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ASYMPTOTIC NORMALITY OF SIMPLE LINEAR RANK STATISTICS UNDER ALTERNATIVES[1]

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0. Summary. Results of Chernoff-Savage (1958) and Govindarajulu-LeCam-Raghavachari (1966) are extended from the two-sample case to the general regression case and, simultaneously, the conditions on the scores-generating function are relaxed. The main results are stated in Section 2 and their proofs are given in Section 5. Sections 3 and 4 contain auxiliary propositions, on which the methods of the present paper are based. Section 6 includes a counterexample, showing that the theorem cannot be extended to discontinuous scores-generating functions.

1. Introduction. Simple linear rank statistics, given by (2.3) below, besides being essential in non-parametric theory also provide a key to solving some problems of general theory such as establishment of asymptotically most powerful tests for some problems. Under the alternative, the distribution of a simple linear rank statistic is determined by the following three entities: first, regression constants $c_1, \cdots, c_N$, second, distribution functions of individual observations $F_1, \cdots, F_N$, third, scores $a(1), \cdots, a(N)$. The scores are usually assumed generated by a function φ, e.g. by (2.4) below. The same might be assumed concerning the regression constants $c_1, \cdots, c_N$. The distribution functions may be derived from a parametric family $F(x, \theta)$, $\theta \varepsilon \Omega$. If $c_i = 1$ or 0, we have the so-called *two-sample* problem.

The central problem concerning simple linear rank statistics is their asymptotic normality either with "natural" parameters (ES, var S) or with some other parameters (μ, σ^2). Under some regularity conditions the answer is positive. And, as may be expected, less regularity in one entity may be counterbalanced by more regularity in the other. The most regular $(c_1, \cdots, c_N)$ are those generated by a linear function, i.e. $c_i = a + ib$, $1 \leqq i \leqq N$; the next condition, in descending restrictivity, is boundedness of $N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 / \sum_{i=1}^N (c_i - \bar{c})^2$; the mildest condition yet used in the literature is the Noether condition $\max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \cdot [\sum_{i=1}^N (c_i - \bar{c})^2]^{-1} \to 0$ (see Hájek (1961), (1962)).

The most regular $(F_1, \cdots, F_N)$ correspond to the null hypothesis $F_1 = \cdots = F_N$; next comes the condition of contiguity (see [7]); finally, we may only assume that the variance of S under $(F_1, \cdots, F_N)$ is of the same order as under $F_1 = \cdots = F_N$; or allow for some rate of degeneration of the variance.

The regularity conditions concerning the scores, generated by a function $\varphi(t)$, $0 < t < 1$, are expressed in terms of smoothness and boundedness of φ.

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An ideal φ is linear; Dwass (1956) succeeded in treating polynomials; Chernoff-Savage (1958) assumed two derivatives, the ith derivative being bounded by $[t(1-t)]^{-i-\frac{1}{2}+\delta}$, $i = 0, 1, 2$, $\delta > 0$, $0 < t < 1$; Govindarajulu-LeCam-Raghavachari (1966) considered a larger class, removing the assumption of existence of a second derivative and relaxing the boundedness.

The first general attempt to treat the asymptotic distribution of simple linear rank statistics in the two-sample case was made by Dwass (1956). As we already mentioned, using the method of U-statistics, he was successful with polynomial scores-generating functions only. Then Chernoff-Savage published their pioneering paper (1958), which was a basis for all further developments excepting those based on the contiguity approach. The Chernoff-Savage conditions have been improved as recent as that by Govindarajulu-LeCam-Raghavachari (1966). In all papers mentioned above, the regularity condition concerning the regression constants (boundedness) and the scores-generating function are rather stringent, whereas the assumption concerning $(F_1, \cdots, F_N)$ are broad enough (non-degeneration of variance only).

In contradistinction to it, the papers by Hájek (1961), (1962) are concerned with situations, where $(F_1, \cdots, F_N)$ is very regular (either null hypothesis or contiguous alternatives) whereas the c's satisfy the Noether condition only and φ is square integrable only. Thus the existing literature has dealt with somewhat extreme cases, leaving open intermediate ones. For example, one would need a theorem concerning discrete φ for $(F_1, \cdots, F_N)$ satisfying milder conditions than contiguity.

Govindarajulu-LeCam-Raghavachari (1966) improved the Chernoff-Savage (1958) method by considering the sample distribution functions as processes and proving some refined limiting properties of these. The methods of the present paper are quite elementary, although rather involved. Two new methodical tools are used, first, the inequality of Theorem 3.1; and, second, the method of projection. The inequality may be used as follows. Supposing Theorem 2.3 has been proved for a certain class $\Phi_0 = \{\varphi_0\}$ of scores-generating functions, it then holds for the class $\Phi = \{\varphi\}$ consisting of functions φ possessing the following property: for every $\epsilon > 0$ there is a $\varphi_0 \varepsilon \Phi_0$ and non-decreasing functions φ_1 and φ_2 such that $\varphi = \varphi_0 + \varphi_1 - \varphi_2$ and $\int_0^1 (\varphi_1^2 + \varphi_2^2)\, dt < \epsilon$. For example, if Φ_0 consists of polynomials (as in Dwass (1956)), Φ encompasses all functions considered in Govindarajulu-LeCam-Raghavachari (1966), and, *a fortiori*, in Chernoff-Savage (1958).

The method of projection was born by observing that many successful methods of deriving asymptotic normality of statistics, which do not have the standard form of a sum of independent random variables, consisted in finding such a standard approximation. This was the case with U-statistics, with the method of Chernoff-Savage (1958) and also with Hájek (1961). The method of projection is based on Lemma 4.1 giving to any statistic $S = s(X_1, \cdots, X_n)$, where the X's are independent, the best approximation of the form $\hat{S} = \sum_{i=1}^N l_i(X_i)$ where the functions l_i may be chosen arbitrarily except for $El_i^2(X_i) < \infty$. In this way the due approximation may be found quite mechanically. This is important namely in more complicated situations, where the intuition and good luck may easily fail. Moreover, since, for best $\hat{S}$, we have $E(S - \hat{S})^2 = \text{var } S - \text{var } \hat{S}$ the remainder $S - \hat{S}$ may be treated in terms of mean square convergence, which is usually easier to handle than the convergence in probability.

The problem of a single simple linear rank statistic may be generalized to a finite set of such statistics (needed in the c-sample problem) or to a continuous parameter set of such statistics (needed in dealing with statistics of Kolmogorov-Smirnov types). The former generalization is touched in Remark 2.4, the latter is out of scope. Also the cases of simple linear statistics for the independence problem (see Bhuchonghul (1964)) as well as the symmetry problem are not considered here.

To keep touch with Hájek (1961), (1962), we use a different notation than Chernoff-Savage: especially we use $\varphi(t)$, $a_N(j)$, S instead of $J(u)$, E_{Nj} and T_N, respectively. They also use the indicators Z_{Ni} of the ith order statistics being included in the first sample instead of ranks. Their formula $\sum_{i=1}^N E_{Ni} Z_{Ni}$ reads $\sum_{i=1}^m a(R_i)$ in our notation, m denoting the number of observations in the first sample and $E_{Ni} = a(i)$, $1 \leqq i \leqq N$. To avoid cumbersome subscripts connected with sequences we give to the limiting assertions the ϵ-form.

2. The main results. Let $X_1, \cdots, X_N$ be independent random variables with continuous distribution functions $F_1, \cdots, F_N$, and let $R_1, \cdots, R_N$ denote the corresponding ranks. Introducing the function

$$u(x) = 1, \quad x \geqq 0, \qquad = 0, \quad x < 0, \tag{2.1}$$

we may write

$$R_i = \sum_{j=1}^N u(X_i - X_j), \qquad 1 \leqq i \leqq N. \tag{2.2}$$

We are interested in asymptotic normality of simple linear rank statistics

$$S = \sum_{i=1}^N c_i a_N(R_i), \tag{2.3}$$

where $c_1, \cdots, c_N$ are arbitrary "regression constants", and $a_N(1), \cdots, a_N(N)$ are "scores" generated by a function $\varphi(t)$, $0 < t < 1$, in either of the following two ways:

$$a_N(i) = \varphi(i/(N+1)), \qquad 1 \leqq i \leqq N, \tag{2.4}$$

$$a_N(i) = E\varphi(U_N^{(i)}), \qquad 1 \leqq i \leqq N, \tag{2.5}$$

where $U_N^{(i)}$ denotes the ith order statistic in a sample of size N from the uniform distribution on $(0, 1)$. Scores given by (2.5) occur in statistics yielding locally most powerful rank tests. Scores (2.4) are distinguished by simplicity. In what follows φ and the scores are assumed fixed, whereas N, $(c_1, \cdots, c_N)$ and $(F_1, \cdots, F_N)$ are considered variable. Let

$$\bar{c} = N^{-1} \sum_{i=1}^N c_i, \qquad \bar{\varphi} = \int_0^1 \varphi(t)\, dt, \qquad H(x) = N^{-1} \sum_{i=1}^N F_i(x). \tag{2.6}$$

THEOREM 2.1. *Consider the statistic S of* (2.3), *where the scores are given either by* (2.4) *or* (2.5). *Assume that* φ *has a bounded second derivative.*

Then for every $\epsilon > 0$ *there exists* K_ϵ *such that*

$$\text{var } S > K_\epsilon \max_{1 \leq i \leq N} (c_i - \bar{c})^2 \tag{2.7}$$

entails

$$\max_{-\infty<x<\infty} |P(S - ES < x(\text{var } S))^{\frac{1}{2}} - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}y^2)\, dy| < \epsilon. \tag{2.8}$$

The assertion remains true, if we replace var S *in* (2.7) *and* 2.8) *by*

$$\sigma^2 = \sum_{i=1}^{N} \text{var } [l_i(X_i)] \tag{2.9}$$

where

$$l_i(x) = N^{-1} \sum_{j=1}^{N} (c_j - c_i) \int [u(y - x) - F_i(y)] \varphi'(H(y))\, dF_j(y) \tag{2.10}$$

with φ' *denoting the derivative of* φ *and* $u(\cdot)$ *given by* (2.1).

In applications we usually assume that the distribution functions differ only slightly. Then the following variation of Theorem 2.1 is useful.

THEOREM 2.2. *Under assumptions of Theorem* 2.1, *for every* $\epsilon > 0$ *there exists a* $\delta_\epsilon > 0$ *such that the joint satisfaction of*

$$\sum_{i-1}^{N} (c_i - \bar{c})^2 > \delta_\epsilon^{-1} \max_{1 \leq i \leq N} (c_i - \bar{c})^2 \tag{2.11}$$

and

$$\max_{i,j,x} |F_i(x) - F_j(x)| < \delta_\epsilon \tag{2.12}$$

entails

$$\sup_x |P(S - ES < xd) - (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}y^2)\, dy| < \epsilon, \tag{2.13}$$

where

$$d^2 = \sum_{i=1}^{N} (c_i - \bar{c})^2 \int_0^1 [\varphi(t) - \bar{\varphi}]^2\, dt. \tag{2.14}$$

DEFINITION. We shall say that $\varphi(t)$, $0 < t < 1$, is *absolutely continuous inside* (0, 1), if for every $\epsilon > 0$ and $0 < \alpha < \frac{1}{2}$ there exists a $\delta = \delta(\epsilon, \alpha)$ such that for each finite set of disjoint intervals (a_k, b_k) the relation

$$\sum_{k=1}^{n} |b_k - a_k| < \delta \quad \text{and} \quad \alpha < a_k, b_k < 1 - \alpha$$

implies

$$\sum_{k=1}^{n} |\varphi(b_k) - \varphi(a_k)| < \epsilon.$$

Thus φ is absolutely continuous inside (0, 1), if it is absolutely continuous on $(\alpha, 1 - \alpha)$ for every $\alpha \varepsilon (0, \frac{1}{2})$.

It is a well-known theorem of calculus that $\varphi(t)$ is absolutely continuous inside (0, 1) if and only if there exists a function $\varphi'(x)$ integrable on every interval $(\alpha, 1 - \alpha)$, $0 < \alpha < \frac{1}{2}$, and such that

$$\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)\, dt, \qquad 0 < a < b < 1. \tag{2.15}$$

We further know that $\varphi'(t)$ represents the derivative of $\varphi(t)$ almost everywhere. However, the converse is not true, i.e. the existence of the derivative a.e. does not generally entail (2.15). Of course if the derivative exists *everywhere* and is integrable on every interval $(\alpha, 1 - \alpha)$, $0 < \alpha < \frac{1}{2}$, then φ is absolutely continuous inside $(0, 1)$. There also may be a finite number of points possessing only the left-hand and right-hand derivatives as, e.g., in $\varphi(t) = |t - \frac{1}{2}|$. Let us emphasize that for $t \to 0$ or 1 the limit of a $\varphi(t)$, which is absolutely continuous inside $(0, 1)$, may not exist or be infinite. Consequently (2.15) may not have even sense for $a = 0$ or $b = 1$.

Consider again the statistic (2.3) where the scores are given either by (2.4) or (2.5).

THEOREM 2.3. *Let* $\varphi(t) = \varphi_1(t) - \varphi_2(t)$, $0 < t < 1$, *where the* $\varphi_i(t)$ *both are non-decreasing, square integrable, and absolutely continuous inside* $(0, 1)$, $i = 1, 2$. *Then for every* $\epsilon > 0$ *and* $\eta > 0$ *there exists* $N_{\epsilon\eta}$ *such that*

$$(2.16) \qquad N > N_{\epsilon\eta}, \qquad \operatorname{var} S > \eta N \max_{1 \leq i \leq N} (c_i - \bar{c})^2$$

entails (2.8).

As in Theorem 2.1, var S *may be replaced by* σ^2 *of* (2.9) *in both* (2.16) *and* (2.8).

THEOREM 2.4. *Under conditions of Theorem* 2.3, *for every* $\epsilon > 0$ *and* $\eta > 0$ *there exist* $N_{\epsilon\eta}$ *and* $\delta_{\epsilon\eta}$ *such that joint satisfaction of*

$$(2.17) \qquad \sum_{i=1}^{N} (c_i - \bar{c})^2 > \eta N \max_{1 \leq i \leq N} (c_i - \bar{c})^2,$$

$$(2.18) \qquad N > N_{\epsilon\eta},$$

$$(2.19) \qquad \max_{i,j,x} |F_i(x) - F_j(x)| < \delta_{\epsilon\eta}$$

entail (2.13).

REMARK 2.1. The theorem also holds, if the scores are given by

$$(2.20) \qquad a_N(i) = \int_{(i-1)/N}^{i/N} \varphi(t)\, dt, \qquad 1 \leq i \leq N.$$

REMARK 2.2. The variance d^2, given by (2.14), is asymptotically equivalent (in the ratio sense) to the variance of S under the null hypothesis $F_i = F$, $1 \leq i \leq N$, and, as a matter of fact, could be replaced by it as well. Theorem 2.1 is as powerful, as far as the c's are concerned, as the corresponding theorem under null hypothesis. In particular, for the two-sample case it provides asymptotic normality even for cases where $m/n \to 0$ or ∞, and also for cases where var S under the alternative is of smaller order then d^2.

REMARK 2.3. (Comparison with Chernoff-Savage and Govindarajulu-LeCam-Raghavachari). If $c_i = 1$, $1 \leq i \leq m$, and $= 0$, if $m < i \leq N$, and if $F_i = F$, $1 \leq i \leq m$, and $F_i = G$, $m < i \leq N$, we have the problem considered by Chernoff-Savage (1958) and by Govindarajulu-LeCam-Raghavachari (1966). In the present paper the condition concerning the function $\varphi(t)$, denoted in the above papers by $J(u)$, is relaxed. Actually, taking a $\varphi(t)$ satisfying the condition $|\varphi'(t)| \leq K[t(1 - t)]^{-3/2+\delta}$, $\delta > 0$, and putting

$$(2.21) \qquad \varphi_1(t) = \varphi(\tfrac{1}{2}) + \int_{\frac{1}{2}}^{t} \max[0, \varphi'(s)]\, ds, \qquad \varphi_2(t) = \int_{\frac{1}{2}}^{t} \max[0, -\varphi'(s)]\, ds$$

we have $\varphi(t) = \varphi_1(t) - \varphi_2(t)$, where φ_1 and φ_2 are non-decreasing, square integrable and absolutely continuous on intervals $(\epsilon, 1 - \epsilon)$, $0 < \epsilon < \frac{1}{2}$. Also if, according to the assumptions of Govindarajulu-LeCam-Raghavachari, $|\varphi'(t)| \leqq f_0 + fg$, where f_0 is integrable, f is integrable and U-shaped, and g is square integrable and U-shaped, the functions (2.21) have the desired properties (see Lemma 2 of [4]).

The present theorem surpasses the corresponding results of [2] and [4] in the following further respects: first, the scores given by (2.5) are considered throughout, second, a simplified variance is given under (2.12) or (2.19), and, finally, it is shown that the asymptotic normality holds with natural parameters.

On the other hand, both papers [2] and [4] show that S is asymptotically normal (μ, σ^2) with

$$\mu = \sum_{i=1}^{N} c_i E[\varphi(H(X_i))]. \tag{2.22}$$

The present author did not succeed in showing that this is still true under conditions of Theorems 2.3 and 2.4. From Theorem 4.2 it follows that ES is replaceable by μ in Theorems 2.1 and 2.2 provided that $\sum_{i=1}^{N} c_i^2$ is bounded by a multiple of $\sum_{i=1}^{N} (c_i - \bar{c})^2$.

In [4], Theorem 1, uniformity is also shown with respect to a class of φ-functions. This is missing in the present paper, though a similar extension is easily possible. The corresponding class would consist of function $\varphi = \varphi_1 - \varphi_2$ such that $\varphi_1, \varphi_2 \,\varepsilon\, \Phi^*$, Φ^* being a class of non-decreasing functions, which are uniformly square integrable, absolutely continuous on intervals $(\epsilon, 1 - \epsilon)$, $0 < \epsilon < \frac{1}{2}$, and compact with respect to the convergence such that $\varphi_k \to \varphi$ is equivalent to $\int_\epsilon^{1-\epsilon} |\varphi_k' - \varphi'| \, dt \to 0$ for every $\epsilon > 0$.

Remark 2.4. (The c-sample problem). Theorem 2.1 contains all of the essentials for providing the c-sample limit theorem considered in [4] and [8]. Let $s_1, \cdots, s_c$ be a decomposition of the set $\{1, \cdots, N\}$. Let $n_j = \text{card } s_j$, $1 \leqq j \leqq c$, and consider statistics

$$S_j = \sum_{i \varepsilon s_j} a_N(R_i), \tag{2.23}$$

where the scores are given either by (2.4) or (2.5). Then any linear combination of the statistics S_j, say

$$S = \sum_{j=1}^{c} \lambda_j S_j \tag{2.24}$$

is of the form (2.3) with

$$c_i = \lambda_j, \quad \text{if} \quad i \,\varepsilon\, s_j, \qquad 1 \leqq j \leqq c. \tag{2.25}$$

Remark 2.5. The constants K_ϵ, δ_ϵ, $N_{\epsilon\eta}$, $\delta_{\epsilon\eta}$ appearing in the above theorems depend on φ.

Remark 2.6. By integration by parts we have

$$\int [u(y - x) - F_i(y)]\varphi'(H(y)) \, dF_j(y) = -\int_{x_0}^{x} \varphi'(H(y)) \, dF_j(y) + \text{const.} \tag{2.26}$$

3. Variance bound for monotone scores. We shall start with two elementary lemmas.

LEMMA 3.1. *Let $\alpha(x_1, \cdots, x_h)$ be a real function, non-decreasing in each argument. Let $X_1, \cdots, X_h$ be arbitrary independent random variables such that $E|\alpha(X_1, \cdots, X_h)| < \infty$.*

Then, for $k < h$, the function

$$\bar{\alpha}(x_1, \cdots, x_k) = E[\alpha(X_1, \cdots, X_h) \mid X_1 = x_1, \cdots, X_k = x_k] \tag{3.1}$$

is non-decreasing in each argument.

Moreover, if $\beta(x_1, \cdots, x_h)$ is another real function, non-decreasing in each argument, then

$$\operatorname{cov}[\alpha(X_1, \cdots, X_h), \beta(X_1, \cdots, X_h)] \geqq 0, \tag{3.2}$$

provided the covariance is well-defined.

PROOF. The first part is trivial. For the second part, if $h = 1$, (3.2) follows from the identity

$$\operatorname{cov}[\alpha(X_1), \beta(X_1)] = \tfrac{1}{2}E\{[\alpha(X_1) - \alpha(Y_1)][\beta(X_1) - \beta(Y_1)]\},$$

where Y_1 is an independent copy of X_1. Generally,

$$\begin{aligned}&\operatorname{cov}[\alpha(X_1, \cdots, X_h), \beta(X_1, \cdots, X_h)]\\ &\quad = \operatorname{cov}[\bar{\alpha}(X_1), \bar{\beta}(X_1)] + \int \operatorname{cov}[\alpha(x, X_2, \cdots, X_h), \beta(x, X_2, \cdots, X_h)]\, dF_1(x)\end{aligned}$$

holds, where $\bar{\alpha}$ (and also $\bar{\beta}$) is defined by (3.1) with $k = 1$. The functions $\bar{\alpha}$ and $\bar{\beta}$ and also the functions $\alpha(x, \cdots)$, and $\beta(x, \cdots)$ are non-decreasing, so that the assertion holds for h, if holding for $h - 1$. The proof is terminated. (Another proof follows from Theorem 2 of [9].)

LEMMA 3.2. *Let $R_1, \cdots, R_N$ be ranks of a sequence of independent observations $X_1, \cdots, X_N$ possessing arbitrary continuous distribution functions $F_1, \cdots, F_N$. Let $a(1), \cdots, a(N)$ be arbitrary scores and $u(x)$ be given by (2.1). Then*

$$\begin{aligned}&E[a(R_i) \mid X_i = x, X_j = y] - E[a(R_i) \mid X_i = x]\\ &\quad = [u(x - y) - F_j(x)] \textstyle\sum_{k=2}^{N} (a(k) - a(k-1))\\ &\qquad \cdot P(R_i = k \mid X_i = x, X_j = x - 1), \quad i \neq j.\end{aligned} \tag{3.3}$$

PROOF. Specify $i = 1$, $j = 2$ in order to simplify notations. Denote by $B(k \mid p_1, \cdots, p_N)$ the probability of k successes in N independent trials with respective probabilities of success $p_1, \cdots, p_N$. Thus $B(\cdot|\cdot)$ denotes the probabilities of the Poisson binomial distribution. We obviously have, in view of (2.2),

$$P(R_1 = k \mid X_1 = x, X_2 = y) = B(k \mid 1, u(x - y), F_3(x), \cdots, F_N(x)) \tag{3.4}$$

and

$$P(R_1 = k \mid X_1 = x) = B(k \mid 1, F_2(x), F_3(x), \cdots, F_N(x)). \tag{3.5}$$

We easily see that

$$
\begin{aligned}
& B(k \mid 1, F_2(x), \cdots, F_N(x)) \\
&\qquad = F_2(x)B(k \mid 1, 1, F_3(x), \cdots, F_N(x) \\
&\qquad\quad + (1 - F_2(x))B(k \mid 1, 0, F_3(x), \cdots, F_N(x)) \qquad (3.6) \\
&\qquad = F_2(x)B(k \mid 1, 1, F_3(x), \cdots, F_N(x)) \\
&\qquad\quad + (1 - F_2(x))B(k + 1 \mid 1, 1, F_3(x), \cdots, F_N(x)).
\end{aligned}
$$

Consequently, in accordance with (3.4) through (3.6),

$$
\begin{aligned}
& P(R_1 = k \mid X_1 = x, X_2 = y) - P(R_1 = k \mid X_1 = x) \\
&\qquad = [u(x - y) - F_2(x)][B(k \mid 1, 1, F_3(x), \cdots, F_N(x)) \\
&\qquad\quad - B(k + 1 \mid 1, 1, F_3(x), \cdots, F_N(x))] \\
&\qquad = [u(x - y) - F_2(x)][P(R_1 = k \mid X_1 = x, X_2 = x - 1) \\
&\qquad\quad - P(R_1 = k + 1 \mid X_1 = x, X_2 = x - 1)],
\end{aligned}
$$

where $x - 1$ could be replaced by any number smaller than x. Now the last relation entails

$$
\begin{aligned}
& E[a(R_1) \mid X_1 = x, X_2 = y] - E[a(R_1) \mid X_1 = x] \\
&= \textstyle\sum_{k=1}^N a(k)[P(R_1 = k \mid X_1 = x, X_2 = y) - P(R_1 = k \mid X_1 = x)] \\
&= [u(x - y) - F_2(x)] \textstyle\sum_{k=1}^N a(k)[P(R_1 = k \mid X_1 = x, X_2 = x - 1) \\
&\quad - P(R_1 = k + 1 \mid X_1 = x, X_2 = x - 1)] \\
&= [u(x - y) - F_2(x)] \textstyle\sum_{k=2}^N [a(k) - a(k - 1)]P(R_1 = k \mid X_1 = x, X_2 = x - 1).
\end{aligned}
$$

Q.E.D.

The inequality derived in the following theorem will remove troubles caused by the fact that the scores-generating function φ may be unbounded in the vicinities of 0 and 1, and may have no bounded second derivative in intervals $(\epsilon, 1 - \epsilon)$, $\frac{1}{2} > \epsilon > 0$.

THEOREM 3.1. (Variance inequality). *Let $X_1, \cdots, X_N$ be independent random variables with arbitrary continuous distribution functions $F_1, \cdots, F_N$. Let $c_1, \cdots, c_N$ be arbitrary constants and $a_1 \leqq \cdots \leqq a_N$ be non-decreasing constants. Let R_i denote the rank of X_i, $1 \leqq i \leqq N$.*

Then, writing a_i and $a(i)$ interchangeably,

$$\operatorname{var}\left[\textstyle\sum_{i=1}^N c_i a(R_i)\right] \leqq 21 \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \textstyle\sum_{i=1}^N (a_i - \bar{a})^2, \qquad (3.7)$$

where

$$\bar{c} = N^{-1} \textstyle\sum_{i=1}^N c_i, \qquad \bar{a} = N^{-1} \textstyle\sum_{i=1}^N a_i.$$

PROOF. Without losing generality, we may assume $\bar{c} = \bar{a} = 0$. Obviously

$$\text{(3.8)} \quad \operatorname{var}\left[\sum_{i=1}^{N} c_i a(R_i)\right] = \sum_{i=1}^{N} c_i^2 \operatorname{var}[a(R_i)] + \sum\sum_{i=1, j=1, i\neq j}^{N} c_i c_j \operatorname{cov}[a(R_i), a(R_j)]$$

and

$$\text{(3.9)} \quad \sum_{i=1}^{N} c_i^2 \operatorname{var}[a(R_i)] \leqq E\left\{\sum_{i=1}^{N} c_i^2 a^2(R_i)\right\} \leqq \max_{1\leqq i\leqq N} c_i^2 E\left\{\sum_{i=1}^{N} a^2(R_i)\right\} = \max_{1\leqq i\leqq N} c_i^2 \sum_{i=1}^{N} a_i^2,$$

since $\sum_{i=1}^{N} a^2(R_i) = \sum_{i=1}^{N} a_i^2$ constantly.

The covariances are more tedious. (2.2) entails that R_i is a non-decreasing function of $-X_j$ (note the minus sign), $j \neq i$, and of X_i. If $X_i = x$ and $X_j = y$ are fixed, both R_i and R_j are non-decreasing functions of $-X_k$, $k \neq i, j$, and the same is true about $a(R_i)$ and $a(R_j)$, since $a_1 \leqq \cdots \leqq a_N$. Consequently, by Lemma 3.1,

$$\text{(3.10)} \quad \operatorname{cov}[a(R_i), a(R_j) \mid X_i = x, X_j = y] \geqq 0, \quad 1 \leqq i \neq j \leqq N.$$

Also $E[a(R_i) \mid X_i = x, X_j = y]$ as well as $-E[a(R_j) \mid X_i = x, X_j = y]$ are both non-decreasing in x and $-y$, $i \neq j$, according to the same lemma. Thus

$$\text{(3.11)} \quad -\operatorname{cov}\{E[a(R_i) \mid X_i, X_j], E[a(R_j) \mid X_i, X_j]\} \geqq 0, \quad 1 \leqq i \neq j \leqq N.$$

Notice that

$$\operatorname{cov}[a(R_i), a(R_j)] = E \operatorname{cov}[a(R_i), a(R_j) \mid X_i, X_j] + \operatorname{cov}\{E[a(R_i) \mid X_i, X_j], E[a(R_j) \mid X_i, X_j]\},$$

and

$$\sum\sum_{i\neq j} \operatorname{cov}[a(R_i), a(R_j)] = -\sum_{i=1}^{N} \operatorname{var}[a(R_i)] \leqq 0,$$

which entails

$$\text{(3.12)} \quad \sum\sum_{i\neq j} E \operatorname{cov}[a(R_i), a(R_j) \mid X_i, X_j] \leqq -\sum\sum_{i\neq j} \operatorname{cov}\{E[a(R_i) \mid X_i, X_j], E[a(R_i) \mid X_i, X_j]\}.$$

Furthermore, in view of (3.10), (3.11) and (3.12),

$$\text{(3.13)} \quad \begin{aligned} &\sum\sum_{i\neq j} c_i c_j \operatorname{cov}[a(R_i), a(R_j)] \\ &\leqq \max_{1\leqq i\leqq N} c_i^2 \Big\{\sum\sum_{i\neq j} E \operatorname{cov}[a(R_i), a(R_j) \mid X_i, X_j] \\ &\qquad - \sum\sum_{i\neq j} \operatorname{cov}\{E[a(R_i) \mid X_i, X_j], E[a(R_j) \mid X_i, X_j]\}\Big\} \\ &\leqq -2 \max_{1\leqq i\leqq N} c_i^2 \sum\sum_{i\neq j} \operatorname{cov}\{E[a(R_i) \mid X_i, X_j], E[a(R_j) \mid X_i, X_j]\}. \end{aligned}$$

The last inequality together with (3.8) and (3.9) implies that (3.7) will be proved, if we show that

$$\text{(3.14)} \quad -\sum\sum_{i\neq j} \operatorname{cov}\{E[a(R_i) \mid X_i, X_j], E[a(R_j) \mid X_i, X_j]\} \leqq 10 \sum_{i=1}^{N} a_i^2.$$

Note the identity

$$\begin{aligned} &\operatorname{cov}\{E[a(R_i)\mid X_i, X_j], E[a(R_j)\mid X_i, X_j]\} \\ (3.15)\qquad &= \operatorname{cov}\{E[a(R_i)\mid X_i]; E[a(R_j)\mid X_i]\} + \operatorname{cov}\{E[a(R_i)\mid X_j], E[a(R_j)\mid X_j] \\ &+ E\{(E[a(R_i)\mid X_i, X_j] - E[a(R_i)\mid X_i])(E[a(R_j)\mid X_i, X_j] \\ &- E[a(R_j)\mid X_j])\}. \end{aligned}$$

We shall now try to obtain bounds for the sums of the right-hand side members of (3.15) over all $j \neq i$. The first two members give the same sum, which may be bounded as follows:

$$\begin{aligned} &-\sum\sum_{i\neq j} \operatorname{cov}\{E[a(R_i)\mid X_i], E[a(R_j)\mid X_i]\} \\ (3.16)\qquad &= \sum_{i=1}^{N} \operatorname{cov}\{E[a(R_i)\mid X_i], E[-\sum_{j\neq i} a(R_j)\mid X_i]\} \\ &= \sum_{i=1}^{N} \operatorname{cov}\{E[a(R_i)\mid X_i], E[a(R_i) - \sum_{j=1}^{N} a_j \mid X_i]\} \\ &= \sum_{i=1}^{N} \operatorname{var}\{E[a(R_i)\mid X_i]\} \leqq \sum_{i=1}^{N} \operatorname{var}[a(R_i)] \leqq \sum_{i=1}^{N} a_i^2. \end{aligned}$$

In the following sum we shall utilize the relation (3.3), the self-evident fact

$$|[u(X_i - X_j) - F_j(X_i)][u(X_j - X_i) - F_i(X_j)]| \leqq 1,$$

and the inequality

$$\begin{aligned} &P(R_i = k \mid X_i = x, X_j = x - 1) \\ &\qquad \leqq P(R_i = k \mid X_i = x) + P(R_i = k - 1 \mid X_i = x), \end{aligned}$$

entailed by (3.4) through (3.6). We thus obtain

$$\begin{aligned} &-\sum\sum_{i\neq j} E\{(E[a(R_i)\mid X_i, X_j] - E[a(R_i)\mid X_i])(E[a(R_j)\mid X_i, X_j] \\ &\quad - E[a(R_j)\mid X_j])\} \\ &\leqq \sum\sum_{i\neq j}\sum_k\sum_h (a_k - a_{k-1})(a_h - a_{h-1})E\{[P(R_i = k\mid X_i) \\ &\quad + P(R_i = k - 1\mid X_i)][P(R_j = h\mid X_j) + P(R_j = h - 1\mid X_j]\} \\ (3.17)\qquad &= \sum_k\sum_h (a_k - a_{k-1})(a_h - a_{h-1})\sum\sum_{i\neq j}[P(R_i = k)P(R_j = h) \\ &\quad + P(R_i = k - 1)P(R_j = h - 1) + P(R_i = k - 1)P(R_j = h) \\ &\quad + P(R_i = k)P(R_j = h - 1)] \\ &\leqq 4\sum_k\sum_h (a_k - a_{k-1})(a_h - a_{h-1}) = 4(a_N - a_1)^2 \\ &\leqq 8\sum_{i=1}^{N} a_i^2. \end{aligned}$$

Here we have used the fact that the events $R_i = k$, $1 \leqq i \leqq N$, are disjoint so that $\sum_{i=1}^{N} P(R_i = k) = 1$. Now (3.14) follows from (3.15), (3.16) used twice, and (3.17).

4. Projection approximation for S with smooth and bounded scores. The central limit theorem is concerned with sequences of sums of independent random variables. Its scope may however be extended to statistics that are asymptotically (as the number of observations increases) equivalent to such sums. Given a statistic $S = s(X_1, \cdots, X_N)$, defined on a sequence of independent observations $X_1, \cdots, X_N$, we shall try to approximate it by a statistic

$$L = \sum_{i=1}^{N} l_i(X_i), \tag{4.1}$$

where the functions l_i may be chosen arbitrarily. Ovbiously, L is a sum of independent random variables $Y_i = l_i(X_i)$, $1 \leqq i \leqq N$. The set $\mathcal{L}$ of statistics L such that $El_i^2(X_i) < \infty$, $1 \leqq i \leqq N$, availed with the usual inner product $(L, L') = ELL'$, is a closed subspace of the Hilbert space of square integrable statistics. The best approximation in the mean square is given by a projection on the subspace $\mathcal{L}$. It is worth noting that the possibility of a successful approximation of a statistic by an L-statistic is usually better than one would intuitively expect. The projection may be obtained explicitly in terms of conditional expectations as is shown in the following.

Projection Lemma 4.1. *Let $X_1, \cdots, X_N$ be independent random variables and $S = s(X_1, \cdots, X_N)$ be a statistic such that $ES^2 < \infty$. Let*

$$\hat{S} = \sum_{i=1}^{N} E(S \mid X_i) - (N-1)ES. \tag{4.2}$$

Then

$$E\hat{S} = ES \tag{4.3}$$

and

$$E(S - \hat{S})^2 = \operatorname{var} S - \operatorname{var} \hat{S}. \tag{4.4}$$

Moreover, if L is given by (4.1) with $El_i^2(X_i) < \infty$, $1 \leqq i \leqq N$, then

$$E(S - L)^2 = E(S - \hat{S})^2 + E(\hat{S} - L)^2. \tag{4.5}$$

Proof. (4.3) is obvious, and (4.4) follows from (4.5), if we take for L the constant $ES = E\hat{S}$ and rearrange the formula. Thus it remains to prove (4.5). Without loss of generality, we may assume that $ES = E\hat{S} = 0$. We obviously have

$$E[(S - \hat{S})(\hat{S} - L)] = \sum_{i=1}^{N} E[E(S - \hat{S} \mid X_i)(E(S \mid X_i) - l_i(X_i))]. \tag{4.6}$$

Now, since $X_1, \cdots, X_N$ are independent,

$$\begin{aligned} E[E(S \mid X_j) \mid X_i] &= ES, && \text{if } j \neq i \\ &= E(S \mid X_i), && \text{if } j = i. \end{aligned}$$

Consequently, since $ES = 0$, $E(\hat{S} \mid X_i) = \sum_{j=1}^{N} E[E(S \mid X_j) \mid X_i] = E(S \mid X_i)$, i.e.

$$E(S - \hat{S} \mid X_i) = 0, \qquad 1 \leqq i \leqq N. \tag{4.7}$$

By inserting (4.7) into (4.6), we obtain $E[(S - \hat{S})(\hat{S} - L)] = 0$, which is equivalent to (4.5). Q.E.D.

If S is a sum of simpler statistics, we may obtain the following upper bound for the residual variance $E(S - \hat{S})^2$:

RESIDUAL VARIANCE LEMMA 4.2. *Let* $S = \sum_{i=1}^N S_i$ *and* $\hat{S} = \sum_{i=1}^N E(S \mid X_i) - (N - 1)ES$. *Then*

$$
\begin{aligned}
E(S - \hat{S})^2 \leqq \textstyle\sum_{i=1}^N E[S_i - E(S_i \mid X_i)]^2 \\
+ \textstyle\sum \sum_{i \neq j} \{E([S_i - E(S_i \mid X_i)][S_j - E(S_j \mid X_j)]) \\
- \textstyle\sum_{k \neq i,j} \operatorname{cov}[E(S_i \mid X_k), E(S_j \mid X_k)]\}.
\end{aligned} \tag{4.8}
$$

PROOF. Utilizing (4.4) we obtain

$$
\begin{aligned}
E(S - \hat{S})^2 = \operatorname{var} S - \operatorname{var} \hat{S} = \textstyle\sum_{i=1}^N [\operatorname{var} S_i - \operatorname{var} \sum_{k=1}^N E(S_i \mid X_k)] \\
+ \textstyle\sum_{i \neq j} \{\operatorname{cov}(S_i, S_j) - \sum_{k=1}^N \operatorname{cov}[E(S_i \mid X_k), E(S_j \mid X_k)]\}.
\end{aligned} \tag{4.9}
$$

Next

$$
\begin{aligned}
\operatorname{var} S_i - \operatorname{var} \textstyle\sum_{k=1}^N E(S_i \mid X_k) &= \operatorname{var} S_i - \textstyle\sum_{k=1}^N \operatorname{var} E(S_i \mid X_k) \\
&\leqq \operatorname{var} S_i - \operatorname{var}(S_i \mid X_i) \\
&= E[S_i - E(S_i \mid X_i)]^2.
\end{aligned} \tag{4.10}
$$

Further, obviously,

$$
\begin{aligned}
\operatorname{cov}(S_i, S_j) &= \operatorname{cov}[E(S_i \mid X_i), E(S_j \mid X_i)] \\
&\quad + \operatorname{cov}[E(S_i \mid X_j), E(S_j \mid X_j)] \\
&\quad + E\{[S_i - E(S_i \mid X_i)][S_j - E(S_j \mid X_j]\}.
\end{aligned} \tag{4.11}
$$

Now, (4.8) is an easy consequence of (4.9) through (4.11).

THEOREM 4.1. *Let the scores-generating function* φ *possess a bounded second derivative. Consider the statistic* $S = \sum_{i=1}^N c_i \varphi(R_i/(N + 1))$ *and put* $\hat{S} = \sum_{i=1}^N E(S \mid X_i) - (N - 1)ES$. *Then there exists a constant* $K = K(\varphi)$ *such that for any* N, $(c_1, \cdots, c_N)$ *and* $(F_1, \cdots, F_N)$

$$
E(S - \hat{S})^2 \leqq KN^{-1} \textstyle\sum_{i=1}^N (c_i - \bar{c})^2. \tag{4.12}
$$

PROOF. Without loss of generality we may assume $\bar{c} = 0$. Put

$$
\rho_i = R_i/(N + 1), \qquad 1 \leqq i \leqq N, \tag{4.13}
$$

so that $S = \sum_{i=1}^N c_i \varphi(\rho_i)$. By Taylor expansion

$$
\begin{aligned}
\varphi(\rho_i) = \varphi[E(\rho_i \mid X_i)] + (\rho_i - E(\rho_i \mid X_i))\varphi'[E(\rho_i \mid X_i)] \\
+ (\rho_i - E(\rho_i \mid X_i))^2 k_i(X_i)
\end{aligned} \tag{4.14}
$$

where $k_i(x)$ is bounded, say $k_i^2(x) < K_2$, $-\infty < x < \infty$, $1 \leqq i \leqq N$. Thus

$S = S_1 + S_2 + S_3$ where

$$S_1 = \sum_{i=1}^N c_i \rho_i \varphi'[E(\rho_i \mid X_i)],$$

$$S_2 = \sum_{i=1}^N c_i \{\varphi[E(\rho_i \mid X_i)] - E(\rho_i \mid X_i)\varphi'[E(\rho_i \mid X_i)]\},$$

$$S_3 = \sum_{i=1}^N c_i [\rho_i - E(\rho_i \mid X_i)]^2 k_i(X_i).$$

Now, denoting the projection of S_i by $\hat{S}_i$, we have $\hat{S} = \hat{S}_1 + \hat{S}_2 + \hat{S}_3$. Now, obviously $\hat{S}_2 = S_2$, so that

$$E(S - \hat{S})^2 \leqq 2E(S_1 - \hat{S}_1)^2 + 2E(S_3 - \hat{S}_3)^2 \leqq 2E(S_1 - \hat{S}_1)^2 + 2ES_3^2. \tag{4.15}$$

Next, by the Schwartz inequality

$$ES_3^2 \leqq \sum_{i=1}^N c_i^2 \sum_{i=1}^N E\{[\rho_i - E(\rho_i \mid X_i)]^4 k_i^2(X_i)\} \leqq K_2 \sum_{i=1}^N c_i^2 \sum_{i=1}^N E[\rho_i - E(\rho_i \mid X_i)]^4.$$

As is seen from (2.2), for $X_i = x$, $R_i = \rho_i(N + 1)$ is a sum of N independent zero-one random variables. Consequently

$$E[\rho_i - E(\rho_i \mid X_i)]^4 \leqq \tfrac{1}{4}(N + 1)^{-2}, \qquad 1 \leqq i \leqq N,$$

and

$$ES_3^2 \leqq K_2(N + 1)^{-1} \sum_{i=1}^N c_i^2. \tag{4.16}$$

Further we shall apply Lemma 4.2 to $S_1 = \sum_{i=1}^N S_{1i}$, $S_{1i} = c_i \rho_i \varphi'[E(\rho_i \mid X_i)]$. Let K_1 be an upper bound for $[\varphi'(t)]^2$, $0 < t < 1$. Then, obviously,

$$E(S_{1i} - E(S_{1i} \mid X_i))^2 \leqq c_i^2(N + 1)^{-2} K_1 E \operatorname{var}(R_i \mid X_i) \leqq \tfrac{1}{4} c_i^2 (N + 1)^{-1} K_1. \tag{4.17}$$

The convariance term in (4.8) is somewhat more complicated. We have

$$\begin{aligned} &E([S_i - E(S_i \mid X_i)][S_j - E(S_j \mid X_j)]) \\ &= E \operatorname{cov}(S_i, S_j \mid X_i, X_j) \\ &\quad + E\{[E(S_i \mid X_i, X_j) - E(S_i \mid X_i)][E(S_j \mid X_i, X_j) - E(S_j \mid X_j)]\}. \end{aligned} \tag{4.18}$$

Now (2.2) entails, $i \neq j$,

$$\begin{aligned} &\operatorname{cov}(S_{1i}, S_{1j} \mid X_i, X_j) \\ &= c_i c_j (N + 1)^{-2} \sum_{k \neq i,j} \{\min [F_k(X_i), F_k(X_j)] - F_k(X_i) F_k(X_j)\} \varphi'(E(\rho_i \mid X_i)) \\ &\quad \cdot \varphi'(E(\rho_j \mid X_j)), \end{aligned}$$

and, consequently,

$$\begin{aligned} &E \operatorname{cov}(S_{1i}, S_{1j} \mid X_i, X_j) \\ &\qquad = c_i c_j (N + 1)^{-2} \sum_{k \neq i,j} \int \int \{\min [F_k(x), F_k(y)] - F_k(x) F_k(y)\} \\ &\qquad\quad \cdot \varphi'(E(\rho_i \mid X_i = x)) \varphi'(E(\rho_i \mid X_j = y)) \, dF_i(x) \, dF_j(y). \end{aligned}$$

The right side may be transformed as follows: Introducing

$$(4.19)\quad l_{ik}(s) = \int [u(x - s) - F_k(x)]\varphi'(E(\rho_i \mid X_i = x))\, dF_i(x),$$

we easily see that

$$\int\int \{\min [F_k(x), F_k(y)] - F_k(x)F_k(y)\}\varphi'(E(\rho_i \mid X_i = x))\varphi'(E(\rho_j \mid X_j = y)) \cdot dF_i(x)\, dF_j(y) = \operatorname{cov} [l_{ik}(X_k), l_{jk}(X_k)]$$

Thus

$$(4.20)\quad E \operatorname{cov} (S_{1i}, S_{1j} \mid X_i, X_j) = c_i c_j (N + 1)^{-2} \textstyle\sum_{k \neq i,j} \operatorname{cov} [l_{ik}(X_k), l_{jk}(X_k)].$$

On the other hand, (3.3) yields

$$\begin{aligned} &E(S_{1i} \mid X_k) - E(S_{1i}) \\ &\quad = c_i \int [E(\rho_i \mid X_k, X_i = x) - E(\rho_i \mid X_i = x)]\varphi'(E(\rho_i \mid X_i = x))\, dF_i(x) \\ &\quad = c_i (N + 1)^{-1} \int [u(x - X_k) - F_k(x)]\varphi'(E(\rho_i \mid X_i = x))\, dF_i(x) \\ &\quad = c_i (N + 1)^{-1} l_{ik}(X_k). \end{aligned}$$

Consequently, $i \neq j$, $k \neq i$, $k \neq j$,

$$(4.21)\quad \operatorname{cov} [E(S_{1i} \mid X_k), E(S_{1j} \mid X_k)] = c_i c_j (N + 1)^{-2} \operatorname{cov} [l_{ik}(X_k), l_{jk}(X_k)]^{\cdot}$$

Again by (3.3),

$$(4.22)\quad E(\rho_i \mid X_i, X_j) - E(\rho_i \mid X_i) = (N + 1)^{-1}[u(X_i - X_j) - F_j(X_i)].$$

Consequently,

$$\begin{aligned} &|E\{[E(S_{1i} \mid X_i, X_j) - E(S_{1i} \mid X_i)][E(S_{1j} \mid X_i, X_j) - E(S_{1j} \mid X_j)]\}| \\ &\quad \leqq K_1 |c_i c_j| (N + 1)^{-2} |E\{[u(X_i - X_j) - F_j(X_i)][u(X_j - X_i) - F_i(X_j)]\}| \\ (4.23)\quad &\quad \leqq K_1 |c_i c_j| (N + 1)^{-2} \{E[u(X_i - X_j) - F_j(X_i)]^2 \\ &\qquad \cdot E[u(X_j - X_i) - F_i(X_j)]^2\}^{\frac{1}{2}} \\ &\quad \leqq \tfrac{1}{4} K_1 |c_i c_j| (N + 1)^{-2}. \end{aligned}$$

Now, combining (4.18) through (4.22), we obtain

$$\begin{aligned} &|E\{[S_{1i} - E(S_{1i} \mid X_i)][S_{1j} - E(S_{1j} \mid X_j)]\} \\ (4.24)\quad &\qquad - \textstyle\sum_{k \neq i,j} \operatorname{cov} [E(S_{1i} \mid X_k), E(S_{1j} \mid X_k)| \\ &\qquad \leqq \tfrac{1}{4} K_1 |c_i c_j| (N + 1)^{-2}. \end{aligned}$$

Applying (4.17) and (4.24) to (4.8), we obtain

$$\begin{aligned} (4.25)\quad E(S_1 - \hat{S}_1)^2 &\leqq \tfrac{1}{4} K_1 (N + 1)^{-1} \textstyle\sum_{i=1}^{N} c_i^2 + \tfrac{1}{4} K_1 (N + 1)^{-2} \sum \sum_{i \neq j} |c_i c_j| \\ &\leqq \tfrac{1}{2} K_1 (N + 1)^{-1} \textstyle\sum_{i=1}^{N} c_i^2. \end{aligned}$$

Finally, (4.15), (4.16) and (4.25) yield

$$E(S - \hat{S}) \leqq (K_1 + \tfrac{1}{2}K_2)(N + 1)^{-1} \sum_{i=1}^N c_i^2.$$

Thus (4.12) is proved with $K = K_1 + \frac{1}{2}K_2$, K_1 and K_2 being upper bounds for the squared first and second derivatives, respectively.

ACKNOWLEDGMENT. In the original paper inequality (4.12) was proved for φ linear only, whereas for φ'' bounded a weaker version of (4.12) with N^{-1} replaced by $N^{-\frac{1}{2}}$ was proved. The present stronger result is based on a suggestion kindly made to the author by Mr. Peter J. Huber.

THEOREM 4.2. *Under conditions of Theorem* 4.1 *there exists a constant* $M = M(\varphi)$ *such that for any* N, $(c_1, \cdots, c_N)$ *and* $(F_1, \cdots, F_N)$

$$(4.26) \qquad E(S - ES - \textstyle\sum_{i=1}^N Z_i)^2 \leqq MN^{-1} \sum_{i=1}^N (c_i - \bar{c})^2$$

and

$$(4.27) \qquad (ES - \mu)^2 \leqq MN^{-1} \textstyle\sum_{i=1}^N c_i^2,$$

where

$$(4.28) \quad Z_i = l_i(X_i) = N^{-1} \textstyle\sum_{j=1}^N (c_j - c_i) \int [u(x - X_i) - F_i(x)] \cdot\varphi'(H(x))\, dF_j(x), \quad 1 \leqq i \leqq N,$$

and

$$\mu = \textstyle\sum_{i=1}^N c_i \int \varphi(H(x))\, dF_i(x),$$

with $H(x) = N^{-1} \sum_{j=1}^N F_j(x)$.

PROOF. We have

$$\hat{S} - E\hat{S} = \textstyle\sum_{i=1}^N \sum_{j=1}^N c_j[E(\varphi(\rho_j) \mid X_i) - E\varphi(\rho_j)].$$

Since, obviously,

$$\textstyle\sum_{j=1}^N \varphi(\rho_j) = \sum_{j=1}^N E(\varphi(\rho_j) \mid X_i) = \sum_{j=1}^N E\varphi(\rho_j) = \sum_{i=1}^N \varphi(i/(N + 1)),$$

we also have

$$(4.29) \quad \hat{S} - E\hat{S} = \textstyle\sum_{i=1}^N \sum_{j=1}^N (c_j - c_i)[E(\varphi(\rho_j) \mid X_i) - E\varphi(\rho_j)].$$

Now (3.3) yields

$$(4.30) \quad \begin{aligned} &E(\varphi(\rho_j) \mid X_i) - E\varphi(\rho_j) \\ &\quad = \int [u(x - X_i) - F_i(x)] \textstyle\sum_{k=2}^N [\varphi(k/(N + 1)) - \varphi((k - 1)/(N+1))] \\ &\qquad \cdot P(R_j = k \mid X_j = x, X_i = x - 1)\, dF_j(x) \\ &\quad = (N + 1)^{-1} \int [u(x - X_i) - F_i(x)]E[\varphi'(\rho_j) \mid X_j = x, X_i = x - 1] \\ &\qquad \cdot dF_j(x) + (N + 1)^{-2}K_2^{\frac{1}{2}}\alpha_i, \qquad |\alpha_i| \leqq 1. \end{aligned}$$

Further,

$$E[\varphi'(\rho_j) \mid X_j = x, X_i = x - 1] = \varphi'(H(x)) + 2N^{-\frac{1}{2}}K_2^{\frac{1}{2}}\beta_i, \quad |\beta_i| \leqq 1,$$

so that

$$E(\varphi(\rho_j) \mid X_i) - E\varphi(\rho_j) = (N+1)^{-1} \int [u(x - X_i) - F_i(x)] \cdot \varphi'(H(x))\, dF_j(x) + 3N^{-3/2} K_2^{\frac{1}{2}} \gamma_i, \quad |\gamma_i| \leqq 1.$$

Consequently, since $EZ_i = 0$,

$$\begin{aligned} E(\hat{S} - E\hat{S} - \textstyle\sum_{i=1}^N Z_i)^2 &= \textstyle\sum_{i=1}^N E(\sum_{j=1}^N (c_j - c_i) 3N^{-3/2} K_2^{\frac{1}{2}} \gamma_i)^2 \\ &\leqq 9N^{-3} K_2 \textstyle\sum_{i=1}^N (\sum_{j=1}^N |c_j - c_i|)^2 \\ &\leqq 18N^{-1} K_2 \textstyle\sum_{i=1}^N (c_i - \bar{c})^2. \end{aligned}$$

This together with (4.12) yields (4.26) with $M = 2K + 36K_2^2$.

Finally,

$$\varphi(\rho_i) = \varphi(H(X_i)) + (\rho_i - H(X_i))\varphi'(H(X_i)) + \tfrac{1}{2}(\rho_i - H(X_i))^2 k_i(X_i)$$

yields, for proper K_4,

$$|E\varphi(\rho_i) - E\varphi(H(X_i))| \leqq K_4 N^{-1}.$$

Therefore,

$$(4.31) \quad |ES - \mu|^2 = (\textstyle\sum_{i=1}^N c_i[E\varphi(\rho_i) - E\varphi(H(X_i))])^2 \leqq K_4^2 N^{-1} \sum_{i=1}^N c_i^2.$$

Obviously, (4.31) provides (4.27) with $M = K_4^2$. The proof is terminated.

5. The proofs to Section 2.

PROOF OF THEOREM 2.1. Consider the random variables Z_i given by (4.28) and σ^2 given by (2.9). Obviously, $EZ_i = 0$ and $\sigma^2 = \sum_{i=1}^N \operatorname{var} Z_i$. Choose a $\epsilon > 0$. Then by the Lindeberg theorem, there exists a $\delta > 0$ such that the relation

$$(5.1) \qquad \sigma^{-2} \textstyle\sum_{i=1}^N \int_{|x| > \delta\sigma} x^2\, dP(Z_i \leqq x) < \delta$$

entails

$$(5.2) \qquad \sup_x |P(\textstyle\sum_{i=1}^N Z_i < x\sigma) - \Phi(x)| < \tfrac{1}{4}\epsilon,$$

where $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^x \exp(-\frac{1}{2}y^2)\, dy$. Furthermore, there exists a $\beta > 0$ such that

$$(5.3) \qquad |\Phi(x) - \Phi(x \pm \beta)| < \tfrac{1}{4}\epsilon, \qquad -\infty < x < \infty,$$

and, in turn,

$$(5.4) \qquad \sup_x |P(\textstyle\sum_{i=1}^N Z_i < x\sigma \pm \beta\sigma) - \Phi(x)| < \tfrac{1}{2}\epsilon.$$

Now we shall show that (2.8) is entailed by

$$(5.5) \quad \operatorname{var} S \geqq [2\delta^{-1} \sup_{0<t<1} |\varphi'(t)| + (2\epsilon^{-\frac{1}{2}}\beta^{-1} + 1)M^{\frac{1}{2}}]^2 \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2,$$

where M is the constant appearing in (4.26). If the scores are generated from

φ by (2.4), Theorem 4.2 entails

$$(5.6) \qquad |\sigma - (\operatorname{var} S)^{\frac{1}{2}}| \leqq M^{\frac{1}{2}} \max_{1 \leqq i \leqq N} |c_i - \bar{c}|,$$

and hence (5.5) entails

$$(5.7) \qquad \delta\sigma \geqq 2 \max |c_i - \bar{c}| \sup |\varphi'(t)|.$$

On the other hand (4.28) implies that always

$$(5.8) \qquad |Z_i| \leqq 2 \max |c_i - \bar{c}| \sup |\varphi'(t)|.$$

Putting together (5.7) and (5.8) we see that the left side of (5.1) is zero for σ satisfying (5.7). Thus (5.5) entails (5.1).

Further, by (4.26) and (5.4)

$$\begin{aligned} P(S - ES < x\sigma) &\leqq P(\textstyle\sum_{i=1}^N Z_i < x\sigma + \beta\sigma) + P(|S - ES - \sum_{i=1}^N Z_i| > \beta\sigma) \\ &\leqq \Phi(x) + \tfrac{1}{2}\epsilon + E(S - ES - \textstyle\sum_{i=1}^N Z_i)^2/\beta^2\sigma^2 \\ &\leqq \Phi(x) + \tfrac{1}{2}\epsilon + M \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2/\beta^2\sigma^2. \end{aligned}$$

Now (5.5) and (5.6) entail

$$\beta^2\sigma^2 \geqq 4\epsilon^{-1} M \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2.$$

Altogether we have

$$(5.9) \qquad P(S - ES < x\sigma) \leqq \Phi(x) + \tfrac{1}{2}\epsilon + \tfrac{1}{4}\epsilon = \Phi(x) + \tfrac{3}{4}\epsilon.$$

Similarly we would prove the opposite inequality leading to

$$(5.10) \qquad \sup_x |P(S - ES < x\sigma) - \Phi(x)| < \tfrac{3}{4}\epsilon.$$

Finally, if $\epsilon < 1$, then (5.5) and (5.6) entail

$$|\sigma - (\operatorname{var} S)^{\frac{1}{2}}| < \beta\sigma,$$

which combined with (5.3) and (5.10) yields

$$(5.11) \qquad \sup_x |P(S - ES < x\, (\operatorname{var} S)^{\frac{1}{2}}) - \Phi(x)| < \epsilon.$$

In the course of the proof we have also proved that (5.10) is satisfied, if var S is replaced by σ^2 in (5.5).

If the scores were generated by (2.5), it suffices to note that, by the Taylor expansion

$$E\varphi(U_N^{(i)}) = \varphi(i/(N + 1)) + \kappa_{iN}$$

where $|\kappa_{iN}| \leqq kN^{-1}$, where k does not depend from i nor from N. Thus, denoting the statistic $\sum_{i=1}^N c_i a_N(R_i)$ by S, if (2.4) holds, and by S', if (2.5) holds, we have

$$E(S - ES - S' + ES')^2 \leqq (kN^{-1} \textstyle\sum_{i=1}^N |c_i - \bar{c}|)^2 \leqq k^2 N^{-1} \sum_{i=1}^N (c_i - \bar{c})^2.$$

Consequently $S - ES$ is equivalent to $S' - ES'$ in asymptotic considerations.

PROOF OF THEOREM 2.2. Obviously

$$\begin{aligned}\int_{x_0}^{y} \varphi'(H(x))\, dF_j(x) &= \int_{x_0}^{y} \varphi'(F_j(x))\, dF_j(x) + R_j(y) \\ &= \varphi(F_j(y)) + \text{const} + R_j(y) \\ &= \varphi(F_i(y)) + \text{const} + R_{ij}(y),\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}|R_{ij}(y)| &\leqq [\sup_t |\varphi''(t)| + \sup_t |\varphi'(t)|] \max_{i,j,x} |F_i(x) - F_j(x)| \\ &= L \max_{i,j,x} |F_i(x) - F_j(x)|, \qquad 1 \leqq i, j \leqq N,\end{aligned} \tag{5.13}$$

where $L = L(\varphi)$ is defined by the last equation. Let

$$V_i = (\bar{c} - c_i)\varphi(F_i(X_i)) \tag{5.14}$$

and note, by inspection of (2.14), that

$$d^2 = \sum_{i=1}^{N} \text{var}\, V_i. \tag{5.15}$$

Now (2.26), (4.28), (5.12) and (5.14) entail

$$Z_i = V_i + \text{const} + R_i^*(X_i)N^{-1} \sum_{i=1}^{N} |c_j - c_i|, \tag{5.16}$$

where R_i^* is bounded by the right side of (5.13). Consequently, by (5.15),

$$\begin{aligned}|\sigma - d| &= |(\text{var} \sum_{i=1}^{N} Z_i)^{\frac{1}{2}} - (\text{var} \sum_{i=1}^{N} V_i)^{\frac{1}{2}}| \\ &\leqq (\sum_{i=1}^{N} \text{var}\,(Z_i - V_i))^{\frac{1}{2}} \\ &\leqq [\sum_{i=1}^{N} (N^{-1} \sum_{j=1}^{N} |c_j - c_i|)^2]^{\frac{1}{2}} L \max_{i,j,x} |F_i(x) - F_j(x)| \\ &\leqq L(2 \sum_{i=1}^{N} (c_i - \bar{c})^2)^{\frac{1}{2}} \max_{i,j,x} |F_i(x) - F_j(x)|.\end{aligned} \tag{5.17}$$

Now we find δ_ϵ as follows. In accordance with Theorem 2.1, we choose $K_{\frac{1}{2}\epsilon}$ and $\alpha > 0$ such that $\sigma^2 > K_{\frac{1}{2}\epsilon} \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2$ entails

$$\sup_x |P(S - ES < x\sigma(1 \pm \alpha) - \Phi(x)| < \epsilon.$$

Then we choose δ_ϵ so that, first, $\delta_\epsilon^{-1} > K_{\frac{1}{2}\epsilon}$, and, second, that (2.12) entails $|\sigma/d - 1| < \alpha$. The last implication is for sufficiently small δ_ϵ guaranteed by (5.17) and (2.14). The rest easily follows.

PROOF OF THEOREM 2.3. We shall start with the following

LEMMA 5.1. *If φ satisfies the conditions of Theorem 2.3, then for any $\alpha > 0$ there exists a decomposition*

$$\varphi(t) = \psi(t) + \varphi_1(t) - \varphi_2(t), \qquad 0 < t < 1, \tag{5.18}$$

such that ψ is a polynomial, φ_1 and φ_2 are non-decreasing, and

$$\int_0^1 \varphi_1^2(t)\, dt + \int_0^1 \varphi_2^2(t)\, dt < \alpha. \tag{5.19}$$

PROOF. Without losing generality, we may assume that $\varphi(t)$ itself is non-decreasing. Take an $\epsilon > 0$ and put

$$\begin{aligned} \varphi_0(t) &= \varphi(\epsilon), && 0 < t < \epsilon, \\ &= \varphi(t), && \epsilon \leqq t < 1 - \epsilon, \\ &= \varphi(1-\epsilon), && 1 - \epsilon < t < 1; \end{aligned} \tag{5.20}$$

$$\varphi_3(t) = \min\,[0, \varphi(t) - \varphi(\epsilon)]; \tag{5.21}$$

$$\varphi_4(t) = \max\,[0, \varphi(t) - \varphi(1-\epsilon)]. \tag{5.22}$$

Obviously,

$$\varphi(t) = [\varphi_0(t) + \varphi_3(t) + \varphi_4(t)], \tag{5.23}$$

where φ_3 and φ_4 are non-decreasing. As $\varphi(t)$ is absolutely continuous on $(\epsilon, 1-\epsilon)$, $\varphi_0(t)$ is absolutely continuous on the whole interval $(0, 1)$. Consequently, there exists a derivative $\varphi_0'(t)$ such that

$$\varphi_0(t) = \varphi(\epsilon) + \int_0^t \varphi_0'(s)\,ds, \qquad 0 \leqq t \leqq 1. \tag{5.24}$$

Further to every $\beta > 0$ there exists a polynomial $q(s)$ such that

$$\int_0^1 |\varphi_0'(s) - q(s)|\,ds < \beta, \tag{5.25}$$

because the set of polynomials is a dense subset of the L_1-space of integrable functions.

Putting

$$\psi(t) = \varphi(\epsilon) + \int_0^t q(s)\,dt, \tag{5.26}$$

$$\varphi_1(t) = \varphi_3(t) + \varphi_4(t) + \int_0^t \max\,[0, \varphi_0'(s) - q(s)]\,ds, \tag{5.27}$$

$$\varphi_2(t) = \int_0^t \max\,[0, q(s) - \varphi_0'(s)]\,ds, \tag{5.28}$$

we easily see that

$$\varphi(t) = \psi(t) + \varphi_1(t) - \varphi_2(t), \tag{5.29}$$

where $\psi(t)$ is a polynomial, $\varphi_1(t)$ and $\varphi_2(t)$ are non-decreasing and (5.20) may be satisfied by taking ϵ and β in the above construction sufficiently small. Q.E.D.

Now take $\epsilon > 0$ and $\eta > 0$. Then choose $\beta > 0$ and $\gamma > 0$ such that

$$|\Phi(x) - \Phi[(x \pm \beta)(1 \pm \gamma)^{-1}]| < \tfrac{1}{4}\epsilon \tag{5.30}$$

and $\alpha > 0$ such that

$$\alpha < \eta \min\,(\gamma^2, \tfrac{1}{4}\beta^2\epsilon)/84. \tag{5.31}$$

Subsequently, we decompose φ according to (5.18) with α satisfying (5.19) and (5.31). We denote

$$\begin{aligned} S_\psi &= \textstyle\sum_{i=1}^N c_i \psi(R_i/(N+1)), \\ S_i &= \textstyle\sum_{i=1}^N c_i \varphi_i(R_i/(N+1)), \qquad i = 1, 2. \end{aligned} \tag{5.32}$$

Obviously $S = S_\psi + S_1 - S_2$. As the functions φ_i are non-decreasing, we have, in view of Theorem 3.1,

$$(5.33)\quad |(\mathrm{var}\, S)^{\frac12} - (\mathrm{var}\, S_\psi)^{\frac12}| \leqq (\mathrm{var}\,(S - S_\psi))^{\frac12} \leqq (\mathrm{var}\, S_1)^{\frac12} + (\mathrm{var}\, S_2)^{\frac12}$$
$$\leqq (42)^{\frac12} \max_{1\leqq i\leqq N} |c_i - \bar c| [\textstyle\sum_{i=1}^N (\varphi_{1\alpha}^2(i/(N+1)) + \varphi_{2\alpha}^2(i/(N+1)))]^{\frac12}.$$

Now, it may be easily shown that for a non-decreasing φ

$$(5.34)\quad \textstyle\sum_{i=1}^N \varphi^2(i/(N+1)) \leqq (N+1)\int_0^1 \varphi^2(t)\,dt \leqq 2N\int_0^1 \varphi^2(t)\,dt.$$

Combining (2.16) and (5.31) through (5.34), we obtain

$$(5.35)\quad |(\mathrm{var}\, S)^{\frac12} - (\mathrm{var}\, S_\psi)^{\frac12} \leqq [\mathrm{var}\,(S - S_\psi)]^{\frac12} \leqq (\mathrm{var}\, S)^{\frac12} \min(\gamma, \tfrac12\beta\epsilon^{\frac12}).$$

Finally, let $K_{\frac12\epsilon} = K_{\frac12\epsilon}(\psi)$ be the constant, the existence of which was established in Theorem 2.1. (Note that ψ is a polynomial, and hence has a bounded second derivative.) Then put $N_{\epsilon\eta} = (1-\gamma)^{-2}\eta^{-1}K_{\frac12\epsilon}$ so that (2.16) in conjunction with (5.35) entails $\mathrm{var}\, S_\psi > K_{\frac12\epsilon}(\psi) \max_{1\leqq i\leqq N}(c_i - \bar c)^2$. Consequently, by Theorem 2.1, (5.30) and (5.35)

$$P(S - ES < x(\mathrm{var}\, S)^{\frac12})$$
$$\leqq P(S_\psi - ES_\psi < (x+\beta)(\mathrm{var}\, S)^{\frac12}) + P(|S - ES - S_\psi + ES_\psi| > \beta(\mathrm{var}\, S)^{\frac12})$$
$$\leqq P(S_\psi - ES_\psi < (x+\beta)(\mathrm{var}\, S)^{\frac12}) + \mathrm{var}\,(S - S_\psi)/\beta^2\, \mathrm{var}\,(S)$$
$$\leqq P(S_\psi - ES_\psi < (x+\beta)(1-\gamma)^{-1}(\mathrm{var}\, S_\psi)^{\frac12}) + \tfrac14\epsilon$$
$$\leqq \Phi((x+\beta)(1-\gamma)^{-1}) + \tfrac34\epsilon \leqq \Phi(x) + \epsilon.$$

Similarly we would prove the opposite inequality needed in (2.8).

The version of the theorem, in which var S is replaced by σ^2, may again be shown by a decomposition of σ^2, corresponding to the decomposition $\varphi = \psi + \varphi_1 - \varphi_2$. If we define σ_ψ^2, σ_1^2 and σ_2^2 by (2.9) and (2.10) where φ is replaced by ψ, φ_1 and φ_2, respectively, we obtain

$$(5.36)\qquad |\sigma - \sigma_\psi| \leqq \sigma_1 + \sigma_2.$$

Now, if φ in (2.10) is non-decreasing then the random variables

$$\int [u(x - X_i) - F_i(x)]\varphi'(H(x))\,dF_j(x), \qquad 1 \leqq j \leqq N,$$

are (see Lemma 3.1) non-negatively correlated. Consequently,

$$\sigma_h^2 \leqq 4 \max_{1\leqq i\leqq N}(c_i - \bar c)^2$$
$$\textstyle\sum_{i=1}^N \mathrm{var}\ [N^{-1}\sum_{j=1}^N \int [u(x - X_i) - F_i(x)]\varphi_h'(H(x))\,dF_j(x)]$$
$$(5.37)\quad \leqq 4\max_{1\leqq i\leqq N}(c_i - \bar c)^2$$
$$\cdot \textstyle\sum_{i=1}^N \mathrm{var}\ (\int [u(x - X_i) - F_i(x)]\varphi_h'(H(x))\,dH(x)$$
$$= 4\max_{1\leqq i\leqq N}(c_i - \bar c)^2 \textstyle\sum_{i=1}^N \mathrm{var}\,\varphi_h(H(X_i))$$
$$\leqq 4N \max_{1\leqq i\leqq N}(c_i - \bar c)^2 \int_0^1 \varphi_h^2(t)\,dt, \qquad h = 1, 2.$$

Further, in view of (5.36), (5.37) and (5.19),

$$|\sigma - \sigma_\psi|^2 \leqq 8N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \alpha.$$

Thus, for α sufficiently small, $|\sigma - \sigma_\psi|$ may be made negligible in comparison with $\eta N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2$. Since, as we see from (5.6) and (5.35), $|(\text{var } S_\psi)^{\frac{1}{2}} - \sigma_\psi|$ as well as $|(\text{var } S)^{\frac{1}{2}} - (\text{var } S_\psi)^{\frac{1}{2}}|$ are under control, too, the difference $|\sigma - (\text{var } S)^{\frac{1}{2}}|$ must be negligible with respect to σ if

$$\sigma^2 > \eta N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2$$

and N is sufficiently large.

The case of the scores given by (2.5) could be treated similarly as in the proof of Theorem 2.1, with help of the inequality

$$\sum_{i=1}^{N} [E\varphi(U_N^{(i)})]^2 \leqq N \int_0^1 \varphi^2(t)\, dt$$

holding for any φ.

Proof of Theorem 2.4. We omit this proof because it consists in the combination of methods used in the proof of Theorems 2.2 and 2.3.

6. A counterexample. The conditions concerning φ in Theorems 2.3 and 2.4 seem to be close to what is necessary, if the other conditions are unchanged. If φ is discontinuous, as it occurs with the median test and in the first stage of the study of the Kolmogorov-Smirnov test, the assertion of the theorems is not valid, as we now show. Particularly, we shall assume that

$$(6.1) \qquad \begin{aligned} \varphi(t) &= 0, \quad 0 < t < \tfrac{1}{2}, \\ &= 1, \quad \tfrac{1}{2} \leqq t < 1, \end{aligned}$$

and that the scores are given by (2.4). Further, let $c_i = 1, 1 \leqq i \leqq \frac{1}{4}N$, and $= 0, \frac{1}{4}N < i \leqq N$; $F_i = F, 1 \leqq i \leqq \frac{1}{4}N$, where F is a uniform distribution on $(\frac{1}{4}, \frac{1}{2})$ and, $F_i = G, \frac{1}{4}N < i \leqq N$, where G is uniform on $(0, \frac{1}{4}) \cup (\frac{1}{2}, 1)$. Then, if N is a multiple of 4,

$$(6.2) \quad S = \sum_{i=1}^{N} c_i \varphi(R_i/(N+1)) = \max \{0, \sum_{i > \frac{1}{4}N} [u(\tfrac{1}{4} - X_i) - \tfrac{1}{3}]\}$$

with u given by (2.1). Since S is a truncated binomial random variable, the order of var S is N, say var $S > bN$, $N \geqq 1$. Further, $\max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 = 9/16$, so that var $S > \eta N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2$ if $\eta \leqq (16/9)b$. On the other hand $P(S = 0) \to \frac{1}{2}$, so that (2.8) is not implied by (2.16) for any $N_{\epsilon\eta}$, provided that $\epsilon < \frac{1}{2}$ and $\eta \leqq b$.

REFERENCES

[1] Bhuchonghul, S. (1964). A class of non-parametric tests for independence in bivariate populations. *Ann. Math. Statist.* **35** 138–149.
[2] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. *Ann. Math. Statist.* **29** 972–994.
[3] Dwass, M. (1956). The large-sample power of rank order tests in the two-sample problem. *Ann. Math. Statist.* **27** 352–374.

[4] GOVINDARAJULU, Z., LECAM, L. and RAGHAVACHARI, M. (1966). Generalizations of theorems of Chernoff and Savage on the asymptotic normality of test statistics. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** 609–638.

[5] HOEFFDING, W. (1948). A class of statistics with asymptotically normal distribution. *Ann. Math. Statist.* **19** 293–325.

[6] HÁJEK, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.* **32** 506–523.

[7] HÁJEK, J. (1962). Asymptotically most powerful rank-order tests. *Ann. Math. Statist.* **33** 1124–1147.

[8] PURI, M. L. (1964). Asymptotic efficiency of a class of c-sample tests. *Ann. Math. Statist.* **35** 102–121.

[9] LEHMANN, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.