A NOTE ON THE BIRKHOFF ERGODIC THEOREM

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We prove the maximal and pointwise ergodic theorem by a method that postpones measure-theoretic concepts to the last. The proof leans heavily on existing proofs (see F. Riesz [3], Kolmogorov [2], Yosida and Kakutani [4] and Gnedenko [1]) but it is simpler and the combinatorial part of the proof has been completely separated from the rather transparent measure-theoretic part.

Let Ω be a set of points, T an invertible transformation from Ω onto itself and $x_0(w)$ any real valued function on Ω. Define $x_i(w) = x_0(T^i(w))$ and $y_{ij} = (x_i + \cdots + x_j)/j - i + 1$; $y_j = y_{0j}$; fix a constant c and consider the sets $E_j = [w: y_k(w) < c, k < j; y_j(w) \geqq c]$ and $G_j = \bigcup_{k=0}^{j} E_k$ (so that $w \varepsilon E_j$ means that the first time the averages $y_k(w)$ reach or exceed c occurs at time $k = j$ and w in G_j means that the averages have reached or exceeded c by time j).

Lemma 1. *The set G_j decomposes into disjoint sets H_{pn}, $p = 0, \cdots, n; n = 0, \cdots, j$, where $y_n \geqq c$ on H_{0n} and $TH_{pn} = H_{p+1.n}$.*

Proof. Let $H_{0j} = E_j$ and $H_{pj} = T^p H_{0j}$, $p = 0, \cdots, j$; then the $\{H_{pj}\}$ are disjoint, since if $j \geqq p > q \geqq 0$ then $y_{-q,-p+j} \geqq c$ on the set H_{pj} while this inequality cannot hold on H_{qj}. Let $H_{0,j-1} = E_{j-1} - \bigcup_{p=0}^{j} H_{pj}$ and $H_{p,j-1} = T^p H_{0,j-1}$; $p = 0, \cdots, j-1$; the $\{H_{p,j-1}\}$ are disjoint from each other as before. If $0 \leqq q < p \leqq j - 1$ and $w \varepsilon H_{qj}$ then $y_{-q,-p+j-1} < c$ while $w \varepsilon H_{p,j-1}$ implies $y_{-q,-p+j-1} \geqq c$ so that $H_{p,j-1} \cap H_{qj} = \varnothing$, on the other hand if $q \geqq p$ then $T^{-p}[H_{p,j-1} \cap H_{qj}] = H_{0,j-1} \cap H_{q-p,j} = \varnothing$, thus $H_{p,j-1}$ is disjoint from the aggregate $\{H_{q,j}\}$. The lemma follows by a finite induction.

Now assume that we have a (not necessarily σ-finite) measure space $(\mu, \mathfrak{F}, \Omega)$:

Theorem 1. *If x_0 is in L_1 and T is also bimeasurable and measure-preserving then*

$$\int_{G_j} x_0 \geqq c\mu G_j \qquad (\textit{maximal ergodic theorem}), \tag{1}$$

$$\lim_n y_n \textit{ exists a.e. } [\mu] \textit{ and is integrable.} \tag{2}$$

Proof. To see (1) we note that $\mu G_j = \sum_{n=0}^{j} (n+1)\mu H_{0n}$ hence

$$\int_{G_j} x_0 = \sum_{n=0}^{j} \sum_{p=0}^{n} \int_{H_{pn}} x_0 = \sum_n \int_{H_{0n}} (x_0 + \cdots + x_n)$$
$$= \sum_n (n+1) \int_{H_{0n}} y_n \geqq c\mu G_j .$$

To prove (2) we observe that if w is in $E = [w; \limsup y_n \geqq c + \epsilon, \liminf y_n \leqq c]$ then TW is also in E hence we may apply (1), and its obvious counterpart with lim sup's replaced by lim inf's, to the space $(\mu, \mathfrak{F} \cap E, E)$ to conclude

$$c\mu E \geqq \int_E x_0 \geqq (c + \epsilon)\mu E.$$

Received 6 November 1964; revised 16 February 1966.

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Thus $\mu E = 0$ and lim y_n exists a.e.; integrability of lim y_n follows from uniform integrability of the x_i's.

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