16)\qquad &\cdot\{\epsilon(Z_i - m/N,\, Z_j - n/N) - \epsilon((i-m)/N,\,(j-n)/N)\} \\
&\leqq \{\epsilon(Z_i - \max(k, m)/N,\, Z_j - \max(l, n)/N) \\
&\qquad - \epsilon(i - \max(k, m)/N,\, j - \max(l, n)/N)\}^2.
\end{aligned}
$$

PROOF. Since $\epsilon(\cdot, \cdot)$ takes only two values, 0 or 1, it suffices to prove that the left side is not $+1$ when the right side is 0.

The right side is 0 implies one of the following:

(i) Both ϵ terms on the right side are 0. In this case it can be seen that the two terms which make up the product on the left could not be $+1$ or -1 simultaneously.

(ii) Both ϵ terms on the right side are 1. In this case all the entries on the left side are 1 and the left side is zero.

These being the only cases, Lemma 3.2 is proved.

LEMMA 3.3. *With the above notation if the set* $\{a_{ij}\}$ *is* Δ *monotone then*

$$E[b(U_1, U_2) - b(R_1/N, R_2/N)]^2 \leqq 3|b_{NN}|N^{-\frac{1}{2}}(N-1)^{-1}[\textstyle\sum_i \sum_j b_{ij}^2]^{\frac{1}{2}}. \tag{3.17}$$

PROOF. Using (3.12) and Lemma 3.2 it follows that

$$\begin{aligned}
&E[b(U_1, U_2) - b(R_1/N, R_2/N)]^2 \\
&\quad = (N(N-1))^{-1}E\{\textstyle\sum\sum_{i\neq j}[b(Z_i, Z_j) - b(i/N, j/N)]^2\} \\
&\quad = (N(N-1))^{-1}\textstyle\sum_k\sum_l\sum_m\sum_n \Delta_{kl}\Delta_{mn} \\
&\qquad \cdot E\{\textstyle\sum\sum_{i\neq j}\{\epsilon(Z_i - k/N, Z_j - l/N) - \epsilon((i-k)/N, (j-l)/N)\}\} \\
&\qquad \cdot\{\epsilon(Z_i - m/N, Z_j - n/N) - \epsilon((i-m)/N, (j-n)/N)\} \\
&\quad \leqq (N(N-1))^{-1}\textstyle\sum_k\sum_l\sum_m\sum_n \Delta_{kl}\Delta_{mn} \\
&\qquad \cdot E\{\textstyle\sum\sum_{i\neq j}\{\epsilon(Z_i - \max(k,m)/N, Z_j - \max(l,n)/N)\} \\
&\qquad - \epsilon(i - \max(k,m))/N, (j - \max(l,n))/N)\}^2 \\
&\quad = \textstyle\sum_k\sum_l\sum_m\sum_n \Delta_{kl}\Delta_{mn}E[\epsilon(U_1 - \max(k,m)/N, \\
&\qquad\quad U_2 - \max(l,n)/N) - \epsilon((R_1 - \max(k,m))/N, \\
&\qquad\quad (R_2 - \max(l,n))/N)]^2 \\
&\quad \leqq 3N^{-\frac{1}{2}}(N-1)^{-1}\textstyle\sum_k\sum_l\sum_m\sum_n \Delta_{kl}\Delta_{mn}[N - \max(k,m)]^{\frac{1}{2}} \\
&\qquad [N - \max(l,n)]^{\frac{1}{2}}.
\end{aligned} \tag{3.18}$$

The last inequality follows from Lemma 3.1.

Finally using the fact that $\Delta_{kl}\Delta_{mn} \geqq 0$ for all k, l, m, n the Cauchy inequality simplifies the upper bound of (3.8) as

$$\begin{aligned}
&E[b(U_1, U_2) - b(R_1/N, R_2/N)]^2 \\
&\quad \leqq 3\,N^{-\frac{1}{2}}(N-1)^{-1}[\textstyle\sum_k\sum_l\sum_m\sum_n \Delta_{kl}\Delta_{mn}]^{\frac{1}{2}} \\
&\qquad \cdot[\textstyle\sum\sum\sum\sum \Delta_{kl}\Delta_{mn}(N - \max(k,m))(N - \max(l,n))]^{\frac{1}{2}} \\
&\quad = 3\,N^{-\frac{1}{2}}(N-1)^{-1}|b_{NN}|[\textstyle\sum\sum b_{ij}^2]^{\frac{1}{2}}.
\end{aligned} \tag{3.19}$$

The last equality follows from (3.11) and (3.15).

PROOF of THEOREM 3.1. From the elementary inequality

$$(x + y + z)^2 \leqq 3x^2 + 3y^2 + 3z^2 \tag{3.20}$$

one obtains

$$\begin{aligned}
&E[a_N(U_1, U_2) - a_N(R_1/N, R_2/N)]^2 \leqq 3E[a_N(U_1, U_2) \\
&\quad - a_N(R_1/N, R_2/N) - a_N(U_1, 1/N) + a_N(R_1/N, 1/N) \\
&\quad - a_N(1/N, U_2) + a_N(1/N, R_2/N)]^2 + 3E[a_N(U_1, 1/N) \\
&\quad - a_N(R_1/N, 1/N)]^2 + 3E[a_N(1/N, U_2) - a_N(1/N, R_2/N)]^2.
\end{aligned} \tag{3.21}$$

Lemma 3.3 when expressed in terms of the a_N function is applicable to the first summand on the right side of (3.21). The inequality (3.9) can be applied to the other two summands and the required upper bound in (3.3) is obtained after some obvious simplifications.

Using the same procedure as above and under the same conditions as in Theorem 3.1, it follows that

$$(3.22) \quad E[a_N(R_1/N, R_2/N) - a_N(U_1, R_2/N)]^2 \leqq C\,(N-1)^{-\frac{1}{2}} \max (a_{ij} - a_{..})^2.$$

REMARK. Note that the joint distribution of $(R_1, \cdots, R_N)$ remains the same under any permutation and thus the statistic (0.1) is invariant under any changes in the subscript labels of the a_{ij}. Hence, Δ-monotonicity condition will be automatically satisfied if we can arrange the numbers a_{ij} in a matrix such that $\Delta_{ij} \geqq 0$, for all i, j. In the one argument case this can be done trivially; however, it is not clear whether an arbitrary set of N^2 real numbers can be arranged Δ-monotonically. It should be noted that if the a_{ij} are generated by a smooth function $a(\lambda, \theta)$ then Δ-monotonicity is equivalent to requiring a constant sign for the determinant

$$\begin{vmatrix} \partial^2 a/\partial\lambda^2 & \partial^2 a/\partial\lambda\,\partial\theta \\ \partial^2 a/\partial\lambda\,\partial\theta & \partial^2 a/\partial\theta^2 \end{vmatrix}.$$

For the present purpose the following condition is sufficient and is less restrictive than the requirement of Δ-monotonicity.

DEFINITION 3.2. A set of N^2 numbers $\{a_{ij}\}$ is said to be piecewise Δ-monotone if a_{ij} can be expressed as

$$a_{ij} = a_{ij}^{(1)} + a_{ij}^{(2)} + \cdots + a_{ij}^{(k)},$$

where the sets $\{a_{ij}^{(l)}\}$, $l = 1, \cdots, k$, are Δ-monotone, and k does not depend on N.

It is clear that by the repeated use of inequality (3.3) one can obtain an upper bound of the order $O(N^{-\frac{1}{2}})$ if the set $\{a_{ij}\}$ is piecewise Δ-monotone. The main idea is to find the regions where Δ_{ij} has the same sign and then express a_{ij} as above. This is similar to defining positive and negative parts of a function.

In order to avoid complication of the notation, the assumption of Δ-monotonicity will be made instead of piecewise Δ-monotonicity, always keeping in mind that the results hold with the latter assumption.

The inequality corresponding to (3.3) for the two sample case can be obtained more economically. Note that the arguments of the function $a_{NM}(R_1/N, S_1/M)$ are independent. This fact can be exploited by conditioning one of the arguments to obtain a slightly sharper upper bound as given by the following theorem. Since the proof is straightforward it is omitted. (To simplify the notation $a_{NM}(\cdot,\cdot)$ will be written henceforth as $a_N(\cdot, \cdot)$.)

THEOREM 3.2. *Let $\{U_i\}$, $i = 1, \cdots, N$, and $\{V_j\}$, $j = 1, \cdots, M$, be two sets of mutually independent random variables, R_i be the rank of U_i among $U_1, \cdots, U_N$ and S_j be the rank of V_j among $V_1, \cdots, V_M$ and let a_N be the step function defined* *previously. Then*

$$(3.23)\quad E[a_N(U_1, V_1) - a_N(U_1, S_1/M)]^2 \leqq [C(NM)^{-1}\textstyle\sum_{i=1}^N \max_j a_{ij}^2]^{\frac{1}{2}}[(NM)^{-1}\sum_i \sum_j (a_{ij} - a_{..})^2]^{\frac{1}{2}};$$

$$(3.24)\quad E[a_N(U_1, V_1) - a_N(R_1/N, S_1/M)]^2 \leqq C\{(NM)^{-1}[\textstyle\sum_{i=1}^N \max_j a_{ij}^2 + \sum_{j=1}^M \max_i a_{ij}^2]\}^{\frac{1}{2}} \cdot\{(NM)^{-1}\sum_i \sum_j (a_{ij} - a_{..})^2\}^{\frac{1}{2}}.$$

4. Asymptotically equivalent statistics. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables having finite variances, defined on a probability space $(\Omega, \mathcal{A}, P)$. The sequence $\{X_n\}$ is said to be asymptotically equivalent to $\{Y_n\}$ in the mean, or simply asymptotically equivalent to $\{Y_n\}$ and the equivalence is denoted by $X_n \sim Y_n$ if

$$(4.1)\qquad E[X_n - Y_n]^2/\text{Var } X_n \to 0 \quad \text{as} \quad n \to \infty.$$

LEMMA 4.1. (a) $X_n \sim Y_n$ *implies*

$$(4.2)\qquad \text{Var } Y_n/\text{Var } X_n \to 1 \quad \textit{as} \quad n \to \infty.$$

(b) *If*
(i) $E[X_n - Y_n]^2 \to 0$
(ii) $\text{Var } X_n$ (*or* $\text{Var } Y_n$) *remains bounded away from* 0 *then*
(1) $X_n \sim Y_n$,
(2) $X_n \to_P X \Leftrightarrow Y_n \to_P X$,
(3) $\mathcal{L}(X_n) \to \mathcal{L}(X) \Leftrightarrow \mathcal{L}(Y_n) \to \mathcal{L}(X)$.

PROOF. Let $\mu_n = EX_n$ and $\eta_n = EY_n$. The convergence (4.1) implies

$$(4.3)\quad \begin{aligned} &E|(X_n - \mu_n)(Y_n - X_n)|/\text{Var } X_n \\ &\quad\leqq \{E(X_n - \mu_n)^2 E(Y_n - X_n)^2\}^{\frac{1}{2}}/\text{Var } X_n \\ &\quad= [E(Y_n - X_n)^2/\text{Var } X_n]^{\frac{1}{2}} \to 0, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence,

$$(4.4)\quad \begin{aligned} &E(Y_n - \mu_n)^2/\text{Var } X_n \\ &\quad= E[Y_n - X_n + X_n - \mu_n]^2/\text{Var } X_n \\ &\quad= E(Y_n - X_n)^2/\text{Var } X_n + 2E(X_n - \mu_n)(Y_n - X_n)/\text{Var } X_n \\ &\qquad+ 1 \to 1, \quad \text{as} \quad n \to \infty. \end{aligned}$$

Further

$$(4.5)\quad (\mu_n - \eta_n)^2/\text{Var } X_n \leqq E(X_n - Y_n)^2/\text{Var } X_n \to 0 \quad \text{as} \quad n \to \infty,$$

and hence

$$(4.6)\quad \text{Var } Y_n/\text{Var } X_n = [E(Y_n - \mu_n)^2 - (\mu_n - \eta_n)^2]/\text{Var } X_n \to 1 \quad \text{as} \quad n \to \infty.$$

This proves the assertion (a). The assertions in (b) are obvious.

It should be noted that $X_n \sim Y_n$ alone is not sufficient for the assertion (2) of (b). Also, some simple examples could be constructed to show that

$$(4.7) \quad X_n \sim Y_n, \qquad Z_n \sim W_n \text{ does not imply that } X_n + Z_n \sim Y_n + W_n .$$

Thus the asymptotic equivalence without conditions (i) and (ii) of (b) needs some caution in its use.

In the present section it will be shown that with certain assumptions regarding the coefficients c_{ij}, condition (b)(i) of Lemma 4.1, holds for the following statistics,

$$(4.8) \quad \begin{aligned} S_N &= \textstyle\sum_{i=1}^N \sum_{j=1}^N c_{ij} a_{R_i R_j}, \\ T_N &= \textstyle\sum\sum_{i\neq j} (c_{ij} - \bar{c}) a_N(U_i, U_j) + \sum_i (c_{ii} - \hat{c}) a_N(U_i, U_i) \\ &\quad + \bar{c} \textstyle\sum\sum_{i\neq j} a_{ij} + \hat{c} \sum_i a_{ii}, \\ S_N^1 &= \textstyle\sum_{i=1}^N \sum_{j=1}^N c_{ij} a_N(U_i, R_j/N), \end{aligned}$$

where

$$(4.9) \qquad \bar{c} = (N(N-1))^{-1} \textstyle\sum\sum_{i\neq j} c_{ij}, \qquad \hat{c} = N^{-1} \sum_i c_{ii} .$$

The two sample variants of the above statistics are

$$(4.10) \quad \begin{aligned} S_N^* &= \textstyle\sum_{i=1}^N \sum_{j=1}^M c_{ij} a_{R_i, S_j}, \\ T_N^* &= \textstyle\sum_{i=1}^N \sum_{j=1}^M (c_{ij} - c_{..}) a_N(U_i, V_j) + c_{..} \sum_{i=1}^N \sum_{j=1}^M a_{ij}, \\ S_N^{**} &= \textstyle\sum_{i=1}^N \sum_{j=1}^M c_{ij}\, a_N(U_i, S_j/M), \end{aligned}$$

where

$$(4.11) \qquad c_{..} = (NM)^{-1} \textstyle\sum_{i=1}^N \sum_{j=1}^M c_{ij} .$$

THEOREM 4.1. *With the same notation as above assume that the coefficients* c_{ij} *satisfy the following conditions uniformly in* N *(and* M *in the two sample case)*

$$\text{(i)} \quad \textstyle\sum_i \sum_j c_{ij}^2 < C_1,$$

$$\text{(ii)} \quad \textstyle\sum_i [\sum_j c_{ij}]^2 < C_2, \quad \sum_j [\sum_i c_{ij}]^2 < C_3 .$$

(a) *In addition to* (i) *and* (ii) *if*

$$\lim\, (NM)^{-1} \textstyle\sum_{i=1}^N \max_j a_{ij}^2 = \lim\, (NM)^{-1} \sum_{j=1}^M \max_i a_{ij}^2 = 0 \quad \text{as} \quad N, M \to \infty .$$

Then

$$(4.12) \quad E[S_N^* - T_N^*]^2 \to 0, \qquad E[S_N^* - S_N^{**}]^2 \to 0 \quad \text{as} \quad N \text{ and } M \to \infty .$$

(b) *In addition to* (i) *and* (ii), *if*

$$(\mathrm{b}_1) \qquad \{a_{ij}\} \text{ are } \Delta\text{-monotone},$$

$$(\mathrm{b}_2) \qquad \lim_{N\to\infty} N^{-1} \max_{1 \leq (i,j) \leq N} (a_{ij} - a_{..})^4 = 0$$

then

$$E[T_N - S_N]^2 \to 0, \qquad E[T_N - S_N^1]^2 \to 0 \quad as \quad N \to \infty. \tag{4.13}$$

REMARKS. The conditions (i) and (ii) of Theorem 4.1 are satisfied in all applications cited in Section 2. These conditions make the coefficients c_{ij} play the role of normalizing constants for the statistics which are represented as sums.

Although the conditions on $\{a_{ij}\}$ in the present form look artificial these will be found satisfied if the step functions a_N satisfy certain uniform integrability conditions.

In the one argument case S_N and T_N reduce to

$$S_N' = \sum_{i=1}^N c_i a_{R_i}, \qquad T_N' = \sum_{i=1}^N (c_i - c.)a_N(U_i) + c.\sum_{i=1}^N a_i, \tag{4.14}$$

where the notation is obvious. In this case,

$$\text{Var } S_N' = (N-1)^{-1}\sum_{i=1}^N (c_i - c.)^2 \sum_{i=1}^N (a_i - a.)^2 \tag{4.15}$$

and

$$E[S_N' - T_N']^2 \leqq \sum_i (c_i - c.)^2 E[a_N(U_1) - a_{R_1}]^2. \tag{4.16}$$

Thus with some mild conditions on the a_i, and almost without any restrictions on the c_i (excepting that they are not all the same), the ratio

$$E[S_N' - T_N']^2/\text{Var } S_N' \to 0 \quad \text{as} \quad N \to \infty. \tag{4.17}$$

Due to this fact, Hájek (1961) could separate the conditions for the asymptotic equivalence of S_N' and T_N' and those for the asymptotic normality of S_N'. Unfortunately, for the present case which involves two arguments, the expression for Var S_N (see A_1 of the Appendix) is too complicated to afford such an elegance.

PROOF OF THEOREM 4.1. From (4.8) it follows that

$$\begin{aligned} T_N - S_N &= \sum\sum_{i \neq j} (c_{ij} - \bar{c})[a_N(U_i, U_j) - a_{R_i,R_j}] \\ &\quad + \sum_i (c_{ii} - \hat{c})[a_N(U_i, U_i) - a_{R_i,R_i}] \\ &= \sum\sum_{i \neq j} (c_{ij} - \bar{c})[a_N(Z_{R_i}, Z_{R_j}) - a_{R_i,R_j}] \\ &\quad + \sum_i (c_{ii} - \hat{c})[a_N(Z_{R_i}, Z_{R_i}) - a_{R_i,R_i}]; \end{aligned} \tag{4.18}$$

so that

$$E[T_N - S_N \mid Z_1, \cdots, Z_N] = 0; \tag{4.19}$$

$$\begin{aligned} E[T_N - S_N \mid Z_1, \cdots, Z_N]^2 &= \text{Var }[S_N - T_N \mid Z_1, \cdots, Z_N] \\ &\leqq C\{E[a_N(Z_{R_1}, Z_{R_2}) - a_{R_1,R_2} \mid Z_1, \cdots, Z_N]^2 \\ &\quad + E[a_N(Z_{R_1}, Z_{R_1}) - a_{R_1,R_1} \mid Z_1, \cdots, Z_N]^2\}, \end{aligned} \tag{4.20}$$

where the last inequality follows from the Lemma A_1 of the Appendix. Taking the expected value on both sides of (4.20), and applying equation (3.8) and

Theorem 3.1 the first convergence in (4.13) follows. The second follows in the same manner. The proof for part (a) of the theorem is along similar lines, and is omitted.

Our main concern for the rest of this section is to find interpretation of the conditions of Theorem 4.1 for practical applications. Usually the choice of rank scores in a given situation is made according to the density function which is suspected to be the underlying one. Thus the numbers $\{a_{ij}\}$, or equivalently the function a_N, is constructed from $a(\lambda, \theta)$ which is related to a density function and the particular nature of the alternatives involved in the problem. For example, suppose F is the distribution function which is thought to be likely for the marginals of both the components of a paired random variable. Then, while testing independence in model (2.1) one may consider

$$a(\lambda, \theta) = \xi(\lambda)\xi(\theta), \tag{4.21}$$

where

$$\xi(\lambda) = f'[F^{-1}(\lambda)]/f[F^{-1}(\lambda)], \qquad 0 < \lambda < 1. \tag{4.22}$$

Returning to the general case suppose a_{ij} are defined by the relation

$$a_{ij} = a(i/(N+1), j/(N+1)), \qquad i, j = 1, \cdots, N; \tag{4.23}$$

then the step function $a_N(\lambda, \theta)$ will approximate the function $a(\lambda, \theta)$.

In what follows, we want to find (1) sufficient conditions on the function $a(\lambda, \theta)$ and (2) some specific constructions of the step function $a_N(\lambda, \theta)$, which together will imply the condition (a) or (b_2) or Theorem 4.1.

Let $a(\lambda, \theta)$ be a nonconstant real valued function on $(0, 1)^2$. Let the step function $a_N(\lambda, \theta)$, which is assumed to be constant over open squares $(i/N, (i+1)/N) \times (j/N, (j+1)/N)$ for $i, j = 1, \cdots, N-1$; be such that

$$a_N(\lambda, \theta) \to a(\lambda, \theta) \quad \text{as} \quad N \to \infty, \tag{4.24}$$

pointwise.

Lemma 4.2. *The conditions* (a) *of Theorem* 4.1 *are satisfied if* $[a_N(\lambda, \theta)]^4$ *are uniformly integrable and the ratio* N/M *is bounded away from* 0 *and* ∞, *while uniform integrability of* $[a_N(\lambda, \theta)]^8$ *suffices for* (b_2).

Proof. Recall that $a_{ij} = a_N(i/N, j/N)$. The uniform integrability of a_N^4 implies

$$\begin{aligned} \{(NM)^{-1} \textstyle\sum_{i=1}^N \max_j a_{ij}^2\}^2 &\leq M^{-2} \max_{i,j} a_{ij}^4 \\ &= NM^{-1} \textstyle\int_{(j-1)/M}^{j/M} \int_{(i-1)/N}^{i/N} a_N^4(\lambda, \theta)\, d\lambda\, d\theta \to 0 \quad \text{as} \quad N, M \to \infty. \end{aligned} \tag{4.25}$$

The rest of the assertions follow in the same manner.

The uniform integrability conditions will be satisfied if $a(\lambda, \theta)$ is piecewise monotone in λ and θ, belongs to the space L_8, and $a_N(\lambda, \theta)$ is defined suitably. The following are two such constructions

$$a_N(\lambda, \theta) = a(i/(N+1), j/(N+1)); \tag{4.26}$$

$$a_N(\lambda, \theta) = N^2 \textstyle\int_{(i-1)/N}^{i/N} \int_{(j-1)/N}^{j/N} a(\lambda, \theta)\, d\lambda\, d\theta, \tag{4.27}$$

where in both cases $(i-1)/N < \lambda \leqq i/N$ and $(j-1)/N < \theta \leqq j/N$.

Lemma A_2 of the Appendix shows that construction (4.26) above gives uniform integrability while from the proof of Lemma 4.2 it is clear that (4.27) also satisfies the conditions (a) and (b_2).

The condition of monotonicity in Lemma A_2, required of $a(\lambda, \theta)$ can be relaxed and replaced by that of piecewise monotonicity, i.e., $a(\lambda, \theta)$ should be expressible as a linear combination of monotone functions. This amounts to restricting oneself to those functions which do not oscillate too much.

Finally, the conditions (i) and (ii) of Theorem 4.1 may be interpreted as those required for the normalization of the sum. The order of c_{ij} depends on how many coefficients are nonzero. For example, if there are N such then c_{ij} is usually $O(N^{-\frac{1}{2}})$.

The above discussion leads to the following statement of the main theorem.

THEOREM 4.2. *Let $U_1, U_2, \cdots, U_N$; $V_1, \cdots, V_N$ be independent uniform* (0, 1) *random variables. Assume that*

(i) $$\sum_i \sum_j c_{ij}^2 < C_1,$$

(ii) $$\sum_i [\sum_j c_{ij}]^2 < C_2, \qquad \sum_i [\sum_j c_{ij}]^2 < C_3,$$

(iii) $a(\lambda, \theta)$ *is piecewise monotone,*

(iv) $a_N(\lambda, \theta)$ *is constructed either from* (4.26) *or from* (4.27).

In addition to the above assumptions, if

(a) $$\int_0^1 \int_0^1 a^4(\lambda, \theta)\, d\lambda\, d\theta < \infty$$

then as $N \to \infty$,

$$E[S_N^* - T_N^*]^2 \to 0. \tag{4.28}$$

(b) *Under more stringent conditions viz.*

(b_1) *the numbers* $a_N(i/N, j/N)$ *are* Δ *monotone,*

(b_2) $$\int_0^1 \int_0^1 a^8(\lambda, \theta)\, d\lambda\, d\theta < \infty,$$

the following holds:

$$E[S_N - T_N]^2 \to 0, \qquad E[S_N^1 - T_N]^2 \to 0 \quad \text{as} \quad N \to \infty. \tag{4.29}$$

REMARK. The statistics T_N and T_N^* could further be modified by replacing $a_N(\cdot, \cdot)$ with $a(\cdot, \cdot)$ in (4.8) and (4.10). If the conditions of Theorem 4.2 hold and if the variances of T_N and T_N^* are bounded away from zero, then by exactly the same argument as used for the proof of Theorem 4.1, it can be seen that the modified forms have the same limiting behavior. In the next section this modification is taken for granted.

5. Asymptotic normality. The results of Section 4 reduce the problem of finding the asymptotic distributions of the rank score statistics S_N, S_N^1, S_N^* to the simpler one dealing with T_N and T_N^*.

In many cases the coefficients c_{ij} take only two values allowing us to use some standard limit theorems. To illustrate this consider example (c) of Section 2 where a rank analog of a serial correlation coefficient is proposed. This statistic can be written in the form of S_N of equation (4.8) where

$$\begin{aligned} c_{ij} &= N^{-\frac{1}{2}} \quad &&\text{if } j = i + 1 \\ &= 0 &&\text{otherwise,} \\ a_{ij} &= (N + 1)^{-2} ij. \end{aligned} \tag{5.1}$$

It may be easily verified, that the conditions of Theorem 4.1 are satisfied. Hence the statistic (2.11) is equivalent to

$$\begin{aligned} T_N &= N^{-\frac{1}{2}} \sum_{i=1}^{N-1} U_i U_{i+1} - N^{-1} N^{-\frac{1}{2}} \sum \sum_{i \neq j} U_i U_j \\ &\quad + N^{-1} N^{-\frac{1}{2}} (N + 1)^{-2} \sum \sum_{i \neq j} ij \\ &= N^{-\frac{1}{2}} \sum_{i=1}^{N-1} (U_i - \tfrac{1}{2})(U_{i+1} - \tfrac{1}{2}) \\ &\quad - N^{-1} N^{-\frac{1}{2}} \sum \sum_{i \neq j} (U_i - \tfrac{1}{2})(U_j - \tfrac{1}{2}) + \tfrac{1}{4} N^{\frac{1}{2}} + o(1) \\ &= N^{-\frac{1}{2}} \sum_{i=1}^{N} (U_i - \tfrac{1}{2})(U_{i+1} - \tfrac{1}{2}) + \tfrac{1}{4} N^{\frac{1}{2}} + o_p(1), \end{aligned} \tag{5.2}$$

where $U_1, \cdots, U_N$ are independent uniform (0, 1) random variables. The asymptotic normality of T_N can be established now by a well known theorem of Hoeffding and Robbins (1948).

For testing independence in a bivariate population, the normal score test statistic (2.2) proposed by Bhuchongkul (1964) can be shown to be asymptotically equivalent to

$$N^{-\frac{1}{2}} \sum_{i=1}^{N} W_i V_i, \tag{5.3}$$

where W_i and V_i, $i = 1, \cdots, N$, are two independent sets of random samples from certain populations. The asymptotic normality of (5.3) follows easily.

6. Appendix.

A_1. *An Upper Bound for the Variance.* As in the text, let $\{U_i\}$, $\{V_j\}$ be the uniform random variables, $\{R_i\}$ and $\{S_j\}$ be the corresponding ranks when ranking is done separately among the U_i and V_j respectively. Let $\{d_{ij}\}$ and $\{e_{ij}\}$ be two sets of constants. It is assumed that the d_{ij} satisfy the following relations uniformly in N (and M, in the two sample case):

$$\sum_i \sum_j d_{ij} = 0, \tag{6.1}$$

$$\sum_i \sum_j d_{ij}^2 < C_1 \tag{6.2}$$

and

$$\sum_i (\sum_j d_{ij})^2 < C_2, \qquad \sum_j (\sum_i d_{ij})^2 < C_3, \tag{6.3}$$

where $\sum_i \sum_j$ stands for the summations over $i, j = 1, \cdots, N$, in the one sample case and $i = 1, \cdots, N; j = 1, \cdots, M$ in the two sample case.

LEMMA A_1. *Under the assumptions* (6.1), (6.2) *and* (6.3).

$$\text{(6.4)} \qquad \operatorname{Var}\left(\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} e_{R_i R_j}\right) \leqq C[E e_{R_1,R_2}^2 + E e_{R_1,R_2}^2]$$

and

$$\text{(6.5)} \qquad \operatorname{Var}\left(\sum_{i=1}^{N} \sum_{j=1}^{M} d_{ij} e_{R_i,S_j}\right) \leqq C E(e_{R_1,S_1}^2).$$

PROOF. Since

$$\text{(6.6)} \qquad \operatorname{Var}\left(\sum_{i=1}^{N} \sum_{j=1}^{N} d_{ij} e_{R_i,R_j}\right) \leqq 2 \operatorname{Var}\left(\sum \sum_{i \neq j} d_{ij} e_{R_i,R_j}\right) + 2 \operatorname{Var}\left(\sum_i d_{ii} e_{R_i,R_i}\right),$$

it suffices to show that the terms on the right side of (6.6) are bounded as indicated in (6.4). Consider

$$\text{(6.7)} \qquad \operatorname{Var}\left(\sum \sum_{i \neq j} d_{ij} e_{R_i,R_j}\right) = E\left(\sum \sum_{i \neq j} (d_{ij} - \bar{d}) e_{R_i,R_j}\right)^2,$$

where $\bar{d} = \sum_i \sum_j d_{ij}/N(N-1)$, $i \neq j$. Conditions (6.1), (6.2) and (6.3) imply that the same relations hold for the deviations $(d_{ij} - \bar{d})$, $i \neq j$. For notational convenience, we write d_{ij} instead of the deviation $(d_{ij} - \bar{d})$. (This notation will be adopted only up to the derivation of an upper bound for (6.7).)

$$\text{(6.8)} \qquad \begin{aligned} E\left(\sum \sum_{i \neq j} d_{ij} e_{R_i,R_j}\right)^2 &= E\left(\sum \sum_{i \neq j} d_{ij} e_{R_i,R_j}\right)\left(\sum \sum_{k \neq l} d_{kl} e_{R_k,R_l}\right) \\ &= D_1 E(e_{R_1,R_2} e_{R_3,R_4}) + D_2 E(e_{R_1,R_2} e_{R_1,R_3}) \\ &\quad + 2D_3 E(e_{R_1,R_2} e_{R_3,R_1}) + D_4 E(e_{R_1,R_2} e_{R_3,R_2}) \\ &\quad + D_5 E(e_{R_1,R_2} e_{R_2,R_1}) + D_6 E(e_{R_1,R_2}^2), \end{aligned}$$

where

$$\text{(6.9)} \qquad \begin{aligned} D_1 &= \sum \sum \sum \sum_{i \neq j \neq k \neq l} d_{ij} d_{kl}, & D_2 &= \sum \sum \sum_{i \neq j \neq l} d_{ij} d_{il}, \\ D_3 &= \sum \sum \sum_{i \neq j \neq k} d_{ij} d_{ki}, & D_4 &= \sum \sum \sum_{i \neq j \neq k} d_{ij} d_{kj}, \\ D_5 &= \sum \sum_{i \neq j} d_{ij} d_{ji}, & D_6 &= \sum \sum_{i \neq j} d_{ij}^2. \end{aligned}$$

All the expected values of the products appearing after the last equality sign in (6.8) are bounded by $E e_{R_1,R_2}^2$. The uniform boundedness of D_5 and D_6 follows from (6.2). That the same holds for D_2 is seen from

$$\text{(6.11)} \qquad \begin{aligned} D_2 &= \sum_i \left[\left(\sum_{j(\neq i)} d_{ij}\right)\left(\sum_{l(\neq i)} d_{il}\right) - \sum_{j(\neq i)} d_{ij}^2\right] \\ &= \sum_i \left(\sum_{j(\neq i)} d_{ij}\right)^2 - \sum \sum_{i \neq j} d_{ij}^2. \end{aligned}$$

The sums D_3, D_4 can be shown to be bounded by the same method. It remains to show the uniform boundedness of D_1. Recalling that d_{ij} above are in fact $(d_{ij} - \bar{d})$,

$$\text{(6.12)} \qquad \left(\sum \sum_{i \neq j} d_{ij}\right)^2 = 0 = D_1 + D_2 + \cdots + D_6,$$

and the uniform boundedness D_1 follows from that for the others. Consider now

$$\text{(6.13)} \qquad \operatorname{Var}\left(\sum_{i=1}^{N} d_{ii} e_{R_i,R_i}\right) = E\left(\sum_{i=1}^{N} (d_{ii} - \tilde{d}) e_{R_i,R_i}\right)^2,$$

where $\tilde{d} = \sum_i d_{ii}/N$. Expanding the square it is seen that

$$\begin{aligned} &\operatorname{Var}\left(\textstyle\sum_{i=1}^N d_{ii}e_{R_i,R_i}\right) \\ (6.14)\qquad &= \textstyle\sum_{i=1}^N (d_{ii} - \tilde{d})^2 Ee^2_{R_1,R_1} \\ &\quad + \textstyle\sum\sum_{i\neq j} (d_{ii} - \tilde{d})(d_{jj} - \tilde{d}) Ee_{R_1,R_1}e_{R_2,R_2} \\ &\leqq 2\textstyle\sum_{i=1}^N d_{ii}^2 Ee^2_{R_1,R_1}, \end{aligned}$$

where the last inequality follows by noting that $\tilde{d}$ is the average of d_{ii}, and thus

$$(6.15)\qquad \left|\textstyle\sum\sum_{i\neq j} (d_{ii} - \tilde{d})(d_{jj} - \tilde{d})\right| = \textstyle\sum_i (d_{ii} - \tilde{d})^2 \leqq \textstyle\sum_{i=1}^N d_{ii}^2.$$

Combining (6.14) with the other bounds the assertion (6.7) follows.

The proof of (6.5) follows along similar lines (in fact simpler), and hence is omitted.

A_2. *Uniform approximation theorem.* The following is the two dimensional version of a lemma due to Hájek ((1961), Lemma 2.1).

Let $\phi(\lambda, \theta)$ be a real valued function defined on $(0, 1)^2$. It is assumed that ϕ is monotone in λ and θ and $\phi \varepsilon L_p$ i.e.

$$(6.16)\qquad \int_0^1\int_0^1 |\phi(\lambda, \theta)|^p \, d\lambda \, d\theta < \infty.$$

Define

$$\begin{aligned} (6.17)\qquad \phi_N(\lambda, \theta) &= \phi(i/(N+1), j/(N+1)); \\ &(i-1)/N < \lambda \leqq i/N, \qquad (j-1)/N < \theta \leqq j/N. \end{aligned}$$

LEMMA A_2. (i) *The functions* ϕ_N^k *are uniformly integrable for* $k = 1, \cdots, p$, *and*

(ii) $\lim_{N\to\infty} \int_0^1\int_0^1 |\phi_N(\lambda, \theta) - \phi(\lambda, \theta)|^k \, d\lambda \, d\theta = 0$ *for* $k = 1, \cdots, p$.

PROOF. It suffices to show that the assertion holds for $k = p$. First assume that $\phi(0, 0) \geqq 0$ and ϕ is nondecreasing so that ϕ^p is also nondecreasing. The uniform integrability will be proved by a successive application of an inequality of Hájek ((1961), Lemma 2.1, expression (2.22) therein, or (6.19) of the present paper)) and Fubini's theorem.

A slight extension of Hájek's inequality may be stated as follows. Suppose $\psi(\lambda)$ defined on $(0, 1)$ is nondecreasing, $\psi(0) \geqq 0$ and $\psi \varepsilon L_p$. Let

$$(6.18)\qquad \psi_N(\lambda) = \psi(i/(N+1)) \quad \text{for} \quad (i-1)/N < \lambda \leqq i/N,$$

and η be the Lebesgue measure on the real line. Then for any measurable subset A of $(0, 1)$

$$(6.19)\qquad \int_A \psi_N^p(\lambda) \leqq \psi^p(\tfrac{3}{4})\eta(A) + 4\int_B \psi^p(\lambda)\, d\lambda$$

where $B = (1 - \eta(A), 1)$.

Let A be an open rectangle $A_1 \times A_2$ and consider the function

$$(6.20)\qquad \xi_N(\lambda, \theta) = \phi(i/(N+1), \theta), \qquad (i-1)/N < \lambda \leqq i/N.$$

Applying inequality (6.19) it follows that

$$\begin{aligned}(6.21)\qquad &\int\int_{A_1\times A_2} \xi_N^{\,p}(\lambda, \theta)\, d\lambda\, d\theta \\ &= \int_{A_2} [\int_{A_1} \xi_N^{\,p}(\lambda, \theta)\, d\lambda]\, d\theta \\ &\leqq \int_{A_2} [\phi^p(\tfrac{3}{4}, \theta)\eta(A_1) + 4 \int_{B_1} \phi^p(\lambda, \theta)\, d\lambda]\, d\theta \\ &= \eta(A_1) \int_{A_2} \phi^p(\tfrac{3}{4}, \theta)\, d\theta + 4 \int_{A_2} \int_{B_1} \phi^p(\lambda, \theta)\, d\lambda\, d\theta,\end{aligned}$$

where $B_1 = (1 - \eta(A_1), 1)$. A similar inequality holds if the first argument $\phi(\lambda, \theta)$ is kept fixed. Rewriting ϕ_N as

$$(6.22)\qquad \phi_N(\lambda, \theta) = \xi_N(\lambda, j/(N + 1)), \qquad (j - 1)/N < \theta \leqq j/N,$$

it follows that

$$\begin{aligned}(6.23)\qquad &\int_{A_1} \int_{A_2} \phi_N^{\,p}(\lambda, \theta)\, d\lambda\, d\theta \\ &\leqq \int_{A_1} [\eta(A_2)\xi_N^{\,p}(\lambda, \tfrac{3}{4}) + 4 \int_{B_2} \xi_N^{\,p}(\lambda, \theta)\, d\theta]\, d\lambda \\ &\leqq \eta(A_1)\eta(A_2)\phi^p(\tfrac{3}{4}, \tfrac{3}{4}) + 4\eta(A_2) \int_{B_1} \phi^p(\lambda, \tfrac{3}{4})\, d\lambda \\ &\quad + 4\eta(A_1) \int_{B_2} \phi^p(\tfrac{3}{4}, \theta)\, d\theta + 16 \int_{B_1} \int_{B_2} \phi^p(\lambda, \theta)\, d\lambda\, d\theta.\end{aligned}$$

This shows that the integrals of $\phi_N^{\,p}(\lambda, \theta)$ are uniformly absolutely continuous and bounded, hence uniformly integrable. Assertion (ii) follows from the L_p convergence.

The restriction $\phi(0, 0) \geqq 0$ can be removed by the consideration of positive and negative parts of the function ϕ. That the assumption of ϕ being nondecreasing can be replaced by that of monotonicity is obvious. This can be further relaxed to piecewise monotonicity by using some elementary inequalities.

Acknowledgments. The author is deeply grateful to Professor S. W. Dharmadhikari for valuable help in weeding out several errors and to the referee for many helpful suggestions.

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