GENERALIZED BAYES DECISION FUNCTIONS, ADMISSIBILITY AND THE EXPONENTIAL FAMILY[1]

By M. Stone

University of Wisconsin and University of Durham, England

1. Introduction and summary. For the experiment $\mathcal{E}$, given by $\mathcal{E} = [x$ (in space X) distributed with probability element $f(x \mid \theta)\, d\mu(x)$, $\{\theta\} = \Theta$ the parameter space, $W(d, \theta)$ a non-negative loss function for decision $d \varepsilon D]$, the definition of a (non-randomized) *strict Bayes decision function* (BDF) can, following Wald [7], be stated: δ_π, *given by the decision function* $d_\pi(x)$, *is a strict* BDF *with respect to the proper prior probability measure* π *on* Θ *if* δ_π *minimizes*

$$R(\pi, \delta) = \int d\pi(\theta) r(\theta, \delta) = \int d\pi(\theta) \int d\mu(x) W[d(x), \theta] f(x \mid \theta)$$

[requiring, of course, that $R(\pi, \delta_\pi) < \infty$]. (As is well known, randomized decision functions can be excluded from standard Bayesian methods. We will not here attempt to justify their exclusion in the below deviations from the standard Bayesian formulations.)

Relaxing the restriction that the prior measure be proper, we have:

Definition 1.1. δ_m is a normal generalized BDF (NGBDF) with respect to the generalized prior m on Θ if δ_m minimizes $R(m, \delta)$ [requiring, of course, that $R(m, \delta_m) < \infty$].

Definition 1.2. δ_m is an extensive generalized BDF (EGBDF) with respect to m if $\delta_m = \{d_m(x)\}$ where $d_m(x)$ minimizes $\int dm(\theta) W(d, \theta) f(x) \mid \theta)$ a.e. μ.

(The use of the epithets "extensive" and "normal" is consistent with their use in Raiffa and Schlaifer [5]. Definition 1.2 differs from the definition of Sacks [6] in not requiring the finiteness of $\int dm(\theta) f(x \mid \theta)$.) Theorem 2.1 shows that if δ_m is an NGBDF then it is also an EGBDF. For some cases in which $\min_\delta R(m, \delta) = \infty$, the following generalization of EGBDF is useful:

Definition 1.3. δ_m^* is a comparative generalized BDF (CGBDF) with respect to m if the quantity $\Delta_m(\delta, \delta_m^*)$ defined by

$$\Delta_m(\delta, \delta_m^*) = \int dm(\theta)[r(\theta, \delta) - r(\theta, \delta_m^*)]$$

is non-negative for all δ.

Admissibility considerations are unaffected by multiplying $W(d, \theta)$ by an arbitrary positive function of θ. So, since the above definitions involve m and W in the composite element $W(d, \theta)\, dm(\theta)$, it is clear that any general sufficient condition for admissibility of δ_m or δ_m^* for m proper may also be stated for m general (that is, possibly improper). (The same point is made by Stein [4], p. 232.) Theorem 3.1 gives such a sufficient condition, suggested by the Lehmann-Blyth

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technique for proving admissibility [1], while Corollary 3.1 is an extension of the well known admissibility of strict BDF's under certain conditions.

The above ideas are applied in Section 4 to the estimation with quadratic loss of the mean of the one dimensional exponential family. The very close links with Karlin's technique [3] are immediately apparent. The application clarifies Karlin's remark (p. 411 of [3]) that his results may be regarded as a refinement of the Lehmann-Blyth technique and also, finally, lends some support to his conjecture (p. 415 of [3]) that a certain condition for admissibility of the contracted estimator γx, $0 < \gamma \leqq 1$, is necessary as well as sufficient. An associated reference is Cheng Ping [2].

2. EGBDF, NGBDF and CGBDF connections. If δ_m is an NGBDF with respect to m, it is clearly a CGBDF with respect to m. In addition:

THEOREM 2.1. *If δ_m is an* NGBDF *with respect to m then it is an* EGBDF *with respect to m.*

PROOF. Suppose δ_m is not an EGBDF. Then there exists a $\delta = \{d(x)\}$ such that

$$\int d\mu(x) \int dm(\theta) W[d(x), \theta] f(x \mid \theta) < \int d\mu(x) \int dm(\theta) W[d_m(x), \theta] f(x \mid \theta)$$

or, since, by Tonelli, non-negativity of W allows inversion of integrations, $R(m, \delta) < R(m, \delta_m)$. Whence, since $R(m, \delta_m) < \infty$, there is a contradiction. So δ_m is an EGBDF.

EXAMPLE 2.1. For $\mathcal{E} = [x$ is binomial (n, θ), $\Theta = [0, 1] = D$, $W(d, \theta) = (d - \theta)^2]$, $\delta_m = \{d_m(x) \equiv x/n\}$ is both an NGBDF and an EGBDF with respect to m given by $dm(\theta) = d\theta/\theta(1 - \theta)$.

EXAMPLE 2.2. For $\mathcal{E} = [x$ is $N(\theta, 1)$, $\Theta = R^1 = D$, $W(d, \theta) = (d - \theta)^2]$, $\delta_m = \{d_m(x) \equiv x\}$ is an EGBDF with respect to m given by $dm(\theta) = d\theta$ but no NGBDF with respect to the same m is definable.

The following examples concern CGBDF's.

EXAMPLE 2.3. In Example 2.2, δ_m is a CGBDF. This result is a special case of Theorem 4.1 below.

EXAMPLE 2.4. Suppose $\mathcal{E}$ has $X = \{x_1, x_2, \cdots\}$, $\Theta = \{\theta_1, \theta_2, \cdots\}$, $D = \{d_1, d_2\}$. Defining

$$\Delta_{ij} = (i + 1)^{-1}(i/(i + 1))^{(j+1)} - (i + 2)^{-1}((i + 1)/(i + 2))^{j+1}$$

if j is odd and $i > j$ *or* if j is even and $i \leqq j$, $\Delta_{ij} = 0$ otherwise, set

$$dm(\theta_j) = 1, \qquad j = 1, 2, \cdots,$$

$$W(d_1, \theta_j) - W(d_2, \theta_j) = \textstyle\sum_i \Delta_{ij},$$

$$f(x_i \mid \theta_j)\, d\mu(x_i) = \Delta_{ij}/\textstyle\sum_i \Delta_{ij}.$$

(It may be proved that $\Delta_{ij}/\sum_i \Delta_{ij} \geqq 0$.) A lengthy analysis shows (a) $\sum_j \Delta_{ij} < 0$, $i = 1, 2, \cdots$, so that the EGBDF is $\delta_m = \{d_m(x_i) \equiv d_1\}$, and (b) $\sum_j (\sum_i \Delta_{ij}) > 0$, so that δ_m cannot also be a CGBDF for the example.

3. Admissibility.

THEOREM 3.1. *Suppose* Θ *has a topology* $\mathfrak{J}$ *of open sets. If*

(i) *given a decision function* δ_0, *the risk function* $r(\theta, \delta)$ *of any* δ *better than or equivalent to* δ_0 *is continuous with respect to* $\mathfrak{J}$;

(ii) *there exists a sequence of generalized priors* $\{m_i\}$ *such that* (a)

$$\liminf_{i\to\infty} m_i(T_0) > 0$$

for all non-null open sets T_0 *in* $\mathfrak{J}$, (b) $\lim_{i\to\infty} \Delta_{m_i}(\delta_0, \delta_i) = 0$ *where* δ_i *is a* CGBDF *with respect to* m_i *then* δ_0 *is admissible.*

PROOF. Suppose δ_0 is inadmissible. Then there exists a δ^+ such that $r(\theta, \delta^+) \leqq r(\theta, \delta_0)$, $\theta \varepsilon \Theta$, and $r(\theta_0, \delta^+) < r(\theta_0, \delta_0)$ for some θ_0. But, by assumption, $r(\theta, \delta^+)$ and $r(\theta, \delta_0)$ are continuous. So there is a neighbourhood of θ_0, say T_0, with $r(\theta, \delta_0) > r(\theta, \delta^+) + \epsilon$, $\epsilon > 0$, $\theta \varepsilon T_0$. Whence $\Delta_{m_i}(\delta_0, \delta^+) \geqq \epsilon m_i(T_0)$, $i = 1, 2, \cdots$. But

$$0 \leqq \Delta_{m_i}(\delta^+, \delta_i) = \Delta_{m_i}(\delta_0, \delta_i) - \Delta_{m_i}(\delta_0, \delta^+). \tag{3.1}$$

As $i \to \infty$, the lim inf of the right-hand-side of (3.1) is negative. The contradiction proves δ_0 admissible.

COROLLARY 3.1. *If* (i) δ_m *is a* CGBDF *with respect to* m, (ii) *the* $r(\theta, \delta)$ *of any* δ *better than or equivalent to* δ_0 *is continuous with respect to* $\mathfrak{J}$, (iii) $m(T_0) > 0$ *for all open non-null sets* T_0 *in* $\mathfrak{J}$ *then* δ_m *is admissible.*

PROOF. In Theorem 3.1, set $m_i = m$ $(i = 1, 2, \cdots,)$ and $\delta_0 = \delta_m$.

4. The exponential family. Our specialization is given by:

$$f(x \mid \theta) = \beta(\omega) \exp(\omega x);$$

$$\theta = \theta(\omega) = E(x \mid \omega);$$

$$\Theta = \{\theta \mid \beta(\omega)^{-1} = \textstyle\int \exp(\omega x)\, d\mu(x) < \infty, E(x^2 \mid \omega) < \infty\};$$

$$\Omega = \{\omega \mid \theta(\omega) \varepsilon \Theta\};$$

$$\underline{\omega} = \liminf \{\omega \mid \omega \varepsilon \Omega\};$$

$$\bar{\omega} = \limsup \{\omega \mid \omega \varepsilon \Omega\}.$$

Then $\Omega - (\underline{\omega}, \bar{\omega})$ is at most $\{\{\underline{\omega}\}, \{\bar{\omega}\}\}$, For $\omega \varepsilon (\underline{\omega}, \bar{\omega})$, $\theta(\omega) = -\beta'(\omega)/\beta(\omega)$. Θ is also an interval (which is given the relative usual topology). Take $W(d, \theta) = (d - \theta)^2$.

LEMMA 4.1. *If* $r(\theta, \delta) < \infty$ *for all* $\theta \varepsilon \Theta$, $r(\theta, \delta)$ *is continuous in* θ.

PROOF. Now $r(\theta, \delta) = E[d(x)^2 \mid \theta] - 2\theta E[d(x) \mid \theta] + \theta^2$. Observe that for $\omega_0 = \omega(\theta_0)$ and $\omega = \omega(\theta)$ in Ω,

$$\begin{aligned} |E[d(x)^2 \mid \theta] - E[d(x)^2 \mid \theta_0]| & \\ \leqq \textstyle\int_{-\infty}^{0} \beta(\omega_0)\, d(x)^2 e^{\omega_0 x}\, &|\beta(\omega_0)^{-1}\beta(\omega) e^{(\omega-\omega_0)x} - 1|\, d\mu(x) \\ + \textstyle\int_0^\infty \beta(\omega)\, d(x)^2 e^{\omega x}\, &|1 - \beta(\omega)^{-1}\beta(\omega_0) e^{-(\omega-\omega_0)x}|\, d\mu(x). \end{aligned}$$

Since $\beta(\omega)$ is continuous in Ω,

$$|\beta(\omega_0)^{-1}\beta(\omega)e^{(\omega-\omega_0)x} - 1| \quad \text{and} \quad |1 - \beta(\omega)^{-1}\beta(\omega_0)e^{-(\omega-\omega_0)x}|$$

are, in $(-\infty, 0)$ and $(0, \infty)$ (respectively), bounded functions of x tending to zero as $\omega \searrow \omega_0$. So, since $\theta(\omega)$ is continuous and monotone increasing and $\int \beta(\omega_0)\, d^2(x)e^{\omega_0 x}\, d\mu(x) < \infty$ by assumption, we have $E[d(x)^2 \mid \theta] \to E[d(x)^2 \mid \theta_0]$ as $\theta \searrow \theta_0$. Similarly as $\theta \nearrow \theta_0$. So $E[d(x)^2 \mid \theta]$ is continuous in θ. An almost similar argument establishes the continuity of $E[d(x) \mid \theta]$ and the lemma follows.

Karlin's sufficient condition [3], K_λ, say, for the admissibility of the estimator $x/(\lambda + 1)$ of θ is that $\int_a^b \beta^{-\lambda}(\omega)\, d\omega \to \infty$ as $a \to \underline{\omega}$ and as $b \to \bar{\omega}$.

THEOREM 4.1. *If K_λ holds, $\delta_\lambda = \{d(x) \equiv x/(\lambda + 1)\}$ is a* CGBDF *with respect to m_λ given by $dm_\lambda = \beta(\omega)^\lambda\, d\omega$.*

PROOF. Writing $\gamma = 1/(\lambda + 1)$,

$$r(\theta, \delta) - r(\theta, \delta_\lambda) = E\{[d(x) - \theta]^2 \mid \theta\} - E\{(\gamma x - \theta)^2 \mid \theta\}$$
$$= T(\omega) + 2E\{[d(x) - \gamma x][\gamma x - \theta] \mid \theta\}$$

where $T(\omega) = E\{[d(x) - \gamma x]^2 \mid \theta\}$. If $\Delta_{m_\lambda}(\delta, \delta_\lambda) = \infty$, $\Delta_{m_\lambda}(\delta, \delta_\lambda) > 0$ automatically. While $\Delta_{m_\lambda}(\delta, \delta_\lambda) < \infty \Rightarrow \int_a^b d\omega\beta(\omega)^\lambda[r(\theta, \delta) - r(\theta, \delta_\lambda)] < \infty$ for arbitrary $a > \underline{\omega}$, $b < \bar{\omega}$, whence by the continuity of $\beta(\omega)$ and $r(\theta, \delta_\lambda)$ in Ω, $\int_a^b d\omega\beta(\omega)^\lambda r(\theta, \delta_\lambda) < \infty$ and $\int_a^b d\omega\beta(\omega)^\lambda r(\theta, \delta) < \infty$; which justify the inversions of order of integration that give

$$(4.1) \quad \int_a^b d\omega\beta(\omega)^\lambda[r(\theta, \delta) - r(\theta, \delta_\lambda)]$$
$$= \int_a^b d\omega\beta(\omega)^\lambda T(\omega) + 2\int d\mu(x)[d(x) - \gamma x]\int_a^b d\omega\beta(\omega)^{\lambda+1}e^{x\omega}(\gamma x - \theta).$$

Reference to pp. 413–414 of [3] shows that, if K_λ holds, the right hand side of (4.1) is non-negative, that is, letting $a \to \underline{\omega}$, $b \to \bar{\omega}$, $\Delta_{m_\lambda}(\delta, \delta_\lambda) \geqq 0$ if K_λ holds.

Define $K_{\lambda,\alpha} =$ '$\int_a^b \beta(\omega)^{-\lambda}e^{-\alpha\omega}\, d\omega \to \infty$ as $a \to \underline{\omega}$ and as $b \to \bar{\omega}$.'

THEOREM 4.2. *If $K_{\lambda,\alpha}$ holds, $\delta_{\lambda,\alpha} = \{d(x) \equiv (x + \alpha)/(\lambda + 1)\}$ is a* CGBDF *with respect to $m_{\lambda,\alpha}$ given by $dm_{\lambda,\alpha} = \beta(\omega)^\lambda e^{\alpha\omega}\, d\omega$.*

PROOF. It is readily verified that

$$(4.2) \quad \Delta_{m_{\lambda,\alpha}}(\delta, \delta_{\lambda,\alpha})$$
$$= \int d\omega\beta_y(\omega)^\lambda \int d\mu_y(y)\beta_y(\omega)e^{\omega y}\{[d(y) - \phi]^2 - [y/(\lambda + 1) - \phi]^2\}$$

where $y = x - \alpha/\lambda$, $\phi = \theta - \alpha/\lambda = E(y \mid \omega)$, $d\mu_y(y) = d\mu(y + \alpha/\lambda)$, $d(y) = d(y + \alpha/\lambda) - \alpha/\lambda$ and $\beta_y(\omega) = \beta(\omega)\exp(\alpha\omega/\lambda)$. Theorem 4.1 then applies to show that the right hand side of (4.2) is non-negative if K_λ holds with $\beta_y(\omega)$ for $\beta(\omega)$, that is, if $K_{\lambda,\alpha}$ holds. So $\Delta_{m_{\lambda,\alpha}}(\delta, \delta_{\lambda,\alpha}) \geqq 0$ if $K_{\lambda,\alpha}$, proving the theorem.

THEOREM 4.3. *If $K_{\lambda,\alpha}$ holds then $\delta_{\lambda,\alpha}$ is admissible.*

PROOF. $r(\theta, \delta_{\alpha,\lambda})$ is readily verified continuous in Ω. Theorem 4.2, Lemma 4.1 and Corollary 3.1 then give the result.

[Theorem 4.3 is also proved in Cheng Ping [2] and is a simple extension of Karlin's Theorem 1. We reprove it to complete and illustrate our approach.]

The possibility of finding a counter example to Karlin's conjecture that K_λ is necessary for the admissibility of δ_λ, involving for some μ a sequence of α values tending to zero for which

(i) $K_{\lambda,\alpha}$ holds but K_λ does not,

(ii) $\lim_{\alpha\to 0} \Delta_{m_{\lambda,\alpha}}(\delta_\lambda, \delta_{\lambda,\alpha}) = 0$

is dismissed by Theorem 4.4 below. [The particular choice $\mu = \{d\mu(x) = x^2 \exp(-1/x)\,dx, 0 < x < 1; d\mu(x) = 0$ otherwise$\}$ does in fact obey $K_{\lambda,\alpha}$ for $\lambda > 0$ and $0 < \alpha < \frac{1}{2}\lambda$ but not K_λ.]

THEOREM 4.4. *If there is a sequence* $\{\alpha_i\}$ *of* α-*values tending to zero, such that* K_{λ,α_i} *holds for* $i = 1, 2, \cdots$ *and* $\lim_{\alpha_i\to 0} \Delta_{m_{\lambda,\alpha_i}}(\delta_\lambda, \delta_{\lambda,\alpha_i}) = 0$ *then* K_λ *holds.*

PROOF. It is verifiable, using integration by parts that '$\Delta_{m_{\lambda,\alpha_i}}(\delta_\lambda, \delta_{\lambda,\alpha_i}) \to 0$' $\Rightarrow$ '$\alpha_i^2 \int d\omega\beta(\omega)^\lambda e^{\alpha_i\omega} \to 0$.'

By Schwarz

$$[\textstyle\int_a^b \beta(\omega)^{-\lambda}\,d\omega][\alpha_i^2 \int_a^b \beta(\omega)^\lambda e^{\alpha\omega}\,d\omega] \geqq [\alpha_i \int_a^b e^{\frac{1}{2}\alpha_i\omega}\,d\omega]^2 = 4[e^{\frac{1}{2}\alpha_i b} - e^{\frac{1}{2}\alpha_i a}]^2. \tag{4.3}$$

If $\bar{\omega} < \infty$ then, for any i, $K_{\lambda,\alpha_i} \Rightarrow \int_a^b \beta(\omega)^{-\lambda}\,d\omega \to \infty$ as $b \to \bar{\omega}$. If $\bar{\omega} = \infty$ then the choice of $b = |\alpha_i|^{-1}$ and fixed a in (4.3) establishes that $\int_a^b \beta(\omega)^{-\lambda}\,d\omega \to \infty$ as $b \to \bar{\omega}$. Similarly $\int_a^b \beta(\omega)^{-\lambda}\,d\omega \to \infty$ as $a \to \underline{\omega}$. Hence K_λ holds.

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REFERENCES

[1] BLYTH, C. R. (1951). On minimax statistical decision procedures and their admissibility. *Ann. Math. Statist.* **22** 22–42.

[2] CHENG, PING (1964). Minimax estimates of parameters of distributions belonging to the exponential family. *Chinese Math.* **5** 277–299.

[3] KARLIN, S. (1958). Admissibility for estimation with quadratic loss. *Ann. Math. Statist.* **29** 406–436.

[4] LECAM, LUCIEN M. and NEYMAN, JERZEY (1965). *Bernoulli 1713, Baves 1763, Laplace 1813.* Proceedings of an International Research Seminar at Statistical Laboratory, Univ. of California. Springer-Verlag, Berlin.

[5] RAIFFA, HOWARD and SCHLAIFER, ROBERT (1961). *Applied Statistical Decision Theory.* Harvard Business School, Boston.

[6] SACKS, J. (1963). Generalized Bayes solutions in estimation problems. *Ann. Math. Statist.* **34** 751–768.

[7] WALD, A. (1950). *Statistical Decision Functions.* Wiley, New York.