THE MARTINGALE VERSION OF A THEOREM OF MARCINKIEWICZ AND ZYGMUND

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1. Introduction. Suppose that $\{d_k\}_{k=1}^{\infty}$ is an orthonormal sequence of independent random variables, i.e., $E(d_k) = 0$, $E(d_k^2) = 1$, $k = 1, 2, \cdots$. When does the assertion that

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} v_k d_k$ exists and is finite almost everywhere

imply

(B) $$\sum_{k=1}^{\infty} v_k^2 < = \infty$$

for any numerical sequence $\{v_k\}_{k=1}^{\infty}$? In fact, since the converse is always true, the question is: *When is* (A) *equivalent to* (B)*?* At least three sufficient conditions for this equivalence are known: (i), uniform boundedness of the random variables $\{d_k\}_{k=1}^{\infty}$, Khinchine and Kolmogorov [5]; (ii), uniform integrability of the sequence $\{d_k^2\}_{k=1}^{\infty}$, Kac and Steinhaus [4]; and (iii), the condition that $E(|d_k|) \geqq \Gamma > 0$ uniformly in k, Marcinkiewicz and Zygmund [6]. The Marcinkiewicz-Zygmund theorem ([6], Theorem 4)—that (A) is equivalent to (B) under condition (iii)— is more general than either (ii) or (i) and is the starting point for the present paper.

We extend the theorem of Marcinkiewicz and Zygmund in two respects. We prove the equivalence of (A) and (B) when (a) the orthonormal system $\{d_k\}_{k=1}^{\infty}$ is a sequence of martingale differences that satisfy a condition analogous to the one given by Marcinkiewicz and Zygmund, and (b) each "coefficient" v_n is, in general, not constant but a function of the past, i.e., a function of $d_1, d_2, \cdots, d_{n-1}$. As in the classical case, one-half of the equivalence problem may be settled immediately: if the sequence of "coefficient" random variables $\{v_k\}_{k=1}^{\infty}$ is such that $\sum_{k=1}^{\infty} v_k^2 < \infty$ on a set A, then the partial sums $S_n = \sum_{k=1}^{n} v_k d_k$ converge to a finite limit almost everywhere on A for any sequence of martingale differences $\{d_k\}_{k=1}^{\infty}$ such that $E(d_k^2 \,\|\, \mathfrak{F}_{k-1}) = 1$ almost everywhere. (See Neveu, [7], Proposition IV.6.2, page 148). Clearly, the converse is false in general since it is false for the original problem. (A counterexample may be found in [6], page 73).

With the provision that all random variables are integrable in the setting just described, the partial sums $S_n = \sum_{k=1}^{n} v_k d_k$ form a martingale. Such martingales occur naturally in the study of certain classical orthogonal series, and, in this connection, a special case of the main theorem of this paper appears in [3]. There, however, the assertions and technique of proof are limited to a class of atomic martingales. The present formulation is the outcome of an attempt to place the

Received 14 November 1966.

results of ([3] Section 3) in the context of what is known about sums of independent variables. However, it now seems more appropriate—in the light of a recent paper by D. L. Burkholder [1]—to consider our result as another contribution toward a theory of martingale transforms.

2. Notation and Definitions. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $\mathfrak{F}_k$, $k = 0, 1, 2 \cdots$ an increasing sequence of σ-fields with $\mathfrak{F}_k \subset \mathfrak{F}$. The symbol $I\{\ \}$ is used to denote the indicator function of the ($\mathfrak{F}$-measurable) set in braces. Sets are said to be equivalent if their indicator functions are equal almost everywhere (a.e.).

Let $\{d_k, \mathfrak{F}_k\}_{k=1}^{\infty}$, or more briefly $\{d_k\}_{k=1}^{\infty}$, be an orthonormal sequence of martingale differences relative to the sequence $\mathfrak{F}_k$, $k \geqq 0$. In what follows, we consider the sequence of partial sums $S = \{S_n\}_{n=1}^{\infty}$,

$$S_n = \sum_{k=1}^{n} v_k d_k, \tag{1}$$

where $\{v_k\}_{k=1}^{\infty}$ is a sequence of random variables subject to the restriction that v_k is measurable with respect to $\mathfrak{F}_{k-1}$. For our purposes, we may assume that all random variables are real valued; the extension of the main result to complex valued variables is straightforward. Following Burkholder [1], we consider S as a *transform* of the martingale $T = \{T_n = \sum_{k=1}^{n} d_k\}_{n=1}^{\infty}$ by the *multiplier sequence* $\{v_k\}_{k=1}^{\infty}$. Here, we conform to the usual custom: the transform S is called a martingale if and only if $E(|v_k d_k|) < \infty$ for all $k \geqq 1$.

In conjunction with sequences of the form (1), we consider various stopping times: a stopping time t is a random variable taking values in the set $\{1, 2, \cdots, +\infty\}$ such that $\varphi_k = I\{t \geqq k\}$ is measurable with respect to $\mathfrak{F}_{k-1}$ for $k \geqq 1$. Usually, a stopping time is defined as the infimmum over a set of integers. If the set in question is empty, we set $t = +\infty$. If A denotes an $\mathfrak{F}$-measurable set, then we write $A_t(\infty) = \{t = +\infty\} \cap A$ and $\{t = +\infty\} = \Omega_t(\infty)$. A transform sequence of the form (1) stopped at t is denoted $S^t = \{S_n^t\}_{n=1}^{\infty}$. The sequence S^t is itself a transform of the martingale T by the multiplier sequence $\varphi_k v_k$:

$$S_n^t = \sum_{k=1}^{n} \varphi_k v_k d_k, \qquad n = 1, 2, \cdots.$$

It is convenient to introduce the notation

$$p = p(M, S) = \inf\{n: |S_n| > M\},$$

the first passage time of the sequence $|S|$ across the level M. The random variable p is a stopping time and, associated with it, we have the sequence of indicator functions $\pi_n = I\{n \leqq p < +\infty\}, n \geqq 1$.

3. A Condition for Martingale Differences. Now let $\{d_k, \mathfrak{F}_k\}_{k=1}^{\infty}$ be any sequence of martingale differences. For the moment, we do not assume that they are orthonormal. The following condition reduces to one given by Marcinkiewicz and Zygmund ([6], Section 3, p. 69) when the random variables $\{d_k\}_{k=1}^{\infty}$ are independent with $E(d_k) = 0$, $k \geqq 1$.

CONDITION $(MZ)^*$:

(i) $0 < (E(d_k^2 \parallel \mathfrak{F}_{k-1}))^{\frac{1}{2}} < +\infty$;

(ii) $E(|d_k| \parallel \mathfrak{F}_{k-1}) \geqq \Gamma(E(d_k^2 \parallel \mathfrak{F}_{k-1}))^{\frac{1}{2}}$

almost everywhere for each $k \geqq 1$. Here Γ is fixed constant subject to the inequality $0 < \Gamma \leqq 1$, and is independent of the index k.

Any sequence of martingale differences $\{d_k\}_{k=1}^{\infty}$ that satisfies condition $(MZ)^*$ may be normalized so that $E(d_k^2 \parallel \mathfrak{F}_{k-1}) = 1$ a.e., $k \geqq 1$. Clearly this involves no loss of generality and the sequence becomes orthonormal in the usual sense. In fact, we incorporate this normalization in the following.

CONDITION (MZ):

(i) $E(d_k^2 \parallel \mathfrak{F}_{k-1}) = 1$ a.e.;

(ii) $P(|d_k| > \lambda \parallel \mathfrak{F}_{k-1}) \geqq \gamma > 0$ for some constants $\lambda > 0, \gamma > 0$ uniformly for k $\geqq 1$.

This condition is equivalent to condition $(MZ)^*$; the proof of equivalence is a corollary of Lemma 1, given below.

Condition (MZ) is applicable in a number of situations. For example, the reader may easily verify that if $\{d_k\}_{k=1}^{\infty}$ is a sequence of independent orthonormal functions such that the L_p-norms $\|d_k\|_p \leqq B < +\infty$ uniformly in k for some p, $2 < p \leqq +\infty$, then condition (MZ) is satisfied. More generally, if the sequence $\{d_k^2\}_{k=1}^{\infty}$ is uniformly integrable, then condition (MZ) holds (the converse, however, is false).

The hypothesis $\| \sup_k |d_k| \|_p < \infty$ for some $1 \leqq p < \infty$ is sometimes used to secure special martingale convergence theorems; for example, see Neveu ([7] Proposition IV.6.2, page 148) and Burkholder ([1] Theorem 4). Condition (MZ) is weaker than this hypothesis in certain situations, although neither condition implies the other in general. For example, if $\{d_k\}_{k=1}^{\infty}$ is a sequence of independent identically distributed random variables—or more generally, any stationary sequence of martingale differences with a trivial tail field—then $\|\sup_k |d_k| \|_p < \infty$ for some $1 \leqq p < \infty$ is equivalent to $\|d_k\|_\infty \leqq B < \infty$ uniformly in k. This in turn, implies condition (MZ) if $\|d_k\|_2 = 1$ for $k \geqq 1$.

For another example of the applicability of condition (MZ), let us consider a class of atomic martingales introduced by Chow [2]. The following definition appears in [3]: an increasing sequence of purely atomic σ-fields $\{\mathfrak{F}_k\}_{k=1}^{\infty}$ is said to be *regular* if for any two atoms E_k belonging to $\mathfrak{F}_k$ and E_{k+1} belonging to $\mathfrak{F}_{k+1}$ with $E_k \supseteq E_{k+1}$, we have $0 < \delta \leqq P(E_{k+1})/P(E_k)$ for some $\delta > 0$ and all $k \geqq 1$.

PROPOSITION 1. *Any martingale* $T = \{T_n = \sum_{k=1}^{n} u_k\}_{n=1}^{\infty}$ *relative to a regular sequence of atomic* σ*-fields may be represented as a transform of an orthonormal sequence* $\{d_k\}_{k=1}^{\infty}$ *that satisfies condition* (MZ). *Furthermore, the sequence* $\{d_k\}_{k=1}^{\infty}$ *is uniformly bounded.*

PROOF. Notice that the definition of regularity also restricts the *upper* bound of all ratios as follows: either $P(E_{k+1})/P(E_k) \leqq 1 - \delta$ or $P(E_{k+1})/P(E_k) = 1$, in which case E_k is also an atom of $\mathfrak{F}_{k+1}$. We eliminate this second possibility by embedding the sequence $\{\mathfrak{F}_k\}_{k=1}^{\infty}$ in a regular sequence $\{\mathfrak{F}_k'\}_{k=1}^{\infty}$ such that

$$P(E_{k+1})/P(E_k) \leqq 1 - \delta.$$

Clearly, this is always possible if the probability space is nonatomic. Furthermore, any martingale T relative to $\{\mathfrak{F}_k'\}_{k=1}^{\infty}$ may be represented in the form (1) with

$$d_k = u_k/(E(u_k^2 \,\|\, \mathfrak{F}_{k-1}'))^{\frac{1}{2}} \quad \text{and} \quad v_k = (E(u_k^2 \,\|\, \mathfrak{F}_{k-1}'))^{\frac{1}{2}}$$

whenever $v_k > 0$. Since the set $\{v_k = 0\}$ is equivalent to $\{u_k = 0\}$, we may define d_k arbitrarily on the set $\{v_k = 0\}$, subject to the restrictions $E(d_k \,\|\, \mathfrak{F}_{k-1}') = 0$, $E(d_k^2 \,\|\, \mathfrak{F}_{k-1}') = 1$, where d_k is measurable with respect to $\mathfrak{F}_k'$. Note here that the above definition involves a slight abuse of notation. If $E(|u_k|^2) = \infty$ for some $k \geqq 1$, strictly speaking, $E(u_k^2 \,\|\, \mathfrak{F}_{k-1})$ is not defined. However, since the sequence of σ-fields in question is regular, $\mathfrak{F}_k$ is necessarily constructed by partitioning each atom of $\mathfrak{F}_{k-1}$ into no more than δ^{-1} sets of positive measure. Therefore, on each atom of $\mathfrak{F}_{k-1}$, u_k is essentially bounded so that $E(u_k^2 \,\|\, \mathfrak{F}_{k-1})$ may be defined in the obvious way. The fact that the orthonormal system satisfies condition (MZ) follows easily from the observation that $\|d_k\|_\infty \leqq \delta^{-\frac{1}{2}}$ for $k \geqq 1$.

The identification of the sequence of partial sums of a Haar series with the sequence of 2^nth partial sums of some Walsh series (see, for example [3]) in effect results from the above construction applied to the increasing sequence of partitions of the unit interval determined by the dyadic rationals. In this case, the orthonormal system $\{d_k\}_{k=1}^{\infty}$ may be taken to be the collection of Rademacher functions.

4. Preliminary Lemmas. The first lemma is due, essentially, to Paley and Zygmund [8].

LEMMA 1. (Paley and Zygmund). *Let $\mathfrak{F}$ be a σ-field, $g \geqq 0$ a random variable, and δ, $0 < \delta < 1$ a fixed constant. If $E(g \,\|\, \mathfrak{F}) \geqq \alpha > 0$ and $E(g^2 \,\|\, \mathfrak{F}) \leqq \beta$ on a set A, then*

$$P(g \geqq \delta\alpha \,\|\, \mathfrak{F}) \geqq (1 - \delta)^2\alpha^2/\beta$$

a.e. *on A.*

PROOF. A virtually equivalent assertion may be found in Zygmund ([9], page 216, 8.26). The proof is short; we give it for completeness. Let I_δ be the indicator function of the set $\{g \geqq \delta\alpha\}$ and $\bar{I}_\delta$ the indicator of its complement. Then

$$E(g\bar{I}_\delta \,\|\, \mathfrak{F}) < \delta\alpha$$

and, therefore, by the Schwartz inequality,

$$(\alpha - \delta\alpha)^2 \leqq (E(gI_\delta \,\|\, \mathfrak{F}))^2 \leqq E(I_\delta g^2 \,\|\, \mathfrak{F})\, E(I_\delta \,\|\, \mathfrak{F}) \leqq \beta\, E(I_\delta \,\|\, \mathfrak{F})$$

where these inequalities are understood to hold a.e. on A.

COROLLARY. *When $E(d_k^2 \,\|\, \mathfrak{F}_{k-1}) = 1$* a.e., *condition $(MZ)^*$ is equivalent to condition (MZ).*

PROOF. Suppose $(MZ)^*$ holds. Let $\beta = 1 = E(d_k^2 \,\|\, \mathfrak{F}_{k-1})$ and $\alpha = \Gamma$, $\delta = \frac{1}{2}$.

Then by Lemma 1,

$$P(|d_k| \geqq \Gamma/2 \parallel \mathfrak{F}_{k-1}) \geqq \Gamma^2/4 > 0$$

so that condition (MZ) holds. Now suppose (MZ) holds for some $\lambda > 0$ and $\gamma > 0$. Let $I_k = I\{|d_k| \geqq \lambda\}$. Then the relation $|d_k| \geqq |d_k|I_k$ implies

$$E(|d_k| \parallel \mathfrak{F}_{k-1}) \geqq E(|d_k|I_k \parallel \mathfrak{F}_{k-1}) \geqq \lambda E(I_k \parallel \mathfrak{F}_{k-1}) \geqq \lambda\gamma > 0$$

a.e., so that $(MZ)^*$ holds with $\lambda\gamma = \Gamma$.

LEMMA 2. *Assume that the sequence of martingale differences* $\{d_k\}_{k=1}^{\infty}$ *satisfies condition* (MZ). *If* $\limsup |v_n d_n| < +\infty$ *on a set* A, *then* $\limsup |v_n| < +\infty$ *almost everywhere on* A.

PROOF. Let λ, γ be the parameters given by condition (MZ). Choose M sufficiently large so that if $I_n = I\{|v_n d_n| \geqq M\}$ then $\sum_{n=1}^{\infty} I_n < +\infty$ a.e. on a subset $A' \subseteq A$ such that $P(A') \geqq (1 - \epsilon)P(A)$. Let $J_n = I\{|v_n| \geqq M/\lambda\}$ and $K_n = I\{|d_n| \geqq \lambda\}$. Then $J_n K_n \leqq I_n$ and

$$\begin{aligned}\sum_{n=1}^{\infty} \gamma\, J_n &\leqq \sum_{n=1}^{\infty} E(J_n K_n \parallel \mathfrak{F}_{n-1}) \\ &\leqq \sum_{n=1}^{\infty} E(I_n \parallel \mathfrak{F}_{n-1}) < +\infty,\end{aligned}$$

a.e. on A by Lévy's strengthened form of the Borel-Cantelli Lemma (see Neveu [7], Corollary, page 151). This implies that

$$\sum_{n=1}^{\infty} J_n < +\infty,$$

a.e. on A', and since $\epsilon > 0$ is arbitrary, we conclude $\limsup |v_n| < +\infty$ a.e. on A.

LEMMA 3. *If* $\sup |S_n| < +\infty$ a.e., *then, given any* $\epsilon > 0$, *there exists a stopping time* t *such that*

(i) $P(\Omega_t(\infty)) \geqq 1 - \epsilon$;
(ii) $\int |S_n{}^t|^2 dP < +\infty$

for all $n = 1, 2, \cdots$.

PROOF. The hypothesis implies that $\limsup |v_n d_n| < +\infty$ a.e. so that by Lemma 2, $\limsup |v_n| < +\infty$ a.e. Let

$$t = \inf\{n : |v_{n+1}| > M\}.$$

The random variable t is a stopping time since $|v_{n+1}|$ is measurable with respect to $\mathfrak{F}_n$. Also $P(\Omega_t(\infty)) \geqq 1 - \epsilon$ provided M is chosen sufficiently large. If $\varphi_k = I\{t \geqq k\}$ then we may write;

$$\int |S_n{}^t|^2 dP = \sum_{k=1}^{n} \int |v_k|^2 \varphi_k dP \leqq n \cdot M^2,$$

for $n = 1, 2, \cdots$, as is required.

LEMMA 4. *If* $\sup |S_n| < +\infty$ a.e. *then given any* $\epsilon > 0, 0 < K \leqq 1$ *there exist a constant* M *and stopping time* r *such that*

(i) $P(\Omega_r(\infty)) \geqq 1 - \epsilon$; *the first passage time* $p(M, S^r)$ *has the properties*
(ii) $E(\pi_n \parallel \mathfrak{F}_{n-1}) \leqq K$ a.e. for $n = 1, 2, \cdots$;
(iii) $\sum_{n=1}^{\infty} \int E(\pi_n \parallel \mathfrak{F}_{n-1}) dP \leqq \epsilon$.

(In the following proof, all inequalities between random variables should be interpreted to hold a.e.)

Proof. The stopping time r is defined as the minimum of four stopping times:

$$r = \min (p, i, d, p'').$$

Let $p = p(M, S) = \inf \{n: |S_n| > M\}$ where M is chosen so that

$$P(\Omega_p(\infty)) \geqq 1 - K\epsilon/4.$$

Let $i = \inf \{n: E(\pi_{n+1} \| \mathfrak{F}_n) > K\}$ where

$$\pi_{n+1} = I\{n + 1 \leqq p < + \infty\}.$$

Notice that $i < p$ on the set $\{i < + \infty\}$ since, if $m \geqq n$, then

$$0 = \int_{\{p=n\}} \pi_{n+1} dP = \int_{\{p=n\}} E(\pi_{n+1} \| \mathfrak{F}_n) dP \geqq \int_{\{p=n,\ i=m\}} E(\pi_{n+1} \| \mathfrak{F}_n) dP \geqq KP(i = m, p = n).$$

Moreover,

$$KP(i < + \infty) \leqq \sum_{n=0}^{\infty} \int_{\{i=n\}} E(\pi_{n+1} \| \mathfrak{F}_n) dP \leqq \sum_{n=0}^{\infty} \int_{\{i=n\}} E(\pi_1 \| \mathfrak{F}_n) dP = \sum_{n=0}^{\infty} \int_{\{i=n\}} \pi_1 dP \leqq P(p < + \infty) \leqq K\epsilon/4.$$

Let $g = \min (p, i)$ and notice that $P(\Omega_g(\infty)) \geqq 1 - \epsilon/2$. Consider the first passage time $p' = p(M, S^g)$ together with the indicator functions $\pi_n' = I\{n \leqq p' < + \infty\}$; we have

$$E(\pi'_{n+1} \| \mathfrak{F}_n) \leqq K \qquad \text{for } n = 0, 1, \cdots.$$

In fact, $g \leqq p$ implies $\pi_n' \leqq \pi_n$ for $n = 1, 2, \cdots$. From this it follows that on the set $\{i = + \infty\}$ we have

$$E(\pi_n' \| \mathfrak{F}_{n-1}) \leqq E(\pi_n \| \mathfrak{F}_{n-1}) \leqq K \qquad \text{for } n \geqq 1.$$

On the set $\{i = N\}$, we have

$$E(\pi_n' \| \mathfrak{F}_{n-1}) \leqq E(\pi_n \| \mathfrak{F}_{n-1}) \leqq K$$

for $n = 1, 2, \cdots, N$ and

$$E(\pi'_{N+k+1} \| \mathfrak{F}_{N+k}) = 0 \qquad \text{for } k \geqq 0,$$

since $i < p$ implies $|S^g_{N+k}| = |S_N| \leqq M$ for $k = 0, 1, \cdots$. Since N is arbitrary, we may combine these inequalities to conclude that $E(\pi_n' \| \mathfrak{F}_{n-1}) \leqq K$ for $n = 1, 2, \cdots$. In summary, the stopping time g and corresponding martingale S^g together satisfy requirements (i) and (ii) of the lemma. From now on, we restrict all considerations to the martingale S^g.

In order to define the stopping time d, notice that

$$\sum_{n=1}^{\infty} \pi_n' = \sum_{n=1}^{\infty} n \cdot I\{p' = n\} < + \infty.$$

It follows from Lévy's version of the Borel-Cantelli Lemma that

$$\sum_{n=1}^{\infty} E(\pi_n' \| \mathfrak{F}_{n-1}) < + \infty.$$

Define the stopping time $d = \inf \{n: \sum_{k=1}^{n+1} E(\pi_k' \parallel \mathfrak{F}_{k-1}) > N\}$ where N is chosen so that

$$P(\Omega_d(\infty)) \geqq 1 - \epsilon/4.$$

Now the argument follows the pattern established above. Define $e = \min (g, d)$ and the martingale S^e. Let $p^* = p(M, S^e)$ and $\pi_n^* = I\{n \leqq p^* < +\infty\}$ for $n = 1, 2, \cdots$. Since $e \leqq g$ implies $\pi_n^* \leqq \pi_n'$, it follows that $E(\pi_n^* \parallel \mathfrak{F}_{n-1}) \leqq K$ for $n = 1, 2, \cdots$. Also $\sum_{n=1}^{\infty} E(\pi_n^* \parallel \mathfrak{F}_{n-1}) \leqq N$ by the argument given above with π_n' in place of π_n and π_n^* in place of π_n'. Therefore, the martingale S^e satisfies requirement (i) of the lemma (since $P(\Omega_e(\infty)) \geqq 1 - 3\epsilon/4$) and requirement (ii). However, at this point, we may conclude only that

$$\sum_{n=1}^{\infty} \int E(\pi_n^* \parallel \mathfrak{F}_{n-1}) dP \leqq N.$$

This difficulty is easily overcome by choosing M sufficiently large. In fact, if requirement (iii) is not fulfilled, choose N_0 so that

$$\sum_{n=N_0+1}^{\infty} \int E(\pi_n^* \parallel \mathfrak{F}_{n-1}) dP \leqq \epsilon/2.$$

Now choose $M' > M$ sufficiently large so that if

$$p'' = p(M', S^e),$$

and

$$\pi_n'' = I\{n \leqq p'' < +\infty\},$$

then

$$\int E(\pi_n'' \parallel \mathfrak{F}_{n-1}) dP \leqq \epsilon/2N_0$$

for $n = 1, 2, \cdots, N_0$. Clearly, $\pi_n'' \leqq \pi_n^*$ for $n = 1, 2, \cdots$ and requirements (i), (ii), and (iii) will be satisfied with

$$r = \min (e, p'') = \min (p, i, d, p'').$$

The proof of the lemma is complete.

5. A Convergence Equivalence. The main theorem may be stated as follows:

THEOREM. *If an orthonormal sequence of martingale differences* $\{d_k\}_{k=1}^{\infty}$ *satisfies condition (MZ), then, for any multiplier sequence* $\{v_k\}_{k=1}^{\infty}$, *the following three sets are equivalent:*

$$A = \{\lim \textstyle\sum_{k=1}^{n} v_k d_k \text{ exists and is finite}\};$$

$$B = \{\textstyle\sum_{k=1}^{\infty} v_k^2 < +\infty\};$$

$$C = \{\textstyle\sum_{k=1}^{\infty} (v_k d_k)^2 < +\infty\}.$$

We prove the theorem is two stages. First, we show that A and B are equivalent. The fact that A essentially contains B is the main contribution of the paper. The converse is known (see [7], Proposition IV.6.2, page 148).

Second, we show that B and C are equivalent. In so doing we use the equivalence of A and B together with a device involving Fubini's theorem that was suggested to us by D. L. Burkholder.

The proof of the equivalence of A and B rests on the possibility of constructing a certain stopping time. This is expressed by the following proposition (which, incidentally, does not involve condition (MZ)).

PROPOSITION 2. *Let* $\{d_k\}_{k=1}^{\infty}$ *be a sequence of martingale differences such that* $E(d_k^2 \,\|\, \mathfrak{F}_{k-1}) = 1$ a.e. *for all* $k \geqq 1$, *and* $\{v_k\}_{k=1}^{\infty}$ *a fixed multiplier sequence. Then the sets A and B are equivalent if and only if for any* $\epsilon > 0$, *there exists a stopping time s and a constant K such that*

(i) $P(A_s(\infty)) \geqq (1 - \epsilon)\, P(A)$;

(ii) $\int |S_n^s|^2 dP \leqq K < +\infty$

for all $n \geqq 1$.

The proof of Proposition 2 consists of a standard stopping time argument quite similar to that used in [7], Proposition IV.6.2, page 148. The details are omitted.

PROOF OF THE THEOREM. (1) *The sets A and B are equivalent.* We proceed to construct the stopping time s mentioned in Proposition 2.

Since A is the set where $\lim_{n\to\infty} S_n$ exists and is finite, there exists an M such that if $p = p(M, S)$ then $P(A_p(\infty)) \geqq (1 - \epsilon/3)P(A)$. The resulting martingale S^p satisfies the hypothesis of Lemma 3, so that if $q = \min(p, t)$ (where t is the stopping time of Lemma 3), then

$$\int |S_n^q|^2 \, dP < +\infty$$

for $n = 1, 2, \cdots$. (However, we assume that

$$\lim_{n\to\infty} \int |S_n^q|^2 \, dP = +\infty;$$

otherwise, there is nothing to prove.) In addition, we may assume that

$$P(A_q(\infty)) \geqq 1 - 2\epsilon/3.$$

Choosing $\epsilon' = \min(\epsilon P(A)/3), (\Gamma^4/128)^2)$ and $K = \Gamma^2/8$ where Γ is the parameter specified in condition (MZ), we apply Lemma 4 to the martingale $S = S^q$. (Henceforth, we drop the superscript q.) We denote by r, p, and M, respectively, the stopping time, first passage time, and constant provided by Lemma 4, and

$$\varphi_k = I\{k \leqq p\}, \qquad \pi_k = I\{k \leqq p < +\infty\}.$$

Fix $n \geqq n_0$ where n_0 is the smallest integer such that

$$\left(\int |S_{n_0}|^2 \, dP\right)^{\frac{1}{2}} \geqq M.$$

Then we compute as follows;

$$M^2 \geqq \int |S_n^r|^2 \varphi_{n+1} \, dP = \int |S_n^r|^2 \, dP - \int_{\{p \leqq n\}} |S_n^r|^2 \, dP$$
$$= \int \sum_{k=1}^n v_k^2 \, d_k^2 \, dP - \sum_{j=1}^n \int_{\{p=j\}} \left|\sum_{k=1}^n v_k \, d_k\right|^2 dP$$

$$= \int \sum_{k=1}^{n} v_k^2 d_k^2 \, dP - \sum_{j=1}^{n} \int_{\{p=j\}} \sum_{k=1}^{n} v_k^2 d_k^2 \, dP$$
$$- 2 \sum_{j=1}^{n} \int_{\{p=j\}} \sum_{1 \leqq k < i \leqq n} v_k d_k v_i d_i \, dP$$
$$= I - II - III.$$

We may write $I - II = \sum_{k=1}^{n} \int v_k^2 d_k^2 \, dP - \sum_{k=1}^{n} \int v_k^2 \pi_k d_k^2 \, dP$ since $v_k^2 d_k^2 = 0$ a.e. on the set $\{p < k\}$. Therefore

$$I - II = \sum_{k=1}^{n} \int (v_k^2 d_k^2 - v_k^2 \pi_k d_k^2) \, dP$$
$$= \sum_{k=1}^{n} \int v_k^2 E(\bar{\pi}_k d_k^2 \parallel \mathfrak{F}_{k-1}) \, dP$$

where $\bar{\pi}_k = 1 - \pi_k$. Since

$$E(\bar{\pi}_k \parallel \mathfrak{F}_{k-1}) = 1 - E(\pi_k \parallel \mathfrak{F}_{k-1}) \geqq 1 - \Gamma^2/8,$$

it follows from condition (MZ) that

$$E(\bar{\pi}_k d_k^2 \parallel \mathfrak{F}_{k-1}) \geqq (\Gamma/2)^2 \Gamma^2/8 = \Gamma^4/32.$$

Consequently,

$$I - II \geqq \Gamma^4/32 \sum_{k=1}^{n} \int v_k^2 \, dP = \Gamma^4/32 \int |S_n^r|^2 \, dP.$$

Now we estimate

$$III = 2 \sum_{j=1}^{n} \int_{\{p=j\}} \sum_{1 \leqq k < i \leqq n} v_k d_k v_i d_i \, dP$$
$$= 2 \int \sum_{1 \leqq k < i \leqq n} v_k d_k \pi_k v_i d_i \pi_i \, dP.$$

Recall that $\pi_k \pi_i = \pi_i$ for $k < i$ and that $\varphi_i \pi_i = \pi_i$, and sum over $k = 1, 2, \cdots, i - 1$ to obtain,

$$III = 2 \sum_{i=2}^{n} \int S_{i-1}^r v_i \varphi_i \pi_i d_i \, dP$$
$$= 2 \sum_{i=2}^{n} \int S_{i-1}^r v_i \varphi_i E(\pi_i d_i \parallel \mathfrak{F}_{i-1}) \, dP.$$

Since $|S_{i-1}^r \varphi_i| \leqq M$, we may apply the Schwartz inequality to estimate

$$|III| \leqq 2M \sum_{i=1}^{n} \int |v_i| |E(d_i \pi_i \parallel \mathfrak{F}_{i-1})| \, dP$$
$$\leqq 2M \sum_{i=1}^{n} \left(\int v_i^2 \, dP\right)^{\frac{1}{2}} \left(\int E^2(d_i \pi_i \parallel \mathfrak{F}_{i-1}) \, dP\right)^{\frac{1}{2}}$$
$$\leqq 2M \left(\sum_{i=1}^{n} \int v_i^2 \, dP\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \int E^2(d_i \pi_i \parallel \mathfrak{F}_{i-1}) \, dP\right)^{\frac{1}{2}}$$
$$\leqq 2 \int |S_n^r|^2 \, dP \left(\sum_{i=1}^{n} \int E^2(d_i \pi_i \parallel \mathfrak{F}_{i-1}) \, dP\right)^{\frac{1}{2}}$$

since $M \leqq (\int |S_n^r|^2 \, dP)^{\frac{1}{2}}$. Now we estimate as follows:

$$\sum_{i=1}^{n} \int E^2(d_i \pi_i \parallel \mathfrak{F}_{i-1}) \, dP \leqq \sum_{i=1}^{n} \int E(d_i^2 \parallel \mathfrak{F}_{i-1}) E(\pi_i \parallel \mathfrak{F}_{i-1}) \, dP$$
$$= \sum_{i=1}^{n} \int E(\pi_i \parallel \mathfrak{F}_{i-1}) \, dP \leqq (\Gamma^4/128)^2.$$

Consequently

$$|III| \leqq \Gamma^4/64 \int |S_n^r|^2 \, dP \quad \text{so that} \quad M^2 \geqq I - II - III$$
$$\geqq \Gamma^4/32 \int |S_n^r|^2 \, dP - \Gamma^4/64 \int |S_n^r|^2 \, dP$$
$$= \Gamma^4/64 \int |S_n^r|^2 \, dP \quad \text{for every} \quad n \geqq n_0.$$

The proof of the theorem is now essentially complete: The stopping time $s = \min(r, q)$ satisfies requirement (i) of Proposition 2 since $P(A_q(\infty)) \geqq (1 - \frac{2}{3}\epsilon)P(A)$ and $P(\Omega_r(\infty)) \geqq 1 - (\epsilon/3)P(A)$.

We have shown that requirement (ii) is satisfied with $K = (64/\Gamma^4)M^2$ for all $n \geqq n_0$. Since the expression in (ii) is an increasing function of n, requirement (ii) must hold for all $n \geqq 1$. Therefore, we may conclude from Proposition 2 that the sets A and B are equivalent.

(2) *The sets B and C are equivalent.* Consider the product probability space $\Omega \times [0, 1]$, where $[0, 1]$ is the usual Lebesgue unit interval. Let $\{r_k\}_{k=1}^{\infty}$ be the collection of Rademacher functions defined on $[0, 1]$. Then, by [5], the sets $C \times [0, 1]$ and $\{\lim_{n\to\infty} \sum_{k=1}^{n} r_k v_k\, d_k$ exists and is finite$\}$ are equivalent with respect to the product measure. From part (1) of the present proof we may conclude that the last mentioned set is equivalent to $B \times [0, 1]$ with respect to the product measure. Therefore, by Fubini's theorem, the sets B and C are equivalent with respect to the original probability measure.

We have shown, therefore, that the sets A, B, and C are equivalent as required and the proof of the theorem is complete.

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