STOCHASTIC POINT PROCESSES: LIMIT THEOREMS[1]

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1. Introduction. Stochastic point processes correspond to our intuitive notion of a countable aggregate of points randomly distributed in R^n (Cartesian n-space). For clarity, the points of the aggregate will be called *particles* and so we shall be concerned with random distributions of particles. The point processes on R^1 that have been most widely studied are the Poisson process, renewal processes, processes with stationary increments and general stationary processes. Khintchine [11] proved a variety of general statements about point processes on R^1 (or random streams as he called them). However most of the published results about point processes on R^1 make essential use of the order properties of the line. The first interesting examples of point processes in higher dimensions seem to be the cluster processes in R^3 introduced in [15] by Neyman and Scott as models for the distribution of clusters of galaxies. Here we begin a systematic study of limit theorems for stochastic point processes for all R^n.

Sections 2 and 3 contain the basic definitions and examples and a lemma fundamental to our later theorems. Sections 3–7 consider a variety of operations which "scramble" a point process. Well distributed processes are introduced as the natural class upon which to perform these operations. Our results lead us to the following heuristic principle:

If one scrambles a point process without introducing any new dependence between particles and if the operation is iterated, then the resulting sequence of scrambled processes converges to a mixture of Poisson processes. This reinforces our notion of the Poisson process as the most random distribution of particles.

The results in this paper are true for all R^n, but for the sake of clarity, I present them in R^2. The generalization of our results to all R^n are immediate. *Thus from this point on, unless explicitly stated otherwise, all statements and proofs refer to point processes on R^2. Furthermore all sets in R^2 are assumed to be bounded Borel sets unless otherwise indicated.*

2. Definitions and examples. Let ω be a countable (finite or denumerably infinite) aggregate of particles in R^2 and let $S \subset R^2$. Then $N(S, \omega)$ denotes the number of particles of ω in S. We generally write $N(S)$ in place of $N(S, \omega)$, the set ω being understood.

Definition 2.1. *A stochastic (or random) point process on R^2* is a triple (M, M_B, P), where (1) M is the class of all countable aggregates of particles in R^2 without limit points, (2) M_B is the smallest Borel algebra on M such that for

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every $S \subset R^2$, $N(S, \omega) = N(S)$ will be measurable, i.e., M_B is the smallest Borel algebra for which the $N(S)$ are random variables, (3) P is a probability measure on M_B. This definition follows [14].

We often denote the process (M, M_B, P) by $[P]$ or N or N_p where N and N_P refer to the counting functions $\{N(S), S \subset R^2\}$. The restriction of our process to a set $S \subset R^2$ is defined in the obvious manner.

It is essentially proved, in [10] and [13], in the more general context of population processes, that there exists a unique measure P on M_B assuming prescribed values for events of the form $\{N(S_1) = r_1, \cdots N(S_n) = r_n\}$ where the S_i are disjoint, provided these values satisfy certain consistency conditions.

Our Definition 3.1 rules out the possibility of more than one particle appearing at a point, e.g., we do not consider the compound Poisson process defined in [6]. This is a technical convenience assumed for purposes of clarity and can easily be done away with.

We say a sequence of point processes $\{[P_n]\}$ *converges* to the point process $[P]$ if for every n-tuple of sets $S_1, \cdots S_m$ and nonnegative integers $K_1, \cdots K_m$ we have

$$\lim_{n\to\infty} P_n\{N(S_1) = K_1, \cdots, N(S_m) = K_m\} = P\{N(S_1) = K_1, \cdots, N(S_m) = K_m\}.$$

Poisson Process. For fixed $\lambda > 0$ and every set S, let the distribution of $N(S)$ be Poisson with mean $\lambda|S|$ and assume that for disjoint sets $S_1, \cdots, S_n$, the $N(S_1), \cdots N(S_n)$ are independent. The process thus defined is called the *Poisson process with mean* λ and its measure is denoted by P_λ.

An interesting question related to the Poisson process follows: Assume $[P]$ is a point process such that for some fixed $\lambda > 0$ and every set S, $N(S)$ is Poisson distributed with mean λ. Is $[P]$ a Poisson Process, i.e., if $S_1, \cdots S_m$ are disjoint, are $N(S_1), \cdots, N(S_m)$ independent? It is often implicitly assumed in the literature that the answer is yes, however, in the appendix to this paper we give an example which shows the answer is no if the S_i are intervals. If the S_i are finite unions of intervals, the answer is yes [16].

A stochastic point process $[P]$ is *stationary* if for every $y \in R^2$ and for every n-tuple of bounded sets $S_1, \cdots, S_n$ we have

$$P\{N(S_1) = k_1, \cdots, N(S_n) = k_n\} = P\{N(S_1 + y) = k_1, \cdots, N(S_n + y) = k_n\}$$

where $S_i + y = \{x + y / x \in S_i\}$. The *mean* of a stationary point process is the expected number of particles in a unit square where the value ∞ is admissible.

3. Fundamental lemma. We consider arrays $\{X_{kl}\}$ of 2-dimensional random variables, $l = 1, 2, \cdots$ and k indexed on the positive integers or positive real numbers, where $X_{kl} = (X_{kl}^1, X_{kl}^2)$. The proof of our lemma is valid for both indexing sets. The variables in question all serve mainly in our applications to specify the location of a particle in one of two given sets. The *row sums* are denoted by $V_k = \sum_l X_{kl}$. $\{X_{kl}\}$ will be called a *null array* if $\text{Sup}_{i,l} P\{X_{kl}^i > 0\} \to 0$, $k \to \infty$, $i = 1, 2$. It is called a *Bernoulli array* if for each k, $\{X_{kl}\}$ is a sequence of independent random variables assuming the values (0, 0), (1, 0) or (0, 1).

Recall that a random variable Y has a *2-dimensional* Poisson distribution with parameters λ_i, denoted by $P(\lambda_1, \lambda_2)$ if

$$P\{Y = (l_1, l_2)\} = e^{-\lambda_1}(\lambda_1^{l_1}/l_1!)e^{-\lambda_2}(\lambda_2^{l_2}/l_2!).$$

The following lemma is a variation of that given in [17].

FUNDAMENTAL LEMMA. *Let $\{X_{kl}\}$ be a Bernouilli null array. Then a necessary and sufficient condition that the distributions of the row sums V_k converge to $P(\lambda_1, \lambda_2)$ is that*

$$\sum_{l=1}^{\infty} P\{X_{kl}^i = 1\} \to \lambda_i, \qquad k \to \infty, \quad i = 1, 2. \tag{3.1}$$

PROOF. Let $1 - (p_{kl}^{10} + p_{kl}^{01}) + p_{kl}^{10}x + p_{kl}^{01}y$ be the generating function for X_{kl}, where $p_{kl}^{ij} = P\{X_{kl} = (i, j)\}$. Then $P_k(x, y) = \prod_{l=1}^{\infty} [1 - (p_{kl}^{10} + p_{kl}^{01}) + p_{kl}^{10}x + p_{kl}^{01}y]$ is the generating function for V_k and

$$\log P_k(x, y) = \sum_{l=1}^{\infty} \log \{1 - [p_{kl}^{10}(1 - x) + p_{kl}^{01}(1 - y)]\}.$$

Since we assumed $\sup_{i,l} P\{X_{kl}^i > 0\} \to 0$ and since $\log (1 - x) = -x - rx$ where $r \to 0$ as $x \to 0$, it follows from (4.1) that

$$\log P_k(x, y) = -(1 - x)\sum_{l=1}^{\infty} (p_{kl}^{10} + r_{kl}p_{kl}^{10}) - (1 - y)\sum_{l=1}^{\infty} (p_{kl}^{01} + r_{kl}p_{kl}^{01}) \tag{3.2}$$

$\to -\lambda_1(1 - x) - \lambda_2(1 - y)$, $k \to \infty$. Thus $P_k(x, y)$ converges to the generating function of $P(\lambda_1, \lambda_2)$. The necessity of (3.1) follows from (3.2) and the fact that $\{X_{kl}\}$ is a null array.

4. Superposition of point processes. We treat first the superposition of a large number of point processes which are uniformly sparse. By the superposition of a finite number of point processes N_i, we mean the process resulting from adding these processes $N = \sum N_i$, regarding them as probabilistically independent, i.e., $N(S)$, the number of particles of the superimposed process in S, is the sum $\sum N_i(S)$ of particles in S for each of the individual processes, where the $N_i(S)$ are independent. For formal definitions in the context of Section 3 see the appendix to [8].

The following theorem carries over the classical result on limits of triangular arrays of random variables, [7], into the realm of point processes.

THEOREM 4. Let $\{N_{kl}\}$, $k = 1, 2, \cdots$, $l = 1, 2, \cdots, k$ be a triangular array of point processes satisfying: for any set S,

(A) $\text{Sup}_l P\{N_{kl}(S) > 0\} \to 0, \quad k \to \infty$,

(B) $\sum_{l=1}^{k} P\{N_{kl}(S) > 1\} \to 0, \quad k \to \infty$

Then a necessary and sufficient condition that $N_k = \sum_{l=1}^{k} N_{kl} \to [P_\lambda]$, (Poisson Process) is that

(C) $\sum_{l=1}^{k} P\{N_{kl}(S) = 1\} \to \lambda|S|$ for every S.

PROOF. (1) *Sufficiency.* We must show that for any pair of disjoint sets S_1 and S_2, the distribution of $[N_{kl}(S_1), N_{kl}(S_2)]$ converges to the 2-dimensional Poisson distribution $P(\lambda|S_1|, \lambda|S_2|)$. Define the Bernoulli array $X_{kl} = [X_{kl}^{(1)}, X_{kl}^{(2)}]$ by

$$
\begin{aligned}
X_{kl} &= [N_{kl}(S_1), N_{kl}(S_2)] \quad \text{if} \quad [N_{kl}(S_1), N_{kl}(S_2)] \\
&= (0, 0) \quad \text{or} \quad (1, 0) \quad \text{or} \quad (0, 1) \\
&= (0, 0) \quad \text{otherwise}
\end{aligned} \tag{4.1}
$$

We verify the conditions of the fundamental lemma. First (A) $\sup_{i,l} P\{X_{kl}^i > 0\} \leqq \sup_l P\{N_{kl}(S_1 \cup S_2) > 0\} \to 0, k \to \infty$. Hence $\{X_{kl}\}$ is null array. Secondly, $\sum_l P\{X_{kl}^i = 1\} = \sum_l (P\{N_{kl}(S_1) = 1\} - P\{N_{kl}(S_1) \geqq 1, N_{kl}(S_2) \geqq 1\})$. But by (B) $\sum_l P\{N_{kl}(S_1) \geqq 1,\ N_{kl}(S_2) \geqq 1\} \leqq \sum_l P\{N_{kl}(S_1 \cup S_2) > 1\} \to 0$ so that using (C) we get

$$
\sum_l P\{X_{kl}^i = 1\} \to \lambda|S_i|, \qquad i = 1, 2. \tag{4.2}
$$

Thus the conditions of the lemma are satisfied and the distribution of $V_k = \sum_l X_{kl}$ converges to $P(\lambda|S_1|, \lambda|S_2|)$. Now let V_k^* be the actual number of particles of N_k in S_1 and S_2, i.e., $V_k^* = [N_k(S_1),\ N_k(S_2)] = \sum [N_{kl}(S_1), N_{kl}(S_2)]$. To finish the proof we must show

$$
\lim_{k\to\infty} P\{V_k^* = (l_1, l_2)\} = \lim_{k\to\infty} P\{V_k = (l_1, l_2)\}. \tag{4.3}
$$

This will follow if we prove $\lim_{k\to\infty} P(V_k^* \neq V_k) = 0$. But

$$
\{V_k \neq V_k^*\} = \{\text{for some } l,\ X_{kl} \neq [N_{kl}(S_1), N_{kl}(S_2)]\},
$$

$\subset \bigcup_l \{X_{kl} \neq [N_{kl}(S_1), N_{kl}(S_2)]\} = \bigcup_l \{N_{kl}(S_1 \cup S_2) > 1\}$ so that by (B) $P\{V_k \neq V_k^*\} \leqq \sum_l P\{N_{kl}(S_1 \cup S_2) > 1\} \to 0$. Q.E.D.

(2) *Necessity.* The only place we used condition (C) in the proof of sufficiency was in (4.2). But by the necessity part of the lemma, the distribution of V_k converges to $P(\lambda|S_1|, \lambda|S_2|)$ only if (C) holds and since the proof of (4.3) does not depend on (C), the necessity is established. Q.E.D.

5. Well-distributed process-mixed Poisson process. The remaining limit theorems treat a wider class of processes than the stationary ones.

A countable aggregate of particles, with no limit points, will be called *well-distributed* (a G-set for short [G stands for gleichverteilung]) with parameter μ, $0 \leqq \mu \leqq \infty$, if for every $v \in R^2$ and every expanding sequence of rectangles $I_1, I_2, \cdots$ centered at v, at least one of whose dimensions tends to infinity, we have $\lim_{n\to\infty} N(I_n)/|I_n| = \mu$. A *well-distributed process* (a *G-Process* for short) is a process whose sample points are well-distributed with probability one. (Different sample points may have different parameters.) The *parameter variable* $\mu(\omega)$ is then a random variable with *parameter distribution* $F(\lambda)$.

THEOREM 5.1. *A sufficient condition for a stationary point process to be well distributed is that it have a finite mean and finite* $1 + \delta$ *moment for some* $\delta > 0$.

The expectation of the parameter distribution equals the mean of the process.

REMARK. In R^1 a finite mean is sufficient.

PROOF. Let I_n be an expanding sequence of rectangles centered about a point p, both of whose dimensions tend to infinity. Then p partitions each I_n into four rectangles each having p as a vertex. Let I_n' be the rectangle with p as its lower left hand vertex and let I_n^2, I_n^3, I_n^4 be the other rectangles going in a clockwise direction. If $N(I_n')/|I_n'|$, $N(I_n^2)/|I_n^2|$, $N(I_n^3)/|I_n^3|$, $N(I_n^4)/|I_n^4|$ all have the same limit, then $(N(I_n)/|I_n|)$ has this limit. We prove $(N(I_n')/|I_n'|)$ has a limit, the proof for the others being identical. Because of the stationarity we can let p be the origin and thus all I_n' are in the first quadrant. Let (a_n, b_n) be the coordinates of the upper right hand vertex of I_n'.

Consider the stationary stochastic process $X(v)$, indexed on the points v of R^2, given by $X(v) = X(x, y) = N(I_{x,y})$ where $I_{x,y}$ is the square whose vertices are (x, y), $(x, y - 1)$, $(x - 1, y)$, $(x - 1, y - 1)$. From our hypothesis it follows that $X(v)$ has finite first and $1 + \delta$ moments. We now apply an ergodic theorem (see [5], Theorem 10, p. 694) to $X(v)$ and get

$$\lim_{a_n\to\infty, b_n\to\infty} (1/a_n b_n) \int_0^{b_n} \int_0^{a_n} X(x, y)\, dx\, dy = \mu \tag{5.1}$$

where μ is a random variable independent of the choice of the sequence of rectangles and $E\mu = EX(x, y) = EN(I_{1,1})$. But we have

$$(1/a_n b_n) \int_{+1}^{a_n} \int_{+1}^{b_n} X(x, y)\, dx\, dy \leqq N(I_n')/a_n b_n = N(I_n')/|I_n'|$$
$$\leqq (1/a_n b_n) \int_0^{a_n+1} \int_0^{b_n+1} X(x, y)\, dx\, dy$$

and therefore by (5.1) $\lim_{n\to\infty} (N(I_n')/|I_n'|) =_{\text{a.e.}} \mu$ and our process is a G-process with parameter variable μ.

In the case where one of the dimensions of the rectangles remains bounded, a minor change in the above argument is required. Q.E.D.

By applying the strong law of large numbers we get

THEOREM 5.2. For the Poisson process $[P_\lambda]$, the parameter variable $\mu = \lambda$ with probability one.

The following mixture of Poisson processes, introduced in R^1 by Dobrushin [3], form the class of limiting processes for our remaining theorems. A stationary point process (M, M_N, P) is a *mixed Poisson process* with mixture distribution $F(\lambda)$, where $F(0) = 0$, if there is a random variable μ, with distribution $F(\lambda)$, defined on M and such that for almost every λ, the given point process conditioned on $\mu = \lambda$ is a Poisson process with mean λ. This means

$$P\{N(S_1) = l_1, \cdots, N(S_n) = l_n\} = \int_0^\infty \prod_{i=1}^n ((\lambda|S_i|)^{l_i}/l_i!) e^{-\lambda|S_i|}\, dF(\lambda).$$

By virtue of theorem 5.2, the mixed Poisson process has μ as its parameter variable and p.d. $F(\lambda)$ equal to its mixture distribution.

6. Random translations. Number the particles of a point process N in an arbitrary order $x_1, x_2, \cdots$. Now let the particles move randomly in time so that at time t, the nth point x_n has moved to $x_n + Y_n(t)$ where the $Y_n(t)$ *are* 2-

dimensional random variables with a common distribution, independent of each other and of the x_n. For each t we can think of this new process N_t of the $\{x_n + Y_n(t)\}$ as a *random translation* of the process N by the random variables $Y_n(t)$. For formal definitions as a special case of cluster processes see [8], [9].

The following theorem was proved by Doob [4] for R^1 and immediately generalizes to all R^n.

THEOREM 6.1. (*Doob*) *If* N *is a Poisson process* $[P_\lambda]$ *and* $\{Y_n(t)\}$ *is an arbitrary set of motions, then for every* t, *the translated process is also a Poisson process with mean* λ.

From this follows that if N is a mixed Poisson process with mixture distribution $F(\lambda)$ and $\{Y_n(t)\}$ is a set of motions, then for every t, the translated process is a mixed Poisson process with mixture distribution $F(\lambda)$. Thus the mixed Poisson processes are *invariant* under any set of motions and we shall prove they are the only processes with this property. Now turning to the study of translated processes we show that under reasonable conditions the mixed Poisson processes are the only limiting processes for N_t as $t \to \infty$. N_∞ is called a limiting process if $N_t \to N_\infty$.

Dobrushin [3], and Maruyama [12] worked on random translations in R^1.

THEOREM 6.2. *If a set of motions* $Y_n(t)$ *satisfy*: (A) *for every* $I \subset R^2$

$$\mathrm{Sup}_{y \varepsilon R^2} F_t(I - y) \to 0 \qquad t \to \infty,$$

then a nasc that for every G-process, with parameter distribution $F(\lambda)$, the limiting process exist and be a mixed Poisson process with mixture distribution $F(\lambda)$ is (B) for any G-set $\{y_1, y_2, \cdots\}$ with parameter λ and any set $I \subset R^2$,

$$\textstyle\sum_i F_t(I - y_i) \to \lambda |I_i|$$

PROOF. (1) *Sufficiency*. First consider a G-process in which some G-set $\{x_1, x_2, \cdots\}$ with parameter λ, has probability one. Let I_1, I_2 be disjoint bounded sets. Since the x_i are not random, the $x_i + Y_i(t)$ are independent of each other. Define a Bernoulli array $\{X_{tl}\}$ by $X_{tl} = (1, 0)$ if $x_l + Y_l(t) \varepsilon I_1$, $= (0, 1)$ if $x_l + Y_l(t) \varepsilon I_2$, $= (0, 0)$ otherwise.

Now since the motions satisfy (A) we have

$$
\begin{aligned}
\sup_{i,l} P\{X_{tl}^i > 0\} &\leqq \sup_l P\{X_{tl}^1 > 0\} + \sup_l P\{X_{tl}^2 > 0\} \\
&= \sup_l F_t(I - x_l) + \sup_l F_t(I_2 - x_l) \qquad (6.1)\\
&\leqq \sup_{x \varepsilon R^2} F_t(I_1 - x) + \sup_{x \varepsilon R^2} F_t(I_2 - x) \\
&\to 0, \qquad t \to \infty.
\end{aligned}
$$

Hence $\{X_{tl}\}$ is a null array. Furthermore, as the $\{x_i\}$ are a G-set we have by (B)

$$(6.2) \qquad \textstyle\sum_l P\{X_{tl}^i = 1\} = \sum_l F_t(I_i - x_l) \to \lambda |I_i|, \qquad t \to \infty.$$

Thus since the conditions of the fundamental lemma are satisfied we have

$$(6.3) \qquad \text{distribution of } \textstyle\sum_l X_{tl} = ([N_y(I_1), N_t(I_2)]) \to P(\lambda|I_1|, \lambda|I_2|).$$

The sufficiency is thus established for G-sets. If we now have any G-process with parameter variable μ and parameter distribution $F(\lambda)$, we condition on $\mu = \lambda$ and integrate the conditional distribution with respect to $F(\lambda)$ making use of the limit for G-sets just proved. Thus N_t will converge to a mixed Poisson process with mixture distribution $F(\lambda)$.

(2) *Necessity.* Assume that for every G-process and a given set of motions $\{Y_n(t)\}$ which satisfy (A), the limiting process exists and is a mixed Poisson process, with appropriate mixture distribution. In particular this is true for G-processes in which one G-set has probability one. Looking at the proof for sufficiency of our theorem, notice that (6.1) holds because the $\{Y_n(t)\}$ satisfy (A). However our lemma then tells us that (6.2) holds only if (6.3) does, i.e., our motions satisfy (B) if we have convergence to the Poisson process. Q.E.D.

The application to the case of stationary processes is clear.

A slight rewording of the results of Breiman [2] shows that a set of motions of the form $Y_n(t) = v_n t$, where the random velocities v_n have absolute values chosen from a distribution with an a.e. continuous density, which is bounded on bounded sets and whose direction with respect to the origin is uniformly distributed between 0 and 2π, satisfies conditions (A) and (B).

THEOREM 6.3. *A* nasc *that a process N such that $EN(I)^{1+\delta} < \infty$ for some rectangle I and some $\delta > 0$ be invariant under all sets of motions is that it be a mixed Poisson Process.*

Sufficiency. We have seen this follows from THEOREM 6.1.

Necessity. Since N is invariant under all sets of motions it is invariant under $Y_n(t) = t$ which means N is stationary. Thus if we take a set of motions satisfying (A) and (B) of Theorem 7.2 and apply the theorem to N, the limiting process N_∞ exists and is mixed Poisson. By the invariance each N_t has the same distributions and therefore this must be true for N_∞. Hence N is a mixed Poisson process. Q.E.D.

7. Deleting particles. We now study the operation of deleting particles from a point process. That is, given a point process N and a number p, $0 \leqq p \leqq 1$, we construct a new point process N_p as follows: a particle of N appears in N_p with probability p and does not appear with probability $1 - p$, with the appearance of distinct particles of N in N_p being independent of each other. For R^1 see [1].

THEOREM 7.1. *If N is a G-process with parameter distribution $F(\lambda)$ and p_k a sequence of numbers, $0 \leqq p_k \leqq 1$, such that $p_k \to 0$, then N_{p_k} converges to a mixed Poisson process with mixture distribution $F(\lambda)$ in the following sense:*

For any collection of bounded disjoint rectangles $I_1, \cdots, I_n$

$$(7.1)\quad P\{N_{p_k}(I_1/p_k) = l_1, \cdots, N_{p_k}(I_n/p_k) = l_n\} \to \int_0^\infty \prod_{i=1}^n ((\lambda|I_i|)^{l_i}/l_i!)e^{-\lambda|I_i|}\, dF(\lambda)$$

where $I/p = \{x/p \mid x \varepsilon I\}$.

PROOF. For notational convenience we prove the theorem for $n = 2$. We first consider a G-process in which a G-set with parameter λ carries probability one.

Choose an arbitrary numbering $\{y_i\}$ of the G-set. Define the Bernoulli array $\{X_{kl}\}$ by $X_{kl} = (1, 0)$ or $(0, 1)$ if y_l is in the process N_{p_k} and appears in I_1/p_k or I_2/p_k respectively, $= (0, 0)$ otherwise. Then $\sum_l X_{kl} = [N_{p_k}(I_1/p_k), N_{p_k}(I_2/p_k)]$.

Now we show that the conditions of the fundamental lemma hold. First $\sup_{i,l} P\{X_{kl}^i > 0\} = p_k \to 0$ hence $\{X_{kl}\}$ is a null array. Secondly by the definition of G-set

$$(7.2) \quad \sum_l P\{X_{kl}^i = 1\} = EN_{p_k}(I_i/p_k) = p_k N(I_i/p_k) = (N(I_i/p_k)/1/p_k) \to \lambda|I_i|, \quad k \to \infty.$$

As the conditions of the fundamental lemma hold we have

$$(7.3) \quad \lim_{k\to\infty} P\{N_{p_k}(I_1/p_k) = l_1, N_{p_k}(I_2/p_k) = l_2\} = \prod_{i=1}^2 e^{-\lambda|I_i|}(\lambda|I_i|)^{l_i}/l_i!$$

Just a remark about (7.3). Since we don't necessarily have $(I_i/p_k) \subset (I_i/p_{k+1})$ we don't have a sequence of rectangles expanding about a point; however the existence of the limit of $(N(I_i/p_k)/1/p_k)$ follows directly from the definition of a G-set.

Now consider a general G-process with parameter variable μ and parameter distribution $F(\lambda)$. Conditioning on $X = \lambda$, integrating this conditional distribution with respect to $F(\lambda)$ and applying the above result we arrive at our theorem. Q.E.D.

APPENDIX

We give here an example due to L. Shepp of a point process such that the number of points in any interval is Poisson distributed with mean λ, but disjoint intervals are not independent. We restrict ourself to a point process S on the interval (0, 1) in R^1. Generalizations are clear. We assume known that a Poisson Process in (0, 1) can be constructed by choosing the number of points according to a Poisson distribution and distributing them independent of each other and uniformly on (0, 1) (see [4]).

Fix (the rate) $\lambda > 0$. Choose n with prob. $e^{-\lambda}\lambda^n/n!$, $n = 0, 1, \cdots$ and let $F_n(x_1, \cdots, x_n) = x_1x_2 \cdots x_n$ for $n \neq 3$ be the cumulative dist. fun. of the n points $t_1, \cdots, t_n$ of S. If n happens to be 3 take

$$(1) \quad F_3(x_1x_2x_3) = x_1x_2x_3 + \epsilon(x_1 - x_2)^2(x_1 - x_3)^2 \cdot(x_2 - x_3)^2 x_1x_2x_3(1 - x_1)(1 - x_2)(1 - x_3).$$

For sufficiently small $\epsilon > 0$, F_3 is a dist. function. Note S is not Poisson distributed. Define

$$(2) \quad \begin{aligned} G_n(a, b, m) &= P_n\{\text{exactly } m \text{ of } t_1, \cdots, t_n \in (a, b)\} \\ &= \tbinom{n}{m} P_n\{t_1, \cdots, t_m \in (a, b)\} \text{ and } t_{m+1}, \cdots, t_n \notin (a, b) \\ &= \tbinom{n}{m} E_n \prod_{j=1}^m (X_b(t_j) - X_a(t_j)) \cdot \prod_{j=m+1}^n (X_a(t_j) + X_1(t_j) - X_b(t_j)) \end{aligned}$$

where

$$X_a(t) = 1 \qquad \text{if} \quad t < a$$
$$= 0 \qquad \text{if} \quad t > a.$$

Note $EX_{a_1}(t_1) \cdots X_{a_n}(t_n) = F(a_1, \cdots, a_n)$. In the expansion of (2) only terms of the form $F(a_1, \cdots, a_n)$ appear where $a_i = a, b$, or 1, for all i. Thus if

$$F(a_1, \cdots, a_n) = a_1 \cdots a_n \tag{3}$$

for all such $a_1, \cdots, a_n$ then $G_n(a, b, m)$ will be just as in the Poisson case. For $n \neq 3$ this is by the choice of F_n as uniform. For $n = 3$, (3) follows from (1). We are done.

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