AN OSCILLATING SEMIGROUP[1]

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1. Introduction. Let I be a countably infinite set. For each $t \geqq 0$, let $P(t) = \{P(t, i, j)\}$ be a stochastic matrix on I, such that $P(t + s) = P(t)P(s)$, $P(0)$ is the identity matrix, and $P(t) \to P(0)$ coordinatewise at $t \to 0$. Then P is called a standard stochastic semigroup on I. As is well known, P has a coordinatewise derivative Q at 0. However, many elements of Q may vanish. In view of this, L. E. Dubins asked me whether $P(t, i, j)/P(t, i, k)$ converged as $t \to 0$. The object of this note is to provide a counterexample.

(1) Theorem. *There is a countable set I, with elements* 0, 1, 2 *and a standard stochastic semigroup P on I, satisfying*

$$\limsup_{t \to 0} P(t, 0, 1)/P(t, 0, 2) = \infty \tag{2}$$

$$\liminf_{t \to 0} P(t, 0, 1)/P(t, 0, 2) = 0.$$

Moreover, there is a Markov chain with stationary transitions P, starting from 0, *all of whose sample functions are step functions. In particular, all elements of Q are finite.*

The construction is given in Section 2, two preliminary facts in Section 3, and the verification in Section 4. Section 5 contains some technical remarks.

2. Construction. The state space I consists of 0, 1, 2, $(1, n, m)$ and $(2, n, m)$ for positive integer n and $m = 1, \cdots, f(n)$. Here $f(n)$ is positive integer to be chosen later. Think of it as large.

Let $0 < q_{n,m} < \infty$, and let

$$c_n = \sum_{m=1}^{f(n)} q_{n,m}^{-1}. \tag{3}$$

The $q_{n,m}$ will be chosen later. Think of them as very large. Let $a_n > 0$, $b_n > 0$, $\sum_{n=1}^{\infty} (a_n + b_n) = 1$,

$$a_{n+1} + a_{n+2} + \cdots = o(a_n), \tag{4}$$

$$b_{n+1} + b_{n+2} + \cdots = o(b_n),$$

$$\limsup_{n \to \infty} a_n/b_n = \infty, \qquad \liminf_{n \to \infty} a_n/b_n = 0. \tag{5}$$

Let τ_0 be exponential with parameter 1, and $\tau_{n,m}$ exponential with parameter $q_{n,m}$, for $n = 1, 2, \cdots, m = 1, \cdots, f(n)$, all independent. In particular,

$$P(\tau_{n,m} \geqq t) = e^{-q_{n,m} t} \quad \text{for} \quad t \geqq 0.$$

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Here is a stochastic process. The process starts in 0, stays there time τ_0, then jumps to $(1, n, 1)$ with probability a_n and to $(2, n, 1)$ with probability b_n, for any $n = 1, 2, \cdots$. Having reached (i, n, m) for $i = 1$ or 2, the process stays there time $\tau_{n,m}$, then jumps to $(i, n, m + 1)$, unless $m = f(n)$, in which case the process jumps to i. Having reached i, the process stays there. As is easily verified, this process is a Markov chain with stationary standard transition, call them P.

(It is easy to make the chain more attractive. Let it stay in 1 or 2 for an independent, exponential time, then return to 0. This complicates the argument, but not much. For some of the details in this modification, see Section 5.)

3. Preliminaries. The elementary proofs of (6) and (8) are omitted.

(6) FACT. $P(\tau_0 + \tau_{n,1} \leqq t) = o(t)$ as $t \to 0$.

Recall (3) and let

$$\sigma_n = \sum_{m=1}^{f(n)} \tau_{n,m}. \tag{7}$$

(8) FACT. σ_n has mean c_n and variance $\sum_{m=1}^{f(n)} 1/q_{n,m}^2$.

4. Verification. The program is to define $q_{n,\cdot}$ inductively on n, and with them a sequence $t_n \downarrow 0$ such that

$$P(t_n, 0, 1) \approx \tfrac{1}{2}a_n t_n \tag{9}$$

and

$$P(t_n, 0, 2) \approx \tfrac{1}{2}b_n t_n. \tag{10}$$

As usual, $x_n \approx y_n$ means $x_n/y_n \to 1$. In view of (5), this establishes (1).

Fix a sequence $\epsilon_n > 0$ with

$$\epsilon_n = o(a_n) \quad \text{and} \quad \epsilon_n = o(b_n). \tag{11}$$

Let $f(1) = 1$, $t_1 = 1$, $q_{1,1} = \frac{1}{2}$. Suppose $f(n)$, t_n, $q_{n,\cdot}$ Chosen for all $n < k$. Recall (3) and (7). By (6), $P(\tau_0 + \sigma_n < t) = o(t)$ for all $n < k$. Choose t_k with $0 < t_k < \frac{1}{2}t_{k-1}$ and

$$P(\tau_0 + \sigma_n < t_k) \leqq \epsilon_k t_k \quad \text{for all} \quad n < k. \tag{12}$$

Let $c_k = \frac{1}{2}t_k$. Now choose $f(k)$ and $q_{k,\cdot}$ so $c_k = \sum_{m=1}^{f(k)} 1/q_{k,m}$, while $\sum_{m=1}^{f(k)} 1/q_{k,m}^2$ is so small that

$$P(\tau_0 + \sigma_k < t_n)/P(\tau_0 + c_k < t_n) \to 1 \quad \text{uniformly in} \quad n \leq k \quad \text{as} \quad k \to \infty. \tag{13}$$

Use (8) to see that (13) can be done.

Now (9) and (10) follow, but only (9) will be argued. To estimate $P(t_n, 0, 1)$, let P_0 be the distribution of the chain starting from 0, and let $A(k, t)$ be the event that, on leaving 0, the chain jumps to $(1, k, 1)$, and is in 1 at time t. Plainly, $P(t, 0, 1) = \sum_{k=1}^{\infty} P_0[A(k, t)]$, and $P_0[A(k, t)] = a_k P[\tau_0 + \sigma_k < t]$. By (13), $P_0[A(n, t_n)] \approx a_n P[\tau_0 + c_n < t_n]$, and $P[\tau_0 + c_n < t_n] = 1 - e^{-(t_n - c_n)} \approx (t_n - c_n) = \frac{1}{2}t_n$. Thus,

$$P_0[A(n, t_n)] \approx \tfrac{1}{2}a_n t_n. \tag{14}$$

Similarly, $\sum_{k=n+1}^{\infty} P_0[A(k, t_n)] \approx \sum_{k=n+1}^{\infty} a_k(t_n - c_k)$. But $t_n - c_k \leqq \frac{1}{2}t_n$, so (4) implies

$$(15) \qquad \sum_{k=n+1}^{\infty} P_0[A(k, t_n)] = o(a_n t_n).$$

Finally, using (12), $\sum_{k=1}^{n-1} P_0[A(k, t_n)] \leqq \sum_{j=1}^{n-1} a_j \epsilon_n t_n \leqq \epsilon_n t_n$. By (11),

$$(16) \qquad \sum_{k=1}^{n-1} P_0[A(k, t_n)] = o(a_n t_n).$$

Combine (14), (15) and (16) to get (9), completing the proof of (1).

5. Technical remarks. Let τ_1 be an exponential random variable, independent of τ_0 and $\{\tau_{n,m}\}$. Modify the chain so that, on first reaching 1 or 2, it stays there time τ_1, then jumps to 0. On returning to 0, the chain restarts afresh. The conclusions and proof of Theorem 1 apply to this chain also, provided that t_k satisfies (17) in addition to (12):

$$(17) \qquad P(\tau_0 + \tau_1 < t_k) \leqq \epsilon_k t_k .$$

Indeed, let $B(k, t)$ be the event that, starting from 0, on leaving 0 the chain jumps to $(1, k, 1)$, and is in 1 at time t, without having returned to 0 on $(0, t)$. Clearly,

$$P_0[B(k, t)] = a_k P(\tau_0 + \sigma_k < t < \tau_0 + \sigma_k + \tau_1);$$

by conditioning on τ_0 and σ_k,

$$a_k P(\tau_0 + \sigma_k < t) P(\tau_1 > t) \leqq P_0[B(k, t)] \leqq a_k P(\tau_0 + \sigma_k < t).$$

Thus, as $t \to 0$, $P_0[B(k, t)] \approx a_k P(\tau_0 + \sigma_k < t)$ uniformly in k. Now, for the chain to be in 1 at time t_n, and to have returned to 0 before time t_n, implies $\tau_0 + \tau_1 < t_n$, an event of probability at most $\epsilon_n t_n$.

For the modified chain, let $\theta_0 = \tau_0$ and let $\theta_1, \theta_2, \cdots$ be the holding times in the first, second, $\cdots$ state visited after 0. Thus, on $B(k, t)$, $\theta_m = \tau_{k,m}$. Given the order of states, the θ's are independent and exponential, so one might expect $P_0(\theta_0 + \theta_1 + \theta_2 \leqq t) = o(t^2)$. Since 1 can be reached from 0 in two jumps but not in one, a similar heuristic argument leads one to expect $P(t, 0, 1) \sim t^2$. Both of these estimates are false. For, letting X_t be the chain at time t, $P(t, 0, 1) = P_0[X_t = 1, \theta_0 + \theta_1 + \theta_2 \leqq t] + P_0[X_t = 1, \theta_0 + \theta_1 + \theta_2 > t]$. The second term is $a_1 P_0[\tau_0 + \tau_{1,1} \leqq t < \tau_0 + \tau_{1,1} + \tau_{1,2}] = \frac{1}{2}a_1 t^2 + o(t^2)$. Similarly for $P(t, 0, 2)$. From (1), $\limsup_{t\to 0} t^{-2} P_0[X_t = 1, \theta_0 + \theta_1 + \theta_2 \leqq t] = \infty$.

REFERENCE

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