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ADMISSIBILITY OF THE SAMPLE MEAN AS ESTIMATE OF THE MEAN OF A FINITE POPULATION

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1. Introduction. In previous papers, the sample mean (1965-II), and later a ratio estimate (1965-IV, Section 5), were shown, with the squared error as the loss function, to be admissible as estimates of the population mean, whatever be the sampling design. The validity of these results was however restricted by the assumption of one particular loss function, thus raising the question whether these estimates remain admissible for other, equally valid, loss functions. In this paper, the restriction on the loss function is removed and the admissibility of the ratio estimate, (which includes the sample mean as a particular case) is shown to hold generally for any loss function, which satisfies certain mild conditions, which would be satisfied by almost any loss function assumed in practice.

2. Notation and definitions. U denotes the population consisting of units $u_1, u_2, \cdots, u_N$; with unit u_i, is associated a variate value x_i, $i = 1, 2, \cdots, N$; $x = (x_1, x_2, \cdots, x_N)$ denotes a point in the sample space R_N; a sample s denotes a subset of U; S denotes the set of all possible samples s; a probability function p is defined on S, such that

$$p(s) \geqq 0 \quad \text{for all} \quad s, \quad \text{and} \quad \sum_{s \varepsilon S} p(s) = 1.$$

Following Godambe and Joshi (1965-I), the pair (S, p) is called the sampling design. A sample s is drawn from S according to p. Then we define,

DEFINITION 2.1. An estimate $e(s, x)$ is a real function e defined on $S \times R_N$, which depends on x, through only those x_i for which $u_i \varepsilon s$.

The above definitions of sampling design and estimate are wide enough to cover all sampling procedures and classes of estimates; for a brief account we refer to Godambe and Joshi (1965-I), Section 5.

Let $V(t)$ be the loss function, where t is the absolute value of the difference between the estimate and the true value. We assume that $V(t)$ is non-decreasing and that it satisfies one more condition, which for convenience will be formulated later (in (29)).

Let T_N be the population mean, i.e.

$$T_N(x) = N^{-1} \sum_{i=1}^{N} x_i. \tag{1}$$

With the loss function $V(t)$, we define admissibility of estimates of $T_N(x)$. For a given sampling design d, let $\bar{S}$ be the subset of S, consisting of all those samples for which $p(s) > 0$. Then,

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DEFINITION 2.2. An estimate $e(s, x)$ is admissible for the population mean $T_N(x)$ in (1), if there exists no estimate $e'(s, x)$, such that,

$$(2) \quad \sum_{s \varepsilon \bar{S}} p(s) V(|e'(s, x) - T_N(x)|) \leqq \sum_{s \varepsilon \bar{S}} p(s) V(|e(s, x) - T_N(x)|)$$

for all $x \varepsilon R_N$, the strict inequality in (2) holding for at least one $x \varepsilon R_N$.

We also define a weaker version of admissibility by,

DEFINITION 2.3. An estimate $e(s, x)$ is weakly admissible for $T_N(x)$, if there exists no estimate $e'(s, x)$, such that the inequality (2) is satisfied for almost all (Lebesgue measure) $x \varepsilon R_N$, the strict inequality in (2) holding on a non-null subset of R_N.

To distinguish the admissibility, defined by Definition 2.2 from weak admissibility, the former will be called strict admissibility. The above definitions of weak and strict admissibility are a simple generalization of the definitions given in a previous paper (1965-III), Theorems 3.1 and 5.1, with the squared error as loss function.

3-I. A Bayes solution. As the sample mean is a particular case of the ratio estimate, we shall prove the result for the latter. Let then $y_i > 0, i = 1, 2, \cdots, N$, be arbitrary positive numbers. We write $i \varepsilon s$ for short, for $u_i \varepsilon s$. Then let,

$$(3) \quad \begin{aligned} y(s) &= \textstyle\sum_{i \varepsilon s} y_i, & Y &= \textstyle\sum_{i=1}^N y_i, \\ \bar{x}_s &= (y(s))^{-1} \textstyle\sum_{i \varepsilon s} x_i, & \bar{X}_N &= Y^{-1} \textstyle\sum_{i=1}^N x_i, \\ \bar{X}_{N-n(s)} &= (Y - y(s))^{-1} \textstyle\sum_{i \not\varepsilon s} x_i, & \bar{e}(s, x) &= YN^{-1} \bar{x}_s. \end{aligned}$$

We shall show that the ratio estimate $\bar{e}(s, x)$, is strictly admissible (Definition 2.2) for the population mean. Note that the estimate $\bar{e}(s, x)$ reduces to the sample mean, if all the $y_i, i = 1, 2, \cdots, N$, are equated to unity, and the whole of the following proof holds for the sample mean, if this substitution is made throughout.

As a first step towards proving the result, we prove

THEOREM 3.1. *The estimate $\bar{e}(s, x)$ in (3), is weakly admissible (Definition 2.3) for the population mean.*

OUTLINE OF THE PROOF. As the proof is rather long we first give its broad outline. Suppose the theorem is false; then there exists an estimate $e'(s, x)$, such that on substituting $\bar{e}(s, x)$ for $e(s, x)$ in the right hand side, the inequality (2) holds a.e. in R_N, with the strict inequality holding on a non-null set. We next consider a prior probability distribution on R_N, such that all the x_i, $i = 1, 2, \cdots, N$, are distributed independently, and normally, with mean θy_i and variance y_i, and θ is distributed with mean zero and variance τ^2. We then work out the estimate which is a Bayes solution wrt the above prior distribution. It is shown that the reduction in risk (expected loss) of the Bayes estimate as compared to $\bar{e}(s, x)$ is $O(\tau^{-2})$. Next, let E be the subset of R_N, on which for at least one $s \varepsilon \bar{S}$, $h(s, x) = e'(s, x) - \bar{e}(s, x) \neq 0$. It is shown that if E is not a null set, then there exists a subset of R_N, on which the integral wrt the assumed prior density, of the loss due to $\bar{e}(s, x)$, falls short of the corresponding integral for $e'(s, x)$ by a quantity $O(\tau^{-1})$. By the minimizing property of the Bayes solution, the excess of the integral over the rest of the sample space is $O(\tau^{-2})$. Hence for all sufficiently large τ, the risk of $\bar{e}(s, x)$ becomes less than that of $e'(s, x)$. But Definition 2.3 implies that the risk of $e'(s, x)$ must be less. The assumption that E is a non-null set thus leads to a contradiction. Hence E must be a null set. From this the result follows. The argument is closely parallel to that in a previous paper (1967). We now give the detailed proof.

PROOF. Suppose the theorem is false; then by Definition 2.3 there exists an estimate $e^*(s, x)$ such that

$$(4)\quad \sum_{s\varepsilon\bar{S}} p(s)V(|e^*(s, x) - T_N(x)|) \leqq \sum_{s\varepsilon\bar{S}} p(s)V(|\bar{e}(s, x) - T_N(x)|)$$

for almost all $x \varepsilon R_N$, the strict inequality holding on a non-null set.

Put,

$$(5)\quad e^*(s, x) = YN^{-1}e'(s, x).$$

Then, since by (1) and (3),

$$(6)\quad T_N(x) = YN^{-1}\bar{X}_N,$$

$$\sum_{s\varepsilon\bar{S}} p(s)V(YN^{-1}|e'(s, x) - \bar{X}_N|) \leqq \sum_{s\varepsilon\bar{S}} p(s)V(YN^{-1}|\bar{e}(s, x) - \bar{X}_N|),$$

in which, the strict inequality holds on a non-null subset of R_N.

We now consider a prior distribution on R_N, such that all the x_i, $i = 1, 2, \cdots, N$, are distributed independently, identically and normally with mean θy_i, and variance y_i, and θ is distributed normally with mean zero and variance τ^2. We determine the estimate $YN^{-1}b(s, x)$ which is a Bayes solution with respect to the assumed prior distribution. Let E_τ denote the expectation with respect to the prior distribution, and for a given sample s, let $B_{\tau,s}$ denote the risk of the Bayes estimate $YN^{-1}b(s, x)$. Then,

$$(7)\quad B_{\tau,s} = E_\tau V(YN^{-1}|b(s, x) - \bar{X}_N|).$$

It is easily verified that $\bar{x}_s$ and $\bar{X}_{N-n(s)}$ are sufficient for θ. Hence for given θ the frequency function on R_N, can be expressed in the form

$$(8)\quad f(x, \theta) = L_1 \cdot L_2 p(\bar{x}_s - \theta) \cdot q(\bar{X}_{N-n(s)} - \theta),$$

where,

$$p(\bar{x}_x - \theta) = [y(s)/2\pi]^{\frac{1}{2}} \cdot \exp[-\tfrac{1}{2}y(s)(\bar{x}_s - \theta)^2],$$

$$q(\bar{X}_{N-n(s)} - \theta) = [(Y - y(s))/2\pi]^{\frac{1}{2}} \exp[-\tfrac{1}{2}(Y - y(s))(\bar{X}_{N-n(s)} - \theta)^2],$$

$$L_1 = L[x_i, i \varepsilon s \mid \bar{x}_s],$$

$$L_2 = L[x_i, i \not\varepsilon s \mid \bar{X}_{N-n(s)}].$$

Hence, integrating wrt the prior distribution, we have from (7),

$$(9)\quad B_{\tau,s} = (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{-\infty}^{\infty} \exp(-\theta^2\tfrac{1}{2}\tau^{-2})\, d\theta \int_{R_N} L_1 \cdot L_2 V_b(s, x) \cdot p(\bar{x}_s - \theta) \cdot q(\bar{X}_{N-n(s)} - \theta)\, dx$$

where dx is written for short for $\prod_{i=1}^{N} dx_i$ and $V_b(s, x)$ for

$$V(YN^{-1}|b(s, x) - \bar{X}_N|).$$

By interchanging the order of integration with respect to θ and x, which is permissible by Fubini's theorem, as the integrand is non-negative for all x and θ, we have

$$(10) \quad B_{\tau,s} = (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{RN} L_1 \cdot L_2 \, dx \int_{-\infty}^{\infty} V_b(s, x) \cdot p(\bar{x}_s - \theta) \cdot q(\bar{X}_{N-n(s)} - \theta) \exp\left(-\tfrac{1}{2}\theta^2\tau^{-2}\right) d\theta$$

Let,

$$(11) \qquad g_s = 1 + (y(s) \cdot \tau^2)^{-1}, \qquad g = 1 + (Y \cdot \tau^2)^{-1}.$$

Then after some simple algebraic reduction, as shown in a previous paper [(1967), equations 23 through 27, read with section 6] we have

$$(12) \quad (2\pi)^{-\frac{1}{2}}\tau^{-1} p(\bar{x}_s - \theta) \cdot q(\bar{X}_{N-n(s)} - \theta) \exp\left(-\tfrac{1}{2}\theta^2\tau^{-2}\right) = F_1 \cdot F_2 \cdot F_3$$

where,

$$F_1 = (2\pi)^{-\frac{1}{2}} g_s^{-\frac{1}{2}} \cdot \tau^{-1} \exp\left(-\tfrac{1}{2}\bar{x}_s^2(\tau^2 g_s)^{-1}\right),$$

$$F_2 = [(Y - y(s))/2\pi Y g]^{\frac{1}{2}}[y(s) \cdot g_s]^{\frac{1}{2}} \cdot \exp\{-\tfrac{1}{2}[Y - y(s)]y(s)g_s(Yg)^{-1}[\bar{X}_{N-n(s)} - x_s g_s^{-1}]^2\},$$

$$F_3 = [Yg/2\pi]^{\frac{1}{2}} \exp\{-\tfrac{1}{2}Yg[[y(s)\bar{x}_s + [Y - y(s)]\bar{X}_{N-n(s)}](Yg)^{-1} - \theta]^2\}.$$

We now substitute the right hand side of (12) in (10) and integrate out wrt θ. F_3 is the only factor which involves θ, and its integral $= 1$. Thus (10) reduces to

$$(13) \qquad B_{\tau,s} = \int_{RN} V_b(s, x) \cdot L_1 \cdot L_2 \cdot F_1 \cdot F_2 \, dx.$$

Now by an orthogonal transformation of co-ordinates in the $[N - n(s)]$-dimensional space of the variates x_i, $i \not\in s$, we obtain that $\bar{X}_{N-n(s)}$, is independent of the other $(N - n(s) - 1)$ transformed variates. Let x' denote the group of the remaining $(N - n(s) - 1)$ transformed variables. Then for each fixed $\bar{X}_{N-n(s)}$,

$$(14) \qquad \int L_2 \, dx' = \int L[x_i, i \not\in s \mid \bar{X}_{N-n(s)}] \, dx' = 1.$$

Let $R_{n(s)}$ denote the space of the variates x_i, $i \in s$, and $R_{N-n(s)}$ the space of the remaining variates x_i, $i \not\in s$. Now by (3),

$$(15) \qquad \bar{X}_N = y(s)Y^{-1}\bar{x}_s + (Y - y(s))Y^{-1}\bar{X}_{N-n(s)}.$$

Hence in the right hand side of (13), the argument of the function $V_b = V(|b(s, x) - \bar{X}_N|)$ is independent of the variables x'. Hence on integrating out wrt x' and using (14), we have from (13),

$$(16) \qquad B_{\tau,s} = \int_{R_{n(s)}} L_1 \cdot F_1 \, dx_s \int_{-\infty}^{\infty} V_b(s, x) \cdot F_2 \, d\bar{X}_{N-n(s)},$$

where $R_{n(s)}$ is the space of the variables x_i, $i \in s$ and dx_s is short for $\prod_{i \in s} dx_s$.

We next determine the estimate $b(s, x)$, which for given x_i, $i \varepsilon s$ minimizes the inner integral in the right hand side of (16). By (15) and (11),

$$\bar{X}_N - b(s, x) = (Y - y(s))Y^{-1}[\bar{X}_{N-n(s)} - \bar{x}_s g_s^{-1}] + \bar{x}_s(Yg_s)^{-1}[Y - y(s) + y(s)\cdot g_s] - b(s, x)$$
$$= (Y - y(s))Y^{-1}[\bar{X}_{N-n(s)} - \bar{x}_s g_s^{-1}] + [\bar{x}_s g g_s^{-1} - b(s, x)].$$

Hence in the inner integral in the right hand side of (16),

$$(17)\quad YN^{-1}[\bar{X}_N - b(s, x)] = (Y - y(s))N^{-1}[\bar{X}_{N-n(s)} - \bar{x}_s g_s^{-1}] + YN^{-1}[\bar{x}_s g g_s^{-1} - b(s, x)].$$

Now transform the variable $\bar{X}_{N-n(s)}$ by putting,

$$(18)\quad t = (Y - y(s))N^{-1}[\bar{X}_{N-n(s)} - \bar{x}_s g_s^{-1}].$$

Let $I_{\tau,s}$ denote the inner integral in (16), and in (17) let,

$$(19)\quad YN^{-1}[b(s, x) - \bar{x}_s g g_s^{-1}] = h(s, x) = h \text{ for short.}$$

It is easily seen that by the transformation (18), $I_{\tau,s}$ is reduced to,

$$(20)\quad I_{\tau,s} = (K/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t - h|) \exp(-\tfrac{1}{2}Kt^2)\, dt$$

where, $K = N^2(Y - y(s))^{-1}y(s)g_s(Yg)^{-1}$.

We shall show that the right hand side of (20) is minimized for $h = 0$. Put

$$I_1 = \int_{-\infty}^{\infty} V(|t - h|) \exp(-\tfrac{1}{2}Kt^2)\, dt$$

(21) and,

$$I_0 = \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt.$$

Suppose $h > 0$. Then,

$$(22)\quad I_1 = \int_{-\infty}^{+h/2} V(|t - h|) \exp(-\tfrac{1}{2}Kt^2)\, dt + \int_{h/2}^{\infty} V(|t - h|) \exp(-\tfrac{1}{2}Kt^2)\, dt.$$

In the right hand side of (22), in the first integrand put $t = \frac{1}{2}h - u$, and in the second put $t = \frac{1}{2}h + u$, u going in each case from zero to ∞. Then,

$$(23)\quad I_1 = \int_0^{\infty} V(\tfrac{1}{2}h + u) \exp[-\tfrac{1}{2}K(\tfrac{1}{2}h - u)^2]\, du + \int_0^{\infty} V(|u - \tfrac{1}{2}h|) \exp[-\tfrac{1}{2}K(\tfrac{1}{2}h + u)^2]\, du.$$

By making similar transformations,

$$I_0 = \int_{-\infty}^{+h/2} V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt + \int_{h/2}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt$$
$$(24)\quad = \int_0^{\infty} V(|u - \tfrac{1}{2}h|) \exp[-\tfrac{1}{2}K(\tfrac{1}{2}h - u)^2]\, du + \int_0^{\infty} V(u + \tfrac{1}{2}h) \exp[-\tfrac{1}{2}K(\tfrac{1}{2}h + u)^2]\, du.$$

Combining (23) and (24),

$$(25)\quad I_1 - I_0 = \int_0^{\infty} [V(u + \tfrac{1}{2}h) - V(|u - \tfrac{1}{2}h|)] \cdot \{\exp[-\tfrac{1}{2}K(\tfrac{1}{2}h - u)^2] - \exp[-\tfrac{1}{2}K(\tfrac{1}{2}h + u)^2]\}\, du.$$

Now throughout the range of integration,

$$u + \tfrac{1}{2}h \geqq |u - \tfrac{1}{2}h|.$$

Since by assumption $V(t)$ is non-decreasing in t, the expression in the first curly bracket in the right hand side of (25) is non-negative. The expression in the second curly bracket is positive. Hence

$$I_1 - I_0 \geqq 0. \tag{26}$$

By a similar calculation (26) is seen to hold, when $h < 0$. Hence $I_{r,s}$ in (20) is minimized for $h = 0$, so that by (19), the Bayes estimate is given by,

$$b(s, x) = gg_s^{-1}\bar{x}_s. \tag{27}$$

We next obtain an upper bound for the improvement in risk of the Bayes estimate $YN^{-1}b(s, x)$ in (27) as compared to the risk of the estimate $YN^{-1}\bar{x}_s$. So far, the only assumption made regarding the loss function $V(t)$ is that it is non-decreasing, i.e. for $t_1 > t_2 \geqq 0$

$$V(t_1) \geqq V(t_2). \tag{28}$$

We now state the further condition to be satisfied by $V(t)$ viz., for arbitrary $K > 0$

$$\int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt < \infty. \tag{29}$$

The significance of this condition may be briefly discussed here. If the loss function is of the form $V(t) = t^c$, $(0 \leqq t < \infty)$, then for it to be non-decreasing we must have $c \geqq 0$. More generally, (29) is satisfied for any loss function of the form

$$V(t) = \sum_{r=1}^{n} A_r t^{c_r} \tag{30}$$

where c_r are positive constants. Thus most loss functions considered in practice, satisfy both the conditions (28) and (29).

Incidentally condition (29) is necessarily satisfied by every loss function which is bounded as required by the axioms of Neumann and Morgenstein (1947).

Let $\bar{A}_{r,s}$ be the risk of the estimate $YN^{-1}\bar{x}_s$, for a given sample s. Then proceeding as from (13) through (16), we have in place of (16) writing $\bar{V}(s, x)$ for $V(YN^{-1}|\bar{x}_N - \bar{X}_s|)$,

$$\bar{A}_{r,s} = \int_{R_{n(s)}} L_1 \cdot F_1\, dx_s \int_{-\infty}^{\infty} \bar{V}(s, x) \cdot F_2\, d\bar{X}_{N-n(s)}. \tag{31}$$

Let $\bar{I}_{r,s}$ be the inner integral in (31). Then on making the transformation (18) and putting,

$$h = YN^{-1}\bar{x}_s(1 - gg_s^{-1}) \tag{32}$$

we have,

$$\begin{aligned} \bar{I}_{r,s} &= (K/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t - h|) \exp(-\tfrac{1}{2}Kt^2)\, dt \\ &= (K/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t|) \exp[-\tfrac{1}{2}K(t + h)^2]\, dt. \end{aligned} \tag{33}$$

We now substitute (33) in (31) and interchange the order of integration with respect to x_s and t. This is permissible by Fubini's theorem as the integrand is non-negative. Then in the space $R_{n(s)}$, we make an orthogonal transformation of co-ordinates, taking $\bar{x}_s$ as an independent coordinate. Let x' denote the group of the remaining $(n(s) - 1)$ transformed variables. Then

$$\int L_1 \, dx' = 1.$$

We thus have from (31),

$$(34) \qquad \bar{A}_{\tau,s} = (K/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t|) \, dt \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}K(t + h)^2\right] \cdot F_1 \, d\bar{x}_s .$$

Now, from (32) by using (11), we get

$$(35) \qquad h = a \cdot \bar{x}_s / \tau^2$$

where

$$a = Y(Ng_s)^{-1}[(y(s))^{-1} - Y^{-1}].$$

Next, in the right hand side of (34), we substitute for F_1 by (12). Note that

$$\begin{aligned} &-\bar{x}_s^2(2\tau^2 g_s)^{-1} - \tfrac{1}{2}K(t + h)^2 \\ &\quad = -\bar{x}_s^2(2\tau^2 g_s)^{-1} - \tfrac{1}{2}K(t + a\bar{x}_s\tau^{-2})^2 \qquad \text{by (35)} \\ &\quad = -(1 + Ka^2 g_s \tau^{-2}) \cdot (2\tau^2 g_s)^{-1}(\bar{x}_s + Ktag_s(1 + Ka^2 g_s \tau^{-2})^{-1})^2 \\ &\qquad - \tfrac{1}{2}Kt^2 \cdot (1 + Ka^2 g_s \tau^{-2})^{-1}. \end{aligned}$$

Making this reduction after substituting for F_1 by (12), and integrating out with respect to $\bar{x}_s$, we obtain that in (34)

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}K(t + h)^2\right] F_1 \, d\bar{x}_s \\ &\qquad = (1 + Ka^2 g_s \tau^{-2})^{-\frac{1}{2}} \cdot \exp\left[-\tfrac{1}{2}Kt^2 \cdot (1 + Ka^2 g_s \tau^{-2})^{-1}\right] \end{aligned}$$

so that (34) yields,

$$\begin{aligned} (36) \quad \bar{A}_{\tau,s} = (2\pi)^{-\frac{1}{2}}[K(1 + Ka^2 g_s \tau^{-2})^{-1}]^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t|) \\ \cdot \exp\left[-\tfrac{1}{2}Kt^2 \cdot (1 + Ka^2 g_s \tau^{-2})^{-1}\right] dt. \end{aligned}$$

Now put

$$K(1 + Ka^2 g_s \tau^{-2})^{-1} = K(1 - \delta),$$

so that,

$$(37) \qquad \delta = Ka^2 g_s \tau^{-2}(1 + Ka^2 g_s \tau^{-2})^{-1} < Ka^2 g_s \tau^{-2}.$$

Hence δ can be made arbitrarily small by increasing τ sufficiently.

Next, by Lemma 3.1 proved at the end of this subsection, (29) implies that for sufficiently small δ,

$$(38) \qquad \bar{A}_{\tau,s} = (K/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t|) \exp\left(-\tfrac{1}{2}Kt^2\right) dt + a_K \cdot \delta + o(\delta)$$

where,

$$a_K = 0, \quad \text{if } V(t) \text{ is constant for } t > 0,$$
$$> 0, \quad \text{otherwise.}$$

The corresponding expression for the Bayes risk $B_{\tau,s}$ in (16) is simply obtained by putting in (35) $h = 0$ so that $a = 0$ and hence by (37) $\delta = 0$, thus giving

$$B_{\tau,s} = (K/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt.$$

Hence

$$(39) \qquad A_{\tau,s} - B_{\tau,s} = a_K \cdot \delta + o(\delta).$$

By (38), $a_K = 0$ only if $V(t)$ is constant for $t > 0$. In this trivial case, all estimates are admissible. Hence excluding it we have $a_K > 0$. We now take τ sufficiently large and thus make δ sufficiently small, so that in (39),

$$o(\delta) \leqq a_K \cdot \delta,$$

and hence

$$A_{\tau,s} - B_{\tau,s} \leqq 2a_K \cdot \delta < 2Ka^2 g_s \cdot a_K \tau^{-2} \qquad \text{by (37).}$$

Here by (20), K and hence a_K, and by (35) a depend on s. Hence putting

$$C = \max_{s \varepsilon \bar{S}} (2Ka^2 g_s \cdot a_K)$$

we have from (39),

$$(40) \qquad A_{\tau,s} - B_{\tau,s} < C\tau^{-2} \qquad \text{for all } \tau > \text{some } \tau_0.$$

(40) gives the upper bound for $(\bar{A}_{\tau,s} - \bar{B}_{\tau,s})$. We show in part II of this section that this upper bound implies weak admissibility of $\bar{x}_s Y N^{-1}$.

We now state and prove the lemma assumed in (38).

LEMMA 3.1. *Conditions* (28) *and* (29) *on the function* $V(t)$ *imply that*

$$(1^*) \qquad \emptyset(k) = K^{\frac{1}{2}} \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt$$

is differentiable in K *for all* $K > 0$ *and its differential coefficient* $\emptyset'(K) < 0$, *except when* $V(t)$ *is constant for all* $t > 0$, *in which case* $\emptyset'(K) = 0$.

PROOF. Take any positive number K_1, such that

$$(2^*) \qquad 0 < K_1 < K.$$

Then by (29),

$$(3^*) \qquad \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}K_1 t^2)\, dt < \infty.$$

Hence,

$$(4^*) \qquad \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2) \exp[\tfrac{1}{2}(K - K_1)t^2]\, dt < \infty.$$

For all sufficiently large t

$$(5^*) \qquad t^2 < \exp[\tfrac{1}{2}(K - K_1)t^2].$$

Hence (4^*) implies that

$$(6^*) \qquad \int_{-\infty}^{\infty} t^2 V(|t|) \exp(-\tfrac{1}{2}Kt^2)\, dt < \infty.$$

In the right hand side of (1^*) put

$$(7^*) \qquad u_K(t) = K^{\frac{1}{2}} \cdot V(|t|) \exp(-\tfrac{1}{2}Kt^2).$$

Then

$$(8^*) \qquad |\partial u_K/\partial K| \leqq \tfrac{1}{2} V(|t|) \exp(-\tfrac{1}{2}Kt^2)(K^{-\frac{1}{2}} + K^{\frac{1}{2}}t^2).$$

In the right hand side of (8^*), put

$$(9^*) \qquad v_K(t) = V(|t|) \exp(-\tfrac{1}{2}Kt^2)(K^{-\frac{1}{2}} + K^{\frac{1}{2}}t^2).$$

Then

$$(10^*) \qquad \partial v_K(t)/\partial K = -\tfrac{1}{2} V(|t|) \exp(-\tfrac{1}{2}Kt^2)(K^{-3/2} + K^{\frac{1}{2}}t^4) \leqq 0.$$

Consider values of K in some arbitrary finite interval $[k_1, k_2]$ where $0 < k_1 < k_2$. By (10^*) and (8^*), for $K \,\varepsilon\, [k_1, k_2]$

$$(11^*) \qquad |\partial u_K/\partial K| \leqq \tfrac{1}{2} V(|t|) \exp(-\tfrac{1}{2}k_1t^2)(k_1^{-\frac{1}{2}} + k_1^{\frac{1}{2}}t^2) = g(t) \quad \text{say}.$$

By (29) and (6^*),

$$(12^*) \qquad \int_{-\infty}^{\infty} g(t)\, dt < \infty.$$

By applying the dominated convergence theorem (see for example, application 3°, Loève, p. 126), it follows from (11^*) and (12^*) that $\emptyset'(K)$ exists and is obtained by differentiating under the integral sign, so that

$$(13^*) \qquad \emptyset'(K) = \int_{-\infty}^{\infty} (\partial u_K(t)/\partial K)\, dt = \tfrac{1}{2} \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt.$$

By assumption, (13^*) holds for $K \,\varepsilon\, [k_1, k_2]$. But since k_1 and k_2 can be arbitrary, (13^*) holds for all $K > 0$.

Next,

$$(14^*) \qquad \int_{kt^2 \leqq 1}^{\infty} \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt + \int_{Kt^2 > 1} \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt = \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt = 0.$$

Then since $V(|t|)$ is non-decreasing in $|t|$,

$$(15^*) \qquad \int_{kt^2 \leqq 1}^{\infty} V(|t|) \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt \leqq V(K^{-\frac{1}{2}}) \int_{kt^2 \leqq 1} \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt,$$

and,

$$(16^*) \qquad \int_{kt^2 > 1} V(|t|) \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt \leqq V(K^{-\frac{1}{2}}) \int_{kt^2 > 1} \exp(-\tfrac{1}{2}Kt^2)(1 - Kt^2)\, dt.$$

The sign of equality holds in both (15*) and (16*), only if $V(t) = V(K^{-\frac{1}{2}}) =$ constant for all $t > 0$. Now combining (15*) and (16*) with (14*) and substituting in (13*), the lemma is proved.

3-II. Weak admissibility. Let $A'_{\tau,s}$ be the risk for the given sample s of the estimate $YN^{-1}e'(s, x)$ in (5). Then as in (10),

$$(41)\quad A'_{\tau,s} = (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{R_N} L_1 \cdot L_2\, dx \int_{-\infty}^{\infty} V'(s, x) p(\bar{x}_s - \theta) q(\bar{X}_{N-n(s)} - \theta) \cdot \exp(-\theta^2/2\tau^2)\, d\theta$$

where $V'(s, x)$ is written for $V(YN^{-1}|\theta'(s, x) - \bar{X}_N|)$. Now, after a little reduction, similar to that in (12), we have

$$(42)\quad p(\bar{x}_s - \theta) q(\bar{X}_{N-n(s)} - \theta) = f_2 \cdot f_3$$

where,

$$f_2 = [(Y - y(s))/(2\pi)]^{\frac{1}{2}} (y(s))^{\frac{1}{2}} \cdot \exp\{-\tfrac{1}{2}[Y - y(s)] y(s) Y^{-1} [\bar{X}_{N-n(s)} - \bar{x}_s]^2\}$$

and,

$$f_3 = (Y/2\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2} Y [[y(s)\bar{x}_s + [Y - y(s)] \cdot \bar{X}_{N-n(s)}] Y^{-1} - \theta]^2\}.$$

These expressions for f_2 and f_3 can be obtained simply from the expressions for F_2 and F_3 in (12) by letting $\tau \to \infty$. Substituting (42) in (41), we get,

$$(43)\quad \begin{aligned} A'_{\tau,s} &= (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{R_{n(s)}} L_1\, dx_s \int_{R_{N-n(s)}} L_2\, d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} V'(s, x) f_2 \cdot f_3 \\ &\qquad \exp(-\theta^2/2\tau^2)\, d\theta \\ &= (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{R_{n(s)}} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} V'(s, x) f_2 \cdot f_3 \\ &\qquad \cdot \exp(-\theta^2/2\tau^2)\, d\theta \end{aligned}$$

by using (14). Similarly the risk $\bar{A}_{\tau,s}$ of $\bar{x}_s$ is given by

$$(44)\quad \bar{A}_{\tau,s} = (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{R_{n(s)}} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} \bar{V}(s, x) f_2 \cdot f_3 \cdot \exp(-\theta^2/2\tau^2)\, d\theta$$

where $\bar{V}(s, x) = V(YN^{-1}|\bar{x}_s - \bar{X}_N|)$ as in (30).

In (43), and (44), excluding the factor $\exp(-\theta^2/2\tau^2)$, θ occurs in the factor f_3 only. It is seen from (42) that

$$(45)\quad \int_{-\infty}^{\infty} f_3\, d\theta = 1.$$

Now put

$$(46)\quad \bar{U}(s, x) = \int_{-\infty}^{\infty} \bar{V}(s, x) f_2 \cdot d\bar{X}_{N-n(s)},$$

and

$$U'(s, x) = \int_{-\infty}^{\infty} V'(s, x) f_2\, d\bar{X}_{N-n(s)}.$$

Note that $\bar{U}(s, x)$ and $U'(s, x)$ depend on x, through only those x_i for which $i \in s$ and thus are estimates according to Definition 2.1.

Then by transforming the variables as in (18) by putting

$$t = (Y - y(s))N^{-1}[\bar{X}_{N-n(s)} - \bar{x}_s],$$

and

$$h = YN^{-1}[e'(s, x) - \bar{x}_s],$$

we get,

$$\bar{U}(s, x) = \int_{-\infty}^{\infty} V(|t|) \exp(-\tfrac{1}{2}bt^2)\, dt = U_0(s), \tag{47}$$

and,

$$U'(s, x) = \int_{-\infty}^{\infty} V(|t - h|) \exp(-\tfrac{1}{2}bt^2)\, dt,$$

where $b = N^2(Y - y(s))^{-1} \cdot y(s)Y^{-1}$.
Hence as shown from (18) through (26),

$$U'(s, x) \geqq \bar{U}(s, x) \quad \text{for all} \quad x \,\varepsilon\, R_N. \tag{48}$$

We may exclude the trivial case of $V(t) = $ constant for all t, $(0 < t < \infty)$ as in that case every estimate whatsoever is strictly admissible. Assuming $V(t)$ to be not constant on $(0, \infty)$ as shown in Lemma 3.2 at the end of this section, the set of values of u, for which $u > h/2$ and $V(u + \frac{1}{2}h) - V(u - \frac{1}{2}h) > 0$ has positive measure. It then follows from (25) that the inequality in (48) holds, whenever,

$$h \neq 0, \quad \text{i.e.} \quad e'(s, x) - \bar{x}_s \neq 0. \tag{49}$$

Obviously, the sign of equality holds in (48) whenever $e'(s, x) = \bar{x}_s$.

Now, there are two possible alternatives, (i) for every sample $s \,\varepsilon\, \bar{S}$, the subset of R_N on which (49) holds is a null subset or (ii) there exists at least one sample $s \,\varepsilon\, \bar{S}$, for which (49) holds on a non-null subset of R_N.

Suppose (ii) is true: For every positive number a, let T_a, denote the subset of $R_{n(s)}$ defined by,

$$x = (x_i, i \,\varepsilon\, s) \,\varepsilon\, T_a, \quad \text{if, and only if,} \tag{50}$$
$$|x_i| \leqq a \quad \text{for all} \quad i \,\varepsilon\, s.$$

Now $\bar{U}(s, x)$ and $U'(s, x)$ depend on x, only through those x_i, for which $i \,\varepsilon\, s$. Hence since the strict inequality in (49) is assumed to hold on a non-null (μ_N) subset of R_N it holds on a non-null $(\mu_{n(s)})$ subset of $R_{n(s)}$.

[NOTE. Here we use the symbol (μ_k) to denote the Lebesgue measure on the k-dimensional subspace R_k.] The existence of the non-null $(\mu_{n(s)})$ subset of $R_{n(s)}$ implies that there exists a positive constant β, $(\beta > 0)$ and a positive number a, such that

$$\int_{T_a} L_1\, dx_s[U'(s, x) - U_0(s)] = \beta.$$

$\int_{T_a} L_1\, dx_s U_0(s)$ is obviously finite as L_1 is bounded and T_a is a finite set. Hence

$$\int_{T_a} L_1\, dx_s U'(s, x) - \int_{T_a} L_1\, dx_s U_0(s) = \beta. \tag{51}$$

Let T_a^c be the complement of T_a. Then from (43) and (44) we have,

$$(52) \qquad A'_{\tau,s} - \bar{A}_{\tau,s} = T_1(2\pi)^{-\frac{1}{2}}\tau^{-1} - T_2(2\pi)^{-\frac{1}{2}}\tau^{-1} + T_3,$$

where

$$(53) \quad T_1 = \int_{T_a} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} V'(s, x)\cdot f_2 \cdot f_3 \cdot \exp(-\theta^2/2\tau^2)\, d\theta,$$

$$(54) \quad T_2 = \int_{T_a} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} \bar{V}(s, x)\cdot f_2 \cdot f_3 \cdot \exp(-\theta^2/2\tau^2)\, d\theta$$

and

$$(55) \quad T_3 = (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{T_a^c} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} [V'(s, x) - \bar{V}(s, x)] \cdot f_2 \cdot f_3 \cdot \exp(-\theta^2/2\tau^2)\, d\theta.$$

Now in the right hand side of (53), the integrand is non-negative and non-decreasing as $\tau \to \infty$. Hence by the monotone convergence theorem, as $\tau \to \infty$,

$$(56) \qquad \begin{aligned} T_1 &\to \int_{T_a} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} V'(s, x) f_2 \cdot f_3 \cdot d\theta \\ &= \int_{T_a} L_1\, dx_s U'(s, x) \qquad \text{by (45) and (46)} \\ &= \int_{T_a} L_1\, dx_s U_0(s) + \beta \qquad \text{by (51).} \end{aligned}$$

Hence for all $\tau \geqq$ some τ_0'

$$(57) \qquad T_1 \geqq \int_{T_a} L_1\, dx_s U_0(s) + \tfrac{1}{2}\beta.$$

Next in the right hand side of (54) since $\exp(-\theta^2/2\tau^2) \leqq 1$, we have,

$$(58) \qquad \begin{aligned} T_2 &\leqq \int_{T_a} L_1\, dx_s \int_{-\infty}^{\infty} dX_{N-n(s)} \int_{-\infty}^{\infty} \bar{V}(s, x) f_2 \cdot f_3\, d\theta \\ &= \int_{T_a} L_1\, dx_s U_0(s) \qquad \text{by (45) and (46).} \end{aligned}$$

Lastly, in (55), by virtue of the minimizing property of the Bayes solution, the integrand is everywhere decreased by replacing $e'(s, x)$ by the Bayes estimate $b(s, x)$ in (27). The resulting integrand being everywhere non-positive the integration can be extended from the set T_a^c to the whole space $R_{n(s)}$. We thus have

$$(59) \qquad \begin{aligned} T_3 &\geqq (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{T_a^c} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} [V_b(s, x) - \bar{V}(s, x)] \cdot f_2 \cdot f_3 \cdot \exp(-\theta^2/2\tau^2)\, d\theta \\ &\geqq (2\pi)^{-\frac{1}{2}}\tau^{-1} \int_{R_{n(s)}} L_1\, dx_s \int_{-\infty}^{\infty} d\bar{X}_{N-n(s)} \int_{-\infty}^{\infty} [V_b(s, x) - \bar{V}(s, x)] \cdot f_2 \cdot f_3 \cdot \exp(-\theta^2/2\tau^2)\, d\theta \\ &= B_{\tau,s} - A_{\tau,s}. \end{aligned}$$

Hence by (40),

$$(60) \qquad T_3 \geqq C/\tau^2 \quad \text{for} \quad \tau \geqq \tau_0.$$

Combining (57), (58) and (60) with (52), we have

$$(61) \qquad A'_{\tau,s} - \bar{A}_{\tau,s} \geqq \tfrac{1}{2}\beta(2\pi)^{-\frac{1}{2}}\tau^{-1} - C/\tau^2 \qquad \text{for all} \quad \tau \geqq \text{ some } \tau_1$$

(61) holds, for the particular sample for which (49) is assumed to hold on a non-null subset of R_N. Denoting this sample by s_0, we write (61) as

$$A'_{\tau,s_0} - \bar{A}_{\tau,s_0} \geqq \tfrac{1}{2}\beta(2\pi)^{-\frac{1}{2}}\tau^{-1} - C/\tau^2 \quad \text{for} \quad \tau \geqq \tau_1. \tag{62}$$

By the minimizing property of the Bayes solution, we have for $s \neq s_0$, $s \in \bar{S}$,

$$A'_{\tau,s} - \bar{A}_{\tau,s} \geqq B_{\tau,s} - \bar{A}_{\tau,s} \geqq -C/\tau^2 \qquad \text{by (40)}. \tag{63}$$

Multiplying both sides of (62) by $p(s_0)$, both sides of (63) by $p(s)$ and summing over all $s \in \bar{S}$, we have,

$$A_\tau' - \bar{A}_\tau \geqq \tfrac{1}{2}\beta\cdot p(s_0)(2\pi)^{-\frac{1}{2}}\tau^{-1} - C/\tau^2 \quad \text{for all} \quad \tau \geqq \tau_1. \tag{64}$$

Here A_τ' and $\bar{A}_\tau$ denote the expected risks of the estimates $YN^{-1}e'(s, x)$ and $YN^{-1}\bar{x}_s$.

Since $s_0 \in \bar{S}$, $p(s_0) > 0$. Hence (64) implies that by making τ sufficiently large, say $\tau \geqq \tau_2$,

$$A_\tau' > \bar{A}_\tau. \tag{65}$$

But this contradicts (6). Hence the assumption (ii) relating to equation (49) must be false. It follows that $e'(s, x) = \bar{x}_s$ a.e. in R_N; hence the strict inequality in (6) and hence in (4) cannot hold on a non-null subset of R_N. From this the weak admissibility of $\bar{e}(s, x)$ follows. This completes the proof of Theorem 3.1.

We shall now state and prove Lemma 3.2 which was used in deriving condition (49).

LEMMA 3.2. *If the loss function $V(t)$ is not constant over $(0, \infty)$, then for any positive constant h, the set of values of t for which*

$$v(t + h) > v(t)$$

has positive measure.

PROOF. Since $V(t)$ is not constant in $(0, \infty)$, there exists at least one number T, such that

$$V(T) > V(0+). \tag{66}$$

Consider values of t in the range

$$\max(T - h, 0) < t \leqq T. \tag{67}$$

Since $V(t)$ is everywhere non-decreasing, for t satisfying (67), we have

$$V(t + h) \geqq V(T) \geqq V(t).$$

Hence $V(t + h) \not> V(t)$ can hold in this range only if, $V(t) = V(T)$.

Obviously there will be an interval of values t, for which $V(t + h) > V(t)$ unless $V(T_1 + 0) = V(T)$, where $T_1 = \max(T - h, 0)$. If $T_1 \neq 0$, then by repeating the same argument with T_1 in place of T, we get $V(T_2 + 0) = V(T)$ where $T_2 = \max(T - 2h, 0)$.

Continuing the argument, we see that the set S of values of t for which

$V(t+h) > V(t)$, can be a null set only if

$$V(T) = V(0+). \tag{68}$$

But (68) contradicts (66). Hence the set S must be of positive measure.

4. Strict admissibility. The rest of the argument proceeds precisely as in Sections 4 and 5 of the previous paper (1965-III). Corresponding to Theorem 4.1 of that paper, we have here

THEOREM 4.1. *If an estimate* $e'(s, x)$ *satisfies* (6), a.e. *in* Q^{a}_{N-k}, *then* $e'(s, x) = \bar{e}(s, x)$ a.e. *in* Q^{a}_{N-k}.

A proof of this theorem for the ratio estimate with the squared error as loss function was given in (1966-IV) section 5, and the same proof holds word by word for the more general loss function.

The argument is then completed by proving as in Theorem 5.1 of (1965-III), that weak admissibility implies strict admissibility. The theorem was proved for a fixed sample size design in (1965-III) and for a varying sample size design in (1967). The proof here needs only slight modifications. Thus in place of (35) in (1965-III), we define the set Q^{a}_{N-k} by

$$\text{if} \quad h_0 > 0, \qquad x \,\varepsilon\, Q^{a]}_{N-m}, \qquad \text{if and only if,} \tag{69}$$

$$Y^{-1} \textstyle\sum_{i=m+1}^{N} x_i \leqq (y(s_0))^{-1} \sum_{i=1}^{m} a_i - Y^{-1} \sum_{i=1}^{m} a_i$$

and if $h_0 < 0$, $x \,\varepsilon\, Q^{a}_{N-m}$, if and only if

$$y^{-1} \textstyle\sum_{i=m+1}^{N} \geqq (y(s_0))^{-1} \sum_{i=1}^{m} a_i - Y^{-1} \sum_{i=1}^{m} a_i .$$

Then for $x \,\varepsilon\, Q^{a}_{N-m}$,

$$|e'(s_0, x) - \bar{X}_N| \geqq |\bar{x}_{s_0} - \bar{X}_N| + h_0 . \tag{70}$$

By Lemma 3.1, there exists a subset of Q^{a}_{N-m}, say Q^{*a}_{N-m} with positive measure, such that for $x \,\varepsilon\, Q^{*a}_{N-m}$,

$$V(|e'(s_0, x) - \bar{X}_N|) > V(\bar{x}_{s_0} - \bar{X}_N). \tag{71}$$

Hence by the requirement of strict admissibility, for each $x \,\varepsilon\, Q^{*a}_{N-m}$, there must be at least one other sample $s \,\varepsilon\, \bar{S}$, for which

$$h(s, x) = e'(s, x) - \bar{x}_s \neq 0.$$

We then partition the set Q^{*a}_{N-m} into sets $L^{a,s}_{N-m}$ as in (36) of (1965-III) and again one of these sets $L^{a,s}_{N-m}$ must be of positive measure. The rest of the proof proceeds on similar lines and we reach the conclusion that unless the set E of points at which $h(s, x) = e'(s, x) - \bar{x}_s$ is empty, there is a contradiction either with Theorem 3.1 or with Theorem 4.1. We thus prove,

THEOREM 5.1. *Weak admissibility of the estimate* $\bar{e}(s, x)$ *implies its strict admissibility.*

This completes the proof of the result which we set out to prove.

REMARK. By a similar argument it is proved that $\bar{x}_s \cdot Y$ is strictly admissible for the population total, and $\bar{x}_s$ for the population weighted mean $\bar{X}_N$.

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