MULTI-SAMPLE ANALOGUES OF SOME ONE-SAMPLE TESTS

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Preface. The results of Part I and Part III were obtained by the second author (cf. Puri (1962)) and those of Part I and Part II by the first author by following different methods (cf. Mehra (1963)). The authors wish to express their sincere thanks to Professors Erich L. Lehmann, Jaroslav Hájek and Edward Paulson for very helpful suggestions and criticisms.

PART I

I.1. Introduction and summary. Consider K treatments in an experiment which yields paired observations, namely (X_{il}, X_{jl}), $l = 1, \cdots, N_{ij}$; $1 \leqq i < j \leqq K$, obtained by N_{ij} independent paired comparisons for each pair (i, j) of treatments and assume that N_{ij} difference scores $Z_l^{(i,j)} = X_{il} - X_{jl}$, $l = 1, \cdots, N_{ij}$, have a common continuous cdf (cumulative distribution function) $\Pi_{ij}(z)$. This is the situation, for example, if in the analysis of an incomplete blocks experiment with each block of size two, one makes the assumption of additivity in the usual analysis of variance model. Then for testing the hypothesis

$$H_0 : \Pi_{ij}(z) + \Pi_{ij}(-z) = 1 \quad \text{and} \quad \Pi_{ij}(z) = \Pi_{i'j'}(z)$$

for any two pairs (i, j) and (i', j') [which states that each of the distributions Π_{ij} of the differences $Z_{ijl} = X_{il} - X_{jl}$, $l = 1, \cdots, N_{ij}$, is symmetric with respect to the origin, and furthermore all distributions Π_{ij} are identical] some rank tests based on the generalizations of the one-sample Chernoff-Savage-Hájek type tests (cf. [9] and [3]) are proposed, their limiting distributions are derived, and their efficiency properties with respect to one another and some of their competitors, viz. the Bradley-Terry test [1], the Durbin test [6] and the classical F test are studied. (For alternative formulations of the null hypothesis, and the study of the special case of the generalization of the one-sample Wilcoxon test, the reader is referred to [16].)

Let $\{J_{N,k}; k = 1, \cdots, N\}$, be a double sequence of numbers satisfying certain conditions to be stated below (Section 2) and let $R_{N,l}^{(i,j)}$ be the rank of $|Z_l^{(i,j)}|$, when the $N = \sum_{i=1}^{k} \sum_{j>i} N_{ij}$ absolute values of the observed differences $|Z_l^{(i,j)}|$, $l = 1, 2, \cdots, N_{ij}$, $1 \leqq i < j \leqq K$, are arranged in the ascending order of

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magnitude in a *combined ranking*. Define

$$\tau_N^{(i,j)} = \sum_{l=1}^{N_{ij}} J_N(R_{N,l}^{(i,j)}/(N+1))\cdot \text{sign}\, Z_l^{(i,j)}, \tag{1.1}$$

where $J_N(u)$ is a step function defined over $(0, 1)$ taking constant values $J_{N,k}$ over the interval $((k-1)/N, k/N]$, i.e., $J_N(u) = J_{N,k} = J(k/N+1)$ for $k - 1/N < u \leqq k/N$. (Note that $\tau_N^{(i,j)} = -\tau_N^{(j,i)}$); $\tau_N^{(i,j)}$ is also expressible as

$$\tau_N^{(i,j)} = \tau_{ij}^{+} + \tau_{ij}^{-} \tag{1.2}$$

where

$$\tau_{ij}^{+} = \sum_{k=1}^{N} J_{N,k} a_{ij,k}^{*}, \qquad \tau_{ij}^{-} = \sum_{k=1}^{N} J_{N,k} a_{ij,k}^{**},$$

with $a_{ij,k}^{*} = 1(a_{ij,k}^{**} = -1)$ if the kth smallest absolute Z in the combined ranking corresponds to a positive (negative) $Z_l^{(i,j)}$, $l = 1, 2, \cdots, N_{ij}$, otherwise $a_{ij,k}^{*} = 0$ $(a_{ij,k}^{**} = 0)$. Consider now, for testing the hypothesis H_0, the statistics of the form

$$L_N = \sum_{i=1}^{K} \{\sum_{j\neq i} (\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}})\}^2/(N^{-1}\sum_{k=1}^{N} J_{N,k}^2)K, \tag{1.3}$$

with the test consisting in rejecting H_0 at level α if L_N exceeds a predetermined number $c_{N,\alpha}$ where $P_{H_0}[L_N \geqq c_{N,\alpha}] = \alpha$. The limit distributions of these statistics as $N \to \infty$, under H_0 and "contiguous" translation alternatives, are derived in Part I under two sets of sufficient conditions—under (a) the assumptions of Hájek [9] and (b) under those of Chernoff and Savage [3] (Section 2). This enables us to determine (Section 3) the asymptotic (Pitman) efficiency of any two statistics belonging to this class relative to each other and, for that matter, relative to any other competing statistic for which the limit distribution is of the same form e.g., Bradley-Terry statistic, the classical $\mathfrak{F}$-statistic and the class of statistics L_N^* described by (1.4).

It turns out, however, that given any statistic belonging to this family, the statistic constructed in exactly the same manner but with $\tau_N^{(i,j)}$ now based on *separate-rankings* of the absolute Z's for each pair (i, j) $(1 \leqq i < j \leqq K)$ is, in the Pitman sense, as efficient as the given statistic. This latter family of statistics is represented by

$$L_N^* = \sum_{i=1}^{K} \{\sum_{j\neq i} (\tau_N^{*(i,j)}/(K\, d_{N_{ij}}^2)^{\frac{1}{2}})\}^2 \tag{1.4}$$

where $d_{N_{ij}}^2 = \sum_{k=1}^{N_{ij}} J_{N_{ij},k}^2$, and

$$\tau_N^{*(i,j)} = \sum_{l=1}^{N_{ij}} J_{N_{ij}}(R_{N_{ij},l}^{*(i,j)}/(N_{ij}+1))\cdot \text{sign}\, Z_l^{(i,j)},$$

$R_{N_{ij},l}^{*(i,j)}$ being the rank of $|Z_l^{(i,j)}|$ when the N_{ij} absolute values $|Z_l^{(i,j)}|$, $l = 1, 2, \cdots, N_{ij}$, are ranked separately for each pair (i, j) $(1 \leqq i < j \leqq K)$. The form of the hypothesis H_0 suggests that it is the "joint-ranking" procedure which is more appropriate. However, if we apply the Pitman criterion, the question as to which of the two procedures—the *joint-ranking* or the *separate-rankings*—is preferable remains unresolved. This question is partially investigated in Part II by considering the local "asymptotic" efficiency as the number of treatments tends to infinity. The results obtained suggest, that for testing against shift in location, a "joint-ranking" statistic L_N is preferable to its counterpart L_N^* based on "separate rankings" except for alternatives for which the Durbin-statistic is relatively Pitman-efficient than the given statistic L_N. It is also shown that for testing against a specified alternative, the "best" rank-order statistic (in the sense of local power) is the one based on the joint ranking procedure.

Part III contains the proof of the asymptotic joint-normality, as $N \to \infty$, of the variables $\tau_N^{(i,j)}$ $(1 \leqq i < j \leqq K)$ under fixed alternatives from which then one can easily derive the limit distribution of L for "contiguous" translation alternatives.

I.2. Limit distributions. Consider the problem of testing H_0 against the alternatives of shift in location. To investigate the asymptotic efficiency of any L_N or L_N^* (or $\mathfrak{F}$-statistic), we obtain in this section their limit distributions, assuming a sequence K_N (defined below) of translation alternatives which approach H_0, as $N \to \infty$, viz.,

$$(2.1) \quad K_N : \Pi_{ij}(z) = \Pi(z + \mu_{ij}N^{-\frac{1}{2}}) \qquad \text{for each pair} \quad (i, j), (1 \leqq i < j \leqq K),$$

where $\Pi(x)$ is a continuous cdf satisfying the symmetry condition $\Pi(z) + \Pi(-z) = 1$ and μ_{ij} are certain constants, not all zero and satisfying $\mu_{ij} = -\mu_{ji}$. Consider now the following two sets of sufficient conditions:

Hájek conditions:

Ω_1 : Assume the existence of a function $J(u)$ defined over $(0, 1)$ such that

$$\text{(i)} \quad \int_0^1 J^2(u)\,du < \infty, \qquad \text{(ii)} \quad \lim_{N\to\infty} \int_0^1 \{J_N(u) - J(u)\}^2\,du = 0.$$

Ω_2 : $\Pi(z)$ possesses a differentiable density $\pi(z)$ such that the function

$$\psi(u) = -\pi'[\Pi^{-1}((1 + u)/2)]/\pi[\Pi^{-1}((1 + u)/2)], \quad 0 < u < 1,$$

satisfies $\int_0^1 \psi^2(u)\,du < \infty$.

Chernoff-Savage type conditions: We introduce some notation. Let $c = K(K - 1)/2$ denote the number of all possible pairs and label them $\alpha = 1, 2, \cdots, c$. Let m_α, n_α be the number of positive and negative $Z^{(\alpha)}$'s (then m_α, n_α are random but $m_\alpha + n_\alpha = N_\alpha$ is non-random). Let $F^{+(\alpha)}(x)(F^{-(\alpha)}(x))$ stand for the conditional distributions of the $|Z^{(\alpha)}|$ given $Z^{(\alpha)} > 0$ $(Z^{(\alpha)} < 0)$ and $F^+_{m_\alpha}(x)(F^-_{n_\alpha}(x))$ the sample cdf's of the absolute values of the positive (negative) $Z^{(\alpha)}$'s. Further let $\lambda_\alpha = m_\alpha/N$, $\mu_\alpha = n_\alpha/N$.

$$(2.2) \qquad H_N(x) = \sum_{\alpha=1}^c [\lambda_\alpha F^+_{m_\alpha}(x) + \mu_\alpha F^-_{n_\alpha}(x)]$$

and

$$(2.2a) \qquad H(x) = \sum_{\alpha=1}^c [\lambda_\alpha F^{+(\alpha)}(x) + \mu_\alpha F^{-(\alpha)}(x)]$$

and denote by Ω_3 and Ω_4 the conditions

Ω_3 : (i) $J(u) = \lim_{N\to\infty} J_N(u)$ exists for $0 < u < 1$ and is not constant;

$$\text{(ii)} \qquad \int_{I_N} [J_N(H_N(x)) - J(H_N(x))]\,dF^+_{m_\alpha}(x) = o_p(N^{-\frac{1}{2}}),$$

$$\int_{I_N} [J_N(H_N(x)) - J(H_N(x))]\, dF_{n_\alpha}^{-}(x) = o_p(N^{-\frac{1}{2}}),$$

where $I_N = \{x : 0 < H_N(x) < 1\}$.

(iii) $J_N(1) = o(N^{\frac{1}{2}})$

(iv)
$$|J(u)| \leqq t[u(1-u)]^{-\frac{1}{2}+\delta},$$
$$|J^{(i)}(u)| = |d^iJ/du^i| \leqq t[u(1-u)]^{-i+\delta},$$

for $i = 1, 2$, for some t and $\delta > 0$.

Ω_4 : (i) The distribution $\Pi(z)$ admits a unimodal density $\pi(z)$ which is bounded in the neighbourhood of the origin.

(ii) $J'[\Pi(x)]\pi(x)$ is bounded.

Let $\chi_t^2(\delta^2)$ stand for the non-central χ^2-variable with t degrees of freedom and the non-centrality parameter δ^2; and let χ_t^2 stand for the corresponding central χ^2-variable. We now state

THEOREM 2.1. *Assume for each N the truth of K_N and that either* (a) *the conditions* (Ω_1, Ω_2) *or* (b) *the conditions* (Ω_3, Ω_4) *are satisfied. Assume further that* $\rho_{ij} = \lim_{N\to\infty} \{N_{ij}/N\}$ *exists and is positive for each pair* (i, j) $(1 \leqq i < j \leqq K)$. *Then the statistic L_N is distributed in the limit, as $N \to \infty$, as a $\chi_{K-1}^2(\delta^2)$ variable with*

$$\delta^2 = (B/K) \textstyle\sum_{i=1}^K \{\sum_{j\neq i} (\rho_{ij}^{\frac{1}{2}}\mu_{ij})\}^2 \tag{2.3}$$

where

$$B = (\textstyle\int_0^1 J(u)\psi(u)\, du)^2/(\int_0^1 J^2(u)\, du) \tag{2.4}$$

under (Ω_1, Ω_2) *and*

$$B = 16(\textstyle\int_0^\infty J'(2\Pi(x) - 1)\{\pi(x)\}^2\, dx)^2/(\int_0^1 J^2(u)\, du) \tag{2.4a}$$

under (Ω_3, Ω_4).

It is easily verified that when both the conditions (Ω_1, Ω_2) and (Ω_3, Ω_4) are satisfied, the two expressions for B above coincide. This holds for most situations of applicational interest (Section 3).

For the special case when $\mu_{ij} = \theta_i - \theta_j$ where not all θ's are equal and $N_{ij} = n$ for each pair (i, j), the non-centrality parameter (2.3) takes the form

$$\delta^2 = (2B/(K-1)) \textstyle\sum_{i=1}^K (\theta_i - \bar{\theta})^2 \tag{2.5}$$

where $\bar{\theta} = \sum_{i=1}^K \theta_i/K$.

The proof of part (a) is based on the following two lemmas, the first of which is an extension of the main theorem of Hájek, based on the notion of "contiguity." This lemma, which enables us to conclude the joint-normality of the variables $\tau_N^{(i,j)}$ $(1 \leqq i < j \leqq K)$ under (Ω_1, Ω_2), is also needed for the results of Part II. The proof of part (b) is based on the more general Theorem 3.1 of Part III.

The statement of Lemma 2.1 concerns a slightly more general model described below: Let $(Z_{\nu 1} \cdots Z_{\nu N_\nu})$, $1 \leqq \nu < \infty$, be a sequence of random vectors, where $N_\nu \to \infty$ as $\nu \to \infty$ and Z's are independent, and denote by $R_{\nu k}$ the rank of $|Z_{\nu k}|$ as the totality of $|Z|$'s are ranked in ascending order of magnitude. Further, let

$$S_\nu = \sum_{k=1}^{N_\nu} d_{\nu k} J_\nu(R_{\nu k}/(N_\nu + 1)) \cdot \text{sign } Z_{\nu k} \tag{2.6}$$

where $d_{\nu k}$, $1 \leqq k \leqq N_\nu$, are certain constants satisfying

$$\lim_{\nu\to\infty} \{(\max_{1\leqq k\leqq N_\nu} d_{\nu k}^2)/(\sum_{k=1}^{N_\nu} d_{\nu k}^2)\} = 0, \tag{2.7}$$

and assume that

$$P[Z_{\nu k} \leqq z/\beta, \sigma] = \Pi((z - \beta c_{\nu k})/\sigma), \tag{2.8}$$

where $\Pi(z)$ is as defined in (2.1), $-\infty < \beta < \infty$, $\sigma > 0$, are unknown parameters and $c_{\nu k}$ are again certain constants satisfying the condition (2.7) with d's replaced by c's and

$$\sup_\nu (\sum_{k=1}^{N_\nu} c_{\nu k}^2) < \infty. \tag{2.8a}$$

Let $\mathcal{L}(Y_\nu/P_\nu) \to N(a_\nu, b_\nu^2)$ denote that the distribution of $b_\nu^{-1}(Y_\nu - a_\nu)$ converges, as $\nu \to \infty$, to $N(0, 1)$ distribution.

LEMMA 2.1. *Suppose that the conditions* (Ω_1, Ω_2) *are satisfied. Then under the model* (2.8), *the statistic* (2.6) *satisfies* $\mathcal{L}(S_\nu) \to N(\eta_\nu, t_\nu^2)$ *where*

$$\eta_\nu = (\beta/\sigma)(\int_0^1 J(u)\psi(u)\, du) \cdot \sum_{k=1}^{N_\nu} d_{\nu k} c_{\nu k}, \tag{2.9}$$
$$t_\nu^2 = (\int_0^1 J^2(u)\, du) \sum_{k=1}^{N_\nu} d_{\nu k}^2.$$

PROOF. Consider a particular distribution Π and let Q_ν and P_ν stand for the probability distributions under $\beta = \beta_0$, $\sigma = \sigma_0$ and $\beta = 0$, $\sigma = \sigma_0$ respectively. The proof below is simply a reconstruction of certain essential steps in Hájek's proof. Let T_ν denote

$$T_\nu = \sum_{k=1}^{N_\nu} d_{\nu\alpha} J[T(|Y_{\nu k}|)] \text{ sign } Z_{\nu\alpha}$$

where $Y_{\nu k} = (Z_{\nu k} - \beta_0)/\sigma_0$ and $T(x) = 2\Pi(x) - 1$, if $x \geqq 0$, and $T(x) = 0$ otherwise. Then, as in [9], one obtains that if after proper normalization, one of the limit exists,

$$\lim_{\nu\to\infty} \mathcal{L}(S_\nu/Q) = \lim_{\nu\to\infty} \mathcal{L}(T_\nu/Q). \tag{2.10}$$

To apply Lemma 4.2 of [9] we have to show now that $\mathcal{L}(T_\nu, L_\nu/P_\nu)$ converges to some bivariate normal distribution. The equation (2.10) and part (iii) of Lemma 4.2 [9] would then give the result forthwith. For this it suffices, on account of the arguments of Section 7 of Wald-Wolfowitz [24], to prove the asymptotic normality of an arbitrary linear combination of T_ν and L_ν, where L_ν is as defined by (4.16) of [9], viz.,

$$\mu_1 T_\nu + \mu_2 L_\nu. \tag{2.11}$$

From equation (5.21) of [9] we know that $P_\nu - \lim_{\nu\to\infty} \{L_\nu + (\gamma^2 d_\nu^2/2) - \gamma S_\nu^*\} = 0$, where $\gamma = (\beta_0/\sigma_0)$, d_ν^2 is defined by (7.6) of [9], and $S_\nu^* = -\sum_{k=1}^{N_\nu} c_{\nu k}\{\pi'(Y_{\nu k})/\pi(Y_{\nu k})\}$, so that (2.11) is asymptotically equivalent in distribution, after proper normalization, to the statistic

$$(2.12)\qquad \sum_{k=1}^{N_\nu} (r_{1k} + r_{2k}) - \tfrac{1}{2}\mu_2\gamma^2 d_\nu^2$$

where $r_{1k} = \mu_1 d_{\nu k} J\{T(|Y_{\nu k}|)\}\cdot \text{sign } Z_{\nu k}$ and $r_{2k} = -\mu_2 c_{\nu k}\gamma\cdot\{\pi'(Y_{\nu k})/\pi(Y_{\nu k})\}$. It is easy to see that the variance σ_ν^2 of (2.12) is given (since the summands r_{1k} and r_{2k} have zero expectations under P_ν) by

$$\sigma_\nu^2 = \mu_1^2(\textstyle\int_0^1 J^2(u)\,du) \sum_{k=1}^{N_\nu} d_{\nu k}^2 + \mu_2^2\gamma^2(\int_0^1 J^2(u)\,du) \sum_{k=1}^{N_\nu} c_{\nu k}^2 + 2\mu_1\mu_2\gamma(\int_0^1 J(u)\psi(u)\,du) \sum_{k=1}^{N_\nu} c_{\nu k}\, d_{\nu k}.$$

We may assume that σ_ν^2 is bounded away from zero (for otherwise the result trivially holds).

Letting now I_A denote the indicator function of the set A, we have for every $\epsilon > 0$

$$(2.13)\qquad \begin{aligned} \sigma_\nu^{-2} \sum_{k=1}^{N_\nu} E\{I_{[|r_{1k}+r_{2k}|\geqq \epsilon\sigma_\nu]}(r_{1k} + r_{2k})^2\} &\leqq \sigma_\nu^{-2} \sum_{k=1}^{N_\nu} E\{I_{[|r_{1k}|\geqq \frac{1}{2}\epsilon\sigma_\nu]} r_{1k}^2\} \\ &+ \sigma_\nu^{-2} \sum_{k=1}^{N_\nu} E\{I_{[|r_{2k}|\geqq \frac{1}{2}\epsilon\sigma_\nu]} r_{1k}^2\} + \sigma_\nu^{-2} \sum_{k=1}^{N_\nu} E\{I_{[|r_{1k}|\geqq \frac{1}{2}\epsilon\sigma_\nu]} r_{2k}^2\} \\ &+ \sigma_\nu^{-2} \sum_{k=1}^{N_\nu} E\{I_{[|r_{2k}|\geqq \frac{1}{2}\epsilon\sigma_\nu]} r_{2k}^2\} \end{aligned}$$

where each summation on the right of (2.13) converges to zero on account of conditions $\Omega_1(1)$, Ω_2, (2.7) and (2.8). Thus the Lindeberg-Feller condition is satisfied, which establishes the asymptotic normality of (2.11); and the proof is complete.

LEMMA 2.2. *Under the conditions of Theorem* 2.1 *with either* (a) (Ω_1, Ω_2) *or* (b) (Ω_3, Ω_4), *the* $c = K(K-1)/2$ *random variables* $\{\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}}\}$, $(1 \leqq i < j \leqq K)$, *are distributed in the limit, as* $N \to \infty$, *as independent* $N(\eta^{(i,j)}, A^2)$ *variables, where* $A = [\int_0^1 J^2(u)\,du]^{\frac{1}{2}}$ *and*

$$(2.14)\qquad \begin{aligned} \eta^{(i,j)} &= \rho_{ij}^{\frac{1}{2}}\mu_{ij}(\textstyle\int_0^1 J(u)\psi(u)\,du) \quad \textit{under} \quad (\Omega_1, \Omega_2), \\ \eta^{(i,j)} &= 4\rho_{ij}^{\frac{1}{2}}\mu_{ij}(\textstyle\int_0^\infty J'[2\Pi(x) - 1]\pi(x)\,d\Pi(x)) \quad \textit{under} \quad (\Omega_3, \Omega_4). \end{aligned}$$

PROOF. The proof of part (a) of this lemma is based on Lemma 2.1, and that of part (b) is given in Part III. Under a labelling $\alpha = 1, 2, \cdots, c$ of the $c = K(K-1)/2$ pairs (i, j) $(1 \leqq i < j \leqq K)$, the statistic (2.6) can be expressed in the present context as

$$(2.15)\qquad \begin{aligned} S_N &= \sum_{\alpha=1}^{c} \sum_{l=1}^{N_\alpha} d_{N,l}^{(\alpha)} J_N(R_{N,l}^{(\alpha)}/(N+1))\cdot \text{sign } Z_l^{(\alpha)} \\ &= \sum_{i=1}^{K} \sum_{j>i} \{\sum_{l=1}^{N_{ij}} d_{N,l}^{(i,j)} J_N(R_{N,l}^{(i,j)}/(N+1))\cdot \text{sign } Z_l^{(i,j)}\}. \end{aligned}$$

For a given pair (i, j), the statistic $\tau_N^{(i,j)}/N^{\frac{1}{2}}$ is obtained from (2.15) by setting $d_{N,l}^{(i,j)} = N^{-\frac{1}{2}}$, $l = 1, 2, \cdots, N_{ij}$, and all other d's equal to zero. The condition (2.7) for this choice of d's is satisfied, so that by Lemma 2.1 $\mathcal{L}(\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}}) \to N(\eta^{(i,j)}, A^2)$ under K_N. Furthermore, a similar argument shows that any arbitrary linear combination of $\{\tau_N^{(i,j)}/N_{ij}^{\frac{1}{2}}, 1 \leqq i < j \leqq K\}$ has normal distribution in the limit. The proof follows.

PROOF OF THEOREM 2.1. It follows from Lemma 2.1 that the variables

$$W_{N,i} = [\sum_{j\neq i} \{N_{ij}^{-\frac{1}{2}} \tau_N^{(i,j)} - \eta^{(i,j)}\}]/AK^{\frac{1}{2}},$$

$i = 1, 2, \cdots, K$, have in the limit a multivariate normal distribution $N(\mathbf{0}, \mathbf{\Lambda})$ where $\mathbf{\Lambda} = \| \delta_{ii'} - 1/K \|$. Now making the analysis of variance transformation

$$U_0 = \sum_{i'=1}^{K} (K^{-\frac{1}{2}}) W_{N,i},$$
$$U_i = \sum_{i'=1}^{K} A_{ii'} W_{N,i'}, \qquad i = 1, 2, \cdots, K-1,$$

where A's are chosen to make the transformation orthogonal and proceeding exactly as in [18], the proof follows.

The following theorem concerns the limiting distribution of the separate-rankings statistic L_N^* defined by (1.4).

THEOREM 2.2. *Under the assumptions of Theorem* 2.1, *the statistic* L_N^* *is distributed in the limit, as* $N \to \infty$, *as a* $\chi_{K-1}^2(\delta^2)$ *variable with* δ^2 *given by* (2.3).

PROOF. Similar to that of Theorem 2.1.

From Theorems 2.1 and 2.2 it follows by letting $\mu_{ij} = 0$ for all pairs (i, j) that L_N and L_N^* are asymptotically distributed, under H_0 as χ_{K-1}^2 variables. This provides a large sample approximation to the critical points $c_{N,\alpha}$ and $c_{N,\alpha}^*$.

I.3. Asymptotic efficiency. In this section we consider some interesting special cases of the statistics L_N and L_N^* and discuss their asymptotic efficiencies relative to each other and the $\mathfrak{F}$-test. If we now let

(3.1)

(i) $J(u) = u$, $0 < u < 1$, then L_N reduces to the rank-sum version of L_N discussed in [16].

(ii) $J(u) = \chi^{-1}(u)$, where χ is the cdf of the chi-distribution with one degree of freedom, we get the multi-sample analogues of the Fisher-Yates-Fraser and Van der Waerden tests of symmetry respectively.

(iii) If we let $J(u)$ = constant, $L_N(L_N^*)$ reduces to the Durbin-statistic.

Let these statistics be denoted by W_N, $L_{N,1}$, $L_{N,2}$ and D_N respectively. Similarly one obtains the counterparts of the above statistics from L_N^*. Let these be denoted by the corresponding starred letters.

Now it is well known [10] that in the situations we are considering the asymptotic efficiency of one test relative to the other is equal to the ratio of their noncentrality parameters. Hence we have (e.g. when $\mu_{ij} = \theta_i - \theta_j$ and $N_{ij} = n$) the efficiencies of $L_{N,1}$, $L_{N,2}$, W_N, D_N and $\mathfrak{F}$-statistics as follows:

$$E_{L_1,\mathfrak{F}} = E_{L_2,\mathfrak{F}} = \sigma_\Pi^2[\int_0^1 \chi^{-1}(u)\psi(u)\,du]^2$$
$$(3.2) \qquad = \sigma_\Pi^2[\int_{-\infty}^{\infty} \pi(x)\,dx/\phi[\Phi^{-1}(\Pi(x))]]^2;$$
$$E_{L_1,W} = E_{L_2,W} = [\int_{-\infty}^{\infty} \pi^2(x)\,dx/\phi[\Phi^{-1}(\Pi(x)]]^2/12[\int_{-\infty}^{\infty} \pi^2(x)\,dx]^2$$
$$E_{L_1,D} = E_{L_2,D} = [\int_{-\infty}^{\infty} \pi^2(x)\,dx/\phi[\Phi^{-1}(\Pi(x))]]^2/4\pi^2(0).$$

TABLE 1

Distribution	E_{L_1}	E_{W,L_1}	$E_{L_1,D}$
Normal	1	$3/\pi \sim .955$	$\pi/2 \sim 1.571$
Uniform	∞	0	∞
Double exponential	$4/\pi \sim 1.273$	$3\pi/8 \sim 1.18$	$2/\pi \sim .637$

Table 1 gives the efficiency comparisons for different densities of the L_1 test, the W test, the D test and the $\mathfrak{F}$ test.

For distributions $\Pi(x)$ which are not covered under the conditions Ω_2, one may define

$$E^*_{S_1,S_2}(\Pi) = \lim_{\sigma\to 0} E_{S_1,S_2}(\Pi_\sigma),$$

if it exists, to be the asymptotic efficiency of S_1 relative to S_2, where Π_σ denotes the convolution of $\Pi(z)$ with $N(0, \sigma^2)$. For $\Pi_\sigma(z)$ the condition Ω_2 is satisfied. This covers the case, for example, of uniform distribution over $[-\theta, \theta]$. It is also interesting to observe that if the form of $\Pi(z)$ is specified, one can be letting $J(u) = \psi(u)$ obtain from the family L_N (or L_N^*) a statistic which is most (Pitman) efficient for the given distribution $\Pi(z)$—for example, by letting

$$\begin{aligned} J(u) &= \chi^{-1}(u) && \text{if } \Pi(z) \text{ is normal;} \\ J(u) &= u && \text{if } \Pi(z) \text{ is logistic;} \qquad (3.3) \\ J(u) &= \text{constant} && \text{if } \Pi(z) \text{ is double exponential.} \end{aligned}$$

Finally, we observe on account of Theorems 2.1 and 2.2 that $E_{L,L^*} = 1$.

DISCUSSION. On account of the last remark above, the question of preference between the *joint-ranking* and *separate-ranking* procedures remains unresolved. It is worth observing that the Pitman efficiency, although satisfactory in most situations, is a rather narrow criterion for comparing the expected performance of two tests, being just a limiting number which compares only their local asymptotic powers as the number of observations tends to infinity. A more comprehensive definition of asymptotic efficiency is discussed by Hodges and Lehmann [11]; but such a comprehensive comparison is often too difficult to carry out in more complex situations. The considerations of Part II, however, based on a comparison of the "asymptotic" efficiencies of the statistics L_N and L_N^* as the number of treatments is allowed to increase, do throw some light on this question.

PART II

II.1. Local "asymptotic" efficiency. In view of the result that the joint-ranking statistic L_N and the separate-ranking statistic L_N^* are equally efficient in the Pitman sense (I.3), the question of the relative merits of these two statistics remains undecided. This part is devoted to an investigation of this question. For reasons stated in the last paragraph of Part I, however, we shall attempt to throw some light on this question by a comparison only of their local "asymptotic" powers, as K, the number of treatments, is allowed to increase indefinitely.

It is shown below that if the number of comparisons $N_{ij}(=n)$ is kept fixed for each pair (i, j), but instead K tends to infinity, both the statistic L_N and the statistic L_N^*, after proper normalization, converge in distribution to the $N(0, 1)$ variable. This enables us to compare their local "asymptotic" powers for each fixed N. We observe that (since $N = n\binom{K}{2}$) as $K \to \infty$, $\{K_N\}$ again provides a sequence of translation alternatives approaching H_0. Let $E(\cdot)$ and $\sigma^2(\cdot)$ stand in the sequel for the expectation and the variance, respectively, with any subscripts indicating the conditions under which these quantities are obtained. We need the following:

Lemma 1.1. *Let* $\chi_r^2 = \chi_r^2(\Delta_r)$ *denote the non-central chi-square variable with* r df *and the non-centrality parameter* Δ_r, *and assume that* $\Delta_r = o(r)$, *as* $r \to \infty$. *Then, as* $r \to \infty$, $\mathcal{L}([\chi_r^2 - E(\chi_r^2)]/\sigma(\chi_r^2)) \to N(0, 1)$.

Proof. The density of χ_r^2 is given by $p_{\Delta_r}(x) = \sum_{k=0}^{\infty} p_k(\Delta_r) f_{r+2k}(x)$, where $p_k(\Delta_r) = (\Delta_r/2)^k \exp\{-(\Delta_r/2\}/k!$ and $f_{r+2k}(x)$ is the probability density of the central χ_{r+2k}^2 variable, so that the characteristic function of $[\chi_r^2 - E(\chi_r^2)]/\sigma(\chi_r^2)$ is given by

$$\begin{aligned} f(t) &= \exp(-it(r+\Delta_r)(2r+4\Delta_r)^{-\frac{1}{2}}) \textstyle\sum_{k=0}^{\infty} p_k(\Delta_r)(1-2it(2r+4\Delta_r)^{-\frac{1}{2}})^{-(r/2+k)} \\ &\sim \{(1-it(2/r)^{\frac{1}{2}})^{-r/2}\exp(-it(r/2)^{\frac{1}{2}})\} \\ &\quad \cdot\exp(-it(\Delta_r(2r)^{-\frac{1}{2}}) \textstyle\sum_{k=0}^{\infty} p_k(\Delta_r)(1-it(2/r)^{\frac{1}{2}})^{-k} \\ &= \{(1-it(2/r)^{\frac{1}{2}})^{-r/2}\exp(-it(r/2)^{\frac{1}{2}})\} \\ &\quad \cdot\exp\{-\tfrac{1}{2}\Delta_r(1+it(2/r)^{\frac{1}{2}})+\tfrac{1}{2}\Delta_r(1-it(2/r)^{\frac{1}{2}})^{-1}\} \end{aligned}$$

where the first term converges to $\exp\{-t^2/2\}$ and the second to unity, as $r \to \infty$, on account of the condition $\Delta_r = o(r)$; the proof is complete.

Remark. In the statement of Lemma 1.1 above we may replace $E(\chi_r^2)$ and $\sigma(\chi_r^2)$ by $r + \Delta_r$ and $(2r + \Delta_r)^{\frac{1}{2}}$ respectively.

Theorem 1.1. *Assume, for each index* N, *the truth of* K_N *with* $\mu_{ij} = \theta_i - \theta_j$ (*where not all* θ*'s are equal*) *and* $N_{ij} = n$ *for all pairs* (i, j) $(1 \leqq i < j \leqq K)$. *Further, assume that*

$$\sup_K K^{-2} \textstyle\sum_{i<j} (\theta_i - \theta_j)^2 < \infty. \tag{1.1}$$

Then under the conditions Ω_1 *and* Ω_2 *of Part* I, $\mathcal{L}(L_N) \to N(\eta, 2(K-1))$ *as* $K \to \infty$, *where*

$$\eta = (K-1) + \delta_K^2, \tag{1.2}$$

with δ_K^2 *given by* (2.5) *of Part* I.

Proof. Let the $C = K(K-1)/2$ pairs (i, j) $(1 \leqq i < j \leqq K)$ be labelled $\alpha = 1, 2, \cdots, C$ (as in the proof of Lemma 2.2 of Part I) in some convenient manner, where if α corresponds to the pair (i, j), $\mu_\alpha = \mu_{ij} = \theta_i - \theta_j$. Then, the

$N = nK(K-1)/2$ c_N's defined for each N and α by

$$c_{N,\alpha,l} = \mu_\alpha/N^{\frac{1}{2}} = (\theta_i - \theta_j)/N^{\frac{1}{2}},$$

$l = 1, 2, \cdots, n$, satisfy the condition (2.7) I, as $K \to \infty$ (and consequently $N \to \infty$). This is easily seen by observing that, for the above c_N's, the left hand side of (2.7)I reduces to

$$(1.3) \quad \lim_{k\to\infty} \{\max_{1\leqq i<j\leqq K} (\theta_i - \theta_j)^2/n \textstyle\sum_{i<j} (\theta_i - \theta_j)^2\}$$
$$\leqq (4/n) \lim_{k\to\infty} \{\max_{1\leqq i\leqq K} (\theta_i - \bar\theta)^2/K \textstyle\sum_{i=1}^K (\theta_i - \bar\theta)^2\} = 0,$$

where $\bar\theta = \sum_{i=1}^K \theta_i/K$. The last inequality follows since $\sum_{i<j} (\theta_i - \theta_j)^2 = K\sum_{i=1}^K (\theta_i - \bar\theta)^2$. On account of (1.1)II and (1.3)II, the conditions (2.7)I and (2.8a)I are satisfied, so that by applying Lemma 2.1 of Part I one obtains the asymptotic normality, after proper normalization, of any statistic of the type (2.15)I (or (2.6)I) for which (2.7)I is satisfied. Consider now any arbitrary linear combination of the variables $v_N^{(i)} = \sum_{j\neq i} V_N^{(i,j)}$, $i = 1, 2, \cdots, K$; viz.,

$$S_N = \textstyle\sum_{i=1}^K \lambda_i V_N^{(i)} = \sum_{i=1}^K (\sum_{j\neq i} \lambda_i V_N^{(i,j)})$$
$$= \textstyle\sum_{i=1}^K \sum_{j>i} (\lambda_i - \lambda_j) V_N^{(i,j)},$$

(using $V_N^{(i,j)} = -V_N^{(j,i)}$), where not all λ's are equal and zero values are permissible. The statistic S_N is obtainable from (2.15)I by letting for each $i = 1, 2, \cdots, K$, $d_{N,l}^{(i,j)} = \lambda_i - \lambda_j$ for $j > i$ and $l = 1, 2, \cdots, n$. With the above choice of d's the left hand side of (2.7)I takes the form

$$\lim_{K\to\infty} \{\max_{1\leqq i<j\leqq K} (\lambda_i - \lambda_j)^2/n \textstyle\sum_{i<j} (\lambda_i - \lambda_j)^2\}$$

which equals zero by the same arguments as used in (1.3)II. Accordingly, by applying Lemma 2.1 of Part I and using, for any K, however large, the same arguments as in Section 7 of Wald and Wolfowitz [24], it follows that, for sufficiently large K, the variables $(nK)^{-\frac{1}{2}}\{V_N^{(i)} - m^{(i)}\}$, $i = 1, 2, \cdots, K$, where $m^{(i)} = \{-(2n)^{\frac{1}{2}}(\theta_i - \bar\theta)^2(\int_0^1 J(u)\psi(u)\,du)\}$, are approximately jointly normally distributed with mean vector zero and the covariance matrix $\mathbf{\Sigma} = \| \delta_{ii'} - (1/K \| \cdot (\int_0^1 J^2(u)\,du)$. Arguments similar to those used in the proof of Theorem 2.1 of Part I, coupled with an application of Lemma 1.1 above and Theorem 5 of Mann and Wald [24] gives the result forthwith; and the proof is complete.

A similar result also holds for the statistic L_N^* defined by (1.4)I:

THEOREM 1.2. *Assume, for each index N, the truth of K_N with $\mu_{ij} = \theta_i - \theta_j$ and $N_{ij} = n$ for all pairs (i, j) $(1 \leqq i < j \leqq K)$. Then, under the conditions of Theorem* 1.1, $\mathcal{L}(L_N^*) \to N(\eta', 2(K-1))$, *as $K \to \infty$, where*

$$(1.4) \qquad \eta' = (K-1) + (\delta_K^{*2}/n\, d_n^2)$$

with δ_K^{*2} and d_n^2 are given by (1.8)II *and* (1.9)II *respectively.*

PROOF. The proof of this theorem can be accomplished by using the central limit theorem for random vectors, Lemma 1.1 and arguments similar to those used in the proof of Theorem 1.1 above.

Theorems 1.1 and 1.2 can be used to compare the local powers of the two statistics L_N and L_N^*, as $K \to \infty$. Let $\theta = (\theta_1, \theta_2, \cdots, \theta_K)$ and let $\{l_{N\alpha}\}$ and $\{l_{N\alpha}^*\}$ be two sequences of numbers determined such that $\lim_{N\to\infty} P_{H_0}[L_N > l_{N\alpha}] = \lim_{N\to\infty} P_{H_0}[L_N^* > l_{N\alpha}^*] = \alpha$. The local powers of the statistics L_N and L_N^* under K_N at level α, are given respectively by

$$\beta_L(n, \alpha, \theta) = P_{K_N}[L_N > l_{N\alpha}] \sim 1 - \Phi\{(l_{N\alpha} - \eta)/(2(K-1))^{\frac{1}{2}}\},$$
$$\beta_{L^*}(n, \alpha, \theta) = P_{K_N}[L_N^* > l_{N\alpha}^*] \sim 1 - \Phi\{(l_{N\alpha}^* - \eta')/(2(K-1))^{\frac{1}{2}}\},$$

for sufficiently large K, on account of Theorems 1.1 and 1.2 above, where $\Phi(x)$ represents the standard normal cdf, $\phi(x)$ the corresponding density and the symbol $\sim$ denotes that the ratio of the two sides tends to one, as $K \to \infty$. Accordingly, from Theorem 1.1 above it follows that

$$(1.5)\quad \beta_L(n, \alpha; \theta) \sim \alpha + 2^{\frac{1}{2}}\{(\textstyle\int_0^1 J(u)\psi(u)\,du)^2/(\int_0^1 J^2(u)\,du)\} \cdot K^{-\frac{3}{2}} \sum_{i=1}^K (\theta_i - \bar{\theta})^2 \cdot \phi(t_\alpha)$$

for sufficiently large K, where t_α is the upper α point of $N(0, 1)$ distribution. To obtain a similar expression for $\beta_{L^*}(n, \alpha, \theta)$, set

$$(1.6)\qquad a_n^{(i,j)} = \lim_{N\to\infty} N^{\frac{1}{2}} E_{K_N}(V_N^{*(i,j)};$$

then following the above reasoning again we obtain

$$(1.7)\qquad \beta_{L^*}(n, \alpha, \theta) \sim \alpha + (\delta_K^{*2}/n\,d_n^2)\phi(t_\alpha)$$

where

$$(1.8)\qquad \delta_K^{*2} = (2^{\frac{1}{2}}/K^{\frac{3}{2}}) \textstyle\sum_{i=1}^K \{\sum_{j\neq i} (a_n^{(i,j)}/K)\}^2,$$

and

$$(1.9)\qquad d_n^2 = \sigma_{H_0}^2(V_n^{*(i,j)}) = \textstyle\sum_{k=1}^n J_{n,k}^2,$$

$J_{n,k} = J_n(k/(n+1))$, $k = 1, 2, \cdots, n$, being the scores on which the definition of the function $\xi_n(u)$, $0 < u < 1$, is based. From (1.5) and (1.7), it follows that for large K the local power for shift alternatives $\beta_L(n, \alpha, \theta)$ will tend to be larger than $\beta_{L^*}(N, \alpha, \theta)$ if and only if

$$\begin{aligned} e_{L,L^*}^{(n)} &= \lim_{K\to\infty} \{(\beta_L(n, \alpha, \theta) - \alpha)(\beta_{L^*}(n, \alpha, \theta) - \alpha)^{-1}\} \\ (1.10)\qquad &= \lim_{K\to\infty} \{(\textstyle\int_0^1 J(u)\psi(u)\,du)^2/(\int_0^1 J^2(u)\,du)\} \sum_{i=1}^K (\theta_i - \bar{\theta})^2 \\ &\quad \cdot [\textstyle\sum_{i=1}^K \{\sum_{j\neq i} (a_n^{(i,j)}/K)\}^2/n\,d_n^2]^{-1} \end{aligned}$$

is larger than unity. The expression $e_{L,L^*}^{(n)}$ may be called the *local ("asymptotic") efficiency* of L_N and L_N^*, as $K \to \infty$, and may be used to throw some light on the question of comparison of L_N and L_N^*. It may be pointed out, however, that for the ratio $e_{L,L^*}^{(n)}$ a meaningful interpretation as in the case of Pitman's formula cannot be given.

II.2. The explicit evaluation of $e_{L,L^*}^{(n)}$. We shall derive in this section an explicit expression for the local "asymptotic" efficiency $e_{L,L^*}^{(n)}$ by evaluating $a_n^{(i,j)}$:

$$a_N^{(i,j)} = \lim_{N\to\infty} N^{\frac{1}{2}} E_{K_N}\{\sum_{l=1}^n J_n(R_{n,l}^{(i,j)}/(n+1)) \text{ sign } Z_l^{(i,j)}\}$$
$$= \lim_{N\to\infty} N^{\frac{1}{2}} \sum_{k=1}^n (n!/(k-1)!(n-k)!) J_{n,k} \int_0^1 [T_N^{(i,j)}(x)]^{k-1}$$
$$\cdot [1 - T_N^{(i,j)}(x)]^{n-k} d[\Pi(x - (\theta_i - \theta_j)N^{-\frac{1}{2}}) - \Pi(x + (\theta_i - \theta_j)N^{-\frac{1}{2}})]$$

where $T_N^{(i,j)}(x) = \Pi(x - (\theta_i - \theta_j)N^{-\frac{1}{2}}) - \Pi(-x - (\theta_i - \theta_j)N^{-\frac{1}{2}})$, if $x \geqq 0$ and $T_N^{(i,j)}(x) = 0$ if $x < 0$. In evaluating the above limit, it is permissible to interchange the operations of limit and integration as is shown by the following:

LEMMA 2.1. *If the distribution $\Pi(x)$ possesses a differentiable density $\pi(x)$ and the condition Ω_2 is satisfied, then the expression $a_n^{(i,j)}$ is given by*

$$(2.1) \quad a_n^{(i,j)} = (\theta_i - \theta_j) \sum_{k=1}^n (n!/(k-1)!(n-k)!) \cdot J_{n,k} \int_0^1 \psi(u) u^{k-1} (1-u)^{n-k} du.$$

PROOF. Since $\Pi(x)$ possesses a differentiable density $\pi(x)$, $a_n^{(i,j)}$ can be written as

$$\lim_{N\to\infty} (\theta_i - \theta_j) \sum_{k=1}^n (n!/(k-1)!(n-k)!) J_{n,k} (A_{k,N} + B_{k,N})$$

where, setting $t_N = (\theta_i - \theta_j)N^{-\frac{1}{2}} (\max_{i<j} (t_N) \to 0$, as $K \to \infty)$.

$$A_{k,N} = \int_0^\infty [T_N(x)]^{k-1} [1 - T_N(x)]^{n-k} (\pi(x - t_N) - \pi(x))(2t_N \cdot \pi(x))^{-1} dT(x)$$

and

$$B_{k,N} = \int_0^\infty [T_N(x)]^{k-1} [1 - T_N(x)]^{n-k} (\pi(x) - \pi(x + t_N))(2t_N \cdot \pi(x))^{-1} dT(x);$$

(in $A_{k,N}$ and $B_{k,N}$ we have suppressed the index (i,j) for convenience). The proof of the lemma will be complete if we show that

$$\lim_{N\to\infty} A_{k,N} = \lim_{N\to\infty} B_{k,N}$$
$$= \tfrac{1}{2} \int_0^\infty [T(x)]^{k-1} [1 - T(x)]^{n-k} (-\pi'(x)/\pi(x)) \, dT(x) = D_k \text{ (say)}$$

where $D_k = \int_0^1 \psi(u) u^{k-1} (1-u)^{n-k} du$, and $T(x) = 2\Pi(x) - 1$, if $x \geqq 0$ and $T(x) = 0$ if $x < 0$. To see this, note that

$$|A_{k,N} - D_k| \leqq \tfrac{1}{2} |\int_0^\infty [T_N(x)]^{k-1} [1 - T_N(x)]^{n-k}$$
$$\cdot \{(\pi(x - t_N) - \pi(x))(t_N \cdot \pi(x))^{-1} - (-\pi'(x)/\pi(x)\} \, dT(x)$$
$$+ \tfrac{1}{2} |\int_0^\infty \{[T_N(x)]^{k-1} [1 - T_N(x)]^{n-k} - [T(x)]^{k-1} [1 - T(x)]^{n-k}\}$$
$$\cdot \pi'(x)/\pi(x) \, dT(x)|$$

where the second term on the right $\to 0$, as $N \to \infty$, using the dominated convergence theorem and the condition Ω_2. Consider now the first term, which cannot exceed

$$\tfrac{1}{2}\left|\int_0^\infty [(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x)^2)^{\frac{1}{2}}](t_N \cdot \pi(x))^{-1}\, dT(x)\right|$$
$$+ \left|\int_0^\infty \{(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x))^{\frac{1}{2}}](-t_N(\pi(x))^{\frac{1}{2}})^{-1} - \pi'(x)/2\pi(x)\}\, dT(x)\right|$$
$$\leqq |t_N|[\int_0^\infty \{[(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x))^{\frac{1}{2}}]t_N^{-1}\}^2\, dx]$$
$$+ 2^{\frac{1}{2}}[\int_0^\infty \{[(\pi(x - t_N))^{\frac{1}{2}} - (\pi(x))^{\frac{1}{2}}](- t_N^{-1}) - \pi'(x)/(2(\pi(x))^{\frac{1}{2}})\}^2\, dx]^{\frac{1}{2}}.$$

The last inequality follows by applying Schwartz inequality to the second term. Both terms on the right tend to zero, as $N \to \infty$, on account of Lemma 4.3 of Hájek [9], since the condition Ω_2 implies the quadratic integrability of the derivative of $(\pi(x))^{\frac{1}{2}}$. This establishes that $\lim_{N\to\infty} A_{k,N} = D_k$. The same argument shows that $\lim_{N\to\infty} B_{k,N} = D_k$. The proof is complete.

Now substituting (2.1)II and the expression for d_n^2 in (1.10)II, we obtain

$$(2.2)\quad e_{L,L^*}^{(n)} = (\int_0^1 J(u)\psi(u)\, du)^2(\int_0^1 J^2(u)\, du)^{-1}$$
$$[(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-1}{k-1} \int_0^1 \psi(u)u^{k-1}(1-u)^{n-k}\, du)^2(n^{-1}\sum_{k=1}^n J_{n,k}^2)^{-1}]^{-1}.$$

One naturally expects the local efficiency $e_{L,L^*}^{(n)}$ to converge to the asymptotic efficiency $E_{L,L^*} = 1$, as $n \to \infty$. However, despite the plausibility of the above statement, we are able to prove it only for the case when $\xi(u)$ is monotone.

THEOREM 2.1. *Suppose that the conditions Ω_1 and Ω_2 of Part* I *are satisfied. Then under the assumption of monotonicity of $J(u)$, $\lim_{n\to\infty} e_{L,L^*}^{(n)} = 1$.*

PROOF. Clearly we need to prove the theorem for non-constant $J(u)$, for otherwise L_N and L_N^* are identical and the result is trivially true. First we observe that, on account of the conditions Ω_1,

$$(2.3)\qquad n^{-1}\textstyle\sum_{k=1}^n J_{n,k}^2 = \int_0^1 J_n^2(u)\, du \to \int_0^1 J^2(u)\, du < \infty,$$

as $n \to \infty$. Further, if we let $p_k(u) = \binom{n-1}{k-1}u^{k-1}(1-u)^{n-k}$,

$$(\int_0^1 \psi(u)[\textstyle\sum_{k=1}^n J_{n,k}p_k(u)]\, du - \int_0^1 \psi(u)J(u)\, du)^2$$
$$(2.4)\qquad \leqq (\int_0^1 \psi^2(u)du)(\int_0^1 \textstyle\sum_{k=1}^n [J_{n,k} - J(u)]^2 p_k(u)du)$$
$$\leqq 2(\int_0^1 \psi^2(u)\, du)(\int_0^1 \textstyle\sum_{k=1}^n [J_n(k/(n+1)) - J_n(u)]^2 p_k(u)\, du$$
$$+ \int_0^1 [J_n(u) - J(u)]^2\, du,$$

by substituting $J_{n,k} = J_n(k/(n+1))$. The proof of the theorem will be complete if we show that the right hand side of (2.4)II tends to zero as $n \to \infty$, and use (2.3)II and (2.4)II in (2.2)II. For this it suffices, on account of Ω_1 and Ω_2, to prove that

$$(2.5)\qquad \lim_{n\to\infty} \int_0^1 \textstyle\sum_{k=1}^n [J_n(k/(n+1)) - J_n(u)]^2 p_k(u)\, du = 0.$$

In order to prove (2.5)II we observe that on account of monotonicity of $J(u)$

and Ω_1, there is no loss of generality in assuming that $J_{n,1} \leqq J_{n,2} \leqq \cdots \leqq J_{n,n}$. Thus, using Lemma 2.1 of Hájek [9] it follows that left hand side of (2.5)II does not exceed

$$(2.6) \qquad 2\cdot 2^{\frac{1}{2}}n^{-\frac{1}{2}} \max_{1 \leqq k \leqq n} |J_{n,k} - \bar{J}_n|[\textstyle\sum_{k=1}^{n} (J_{n,k} - \bar{J}_n)^2/n]^{\frac{1}{2}}$$

where $\bar{J}_n = \sum_{k=1}^{n} J_{n,k}/n$. Now the expression (2.6)II tends to zero as $n \to \infty$, since

$$n^{-1}\textstyle\sum_{k=1}^{n} (J_{n,k} - \bar{J}_n)^2 \leqq n^{-1}\sum_{k=1}^{n} J_{n,k}^2 \to \int_0^1 J^2(u)\, du < \infty$$

on account of (2.3) and

$$n^{-1} \max_{1 \leqq k \leqq n} |J_{n,k}|^2 = \max_{1 \leqq k \leqq n} \textstyle\int_{[(k-1)/n, k/n]} J_n^2(u)\, du \to 0$$

as $n \to \infty$, on account of uniform integrability of the functions $J_n^2(u)$, a consequence of the conditions Ω_1; and the proof is complete.

For distribution functions $\Pi(z)$ not satisfying the differentiability conditions of Theorem 2.1 of Part I, one may define the local efficiency of L_N relative to L_N^* (as $K \to \infty$) in the same manner as the asymptotic relative efficiency $E_{S_1,S_2}^*(\pi)$ was defined in Part I, viz.,

$$(2.7) \qquad e_{L,L^*}^{*(n)}(\pi) = \lim_{\sigma \to 0} e_{L,L^*}^{(n)}(\pi_\sigma),$$

provided the limit exists. It is interesting to note that $\lim_{n\to\infty} e_{L,L^*}^{*(n)}(\pi)$ may or may not be equal to 1, as is illustrated by the case when $\pi(z)$ is the cdf of the uniform distribution over $(-t, t)$ (see II.3).

II.3. Special cases. In this section, we shall evaluate the local efficiency $e_{L,L^*}^{(n)}$ for some well known distributions and the special choices of the functions $J_n(u)$ and $J(u)$ considered in Section I.3:

Wilcoxon-statistics. By substituting $\xi(u) = u$, $0 < u < 1$, and $J_{n,k} = (k/(n+1))$, $k = 1, 2, \cdots, n$, in (2.2)II we obtain

$$(3.1) \qquad e_{W,W^*}^{(n)}(\pi) = \tfrac{1}{2}(n+1)(2n+1)(\textstyle\int_0^1 u\psi(u)\, du)^2 \cdot (\int_0^1 \psi(u)[(n-1)u + 1)\, du)^{-2},$$

so that from (3.3)I it follows that

$$(3.2) \qquad \begin{aligned} e_{W,W^*}^{(n)}(\text{Normal}) &= (n+1)(2n+1)/2(n-1+2^{\frac{1}{2}})^2 > 1, \\ e_{W,W^*}^{(n)}(\text{Logistic}) &= (2n+2)/(2n+1) > 1, \\ e_{W,W^*}^{(n)}(\text{Double exponential}) &= (2n+1)/(2n+2) < 1, \\ e_{W,W^*}^{n}(\text{Cauchy}) &= (2n+1)/(2n+2) < 1. \end{aligned}$$

For evaluating $e_{W,W^*}^{*(n)}$ (uniform), defined by (2.7)II, we note that the density $\pi_\sigma(z)$ of the distribution $\Pi_\sigma(z)$, the convolution of $R(-\frac{1}{2}, \frac{1}{2})$ and $N(0, 1)$ distributions, is given by

$$(3.3) \qquad \pi_\sigma(z) = \Phi((2z+1)/2\sigma) - \Phi((2z-1)/2\sigma),$$

where Φ is the standard normal cdf, so that from (3.1)II

$$\begin{aligned}
e^{*(n)}_{W,W^*}\text{ (uniform)} &= \lim_{\sigma\to 0} e^{(n)}_{W,W^*}(\pi) \\
&= \lim_{\sigma\to 0} \tfrac{1}{2}(n+1)(2n+1)\left(\int_0^1 u\psi_\sigma(u)\,du\right)^2 \\
&\quad \cdot\left[\left(\int_0^1 \psi_\sigma(u)[(n-1)u+1]\,du\right)^2\right]^{-1} \qquad (3.4)\\
&= \lim_{\sigma\to 0} \tfrac{1}{2}(n+1)(2n+1)\left(\int_{-\infty}^{\infty} \psi_\sigma^2(x)\,dx\right)^2 \\
&\quad \cdot\left(\pi_\sigma(0) + (n-1)\int_{-\infty}^{\infty} \pi_\sigma^2(x)\,dx\right)^{-2} \\
&= (n+1)(2n+1)(2n^2)^{-1} > 1;
\end{aligned}$$

the last equality follows by interchanging the limit and integration, which is permissible since $|\pi_\sigma(x)| < 2$ for $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ and $|\pi_\sigma(x)|$ is bounded by a Lebesgue integrable function for $x < -\frac{1}{2}$ and $x > \frac{1}{2}$. We note that the local efficiency expressions (3.2)II and (3.4)II converge to 1, as $n \to \infty$.

Absolute-normal-score statistics. By letting $J(u) = \chi^{-1}(u)$, $0 < u < 1$, in (2.2)II we obtain the local efficiencies $e^{(n)}_{L_1,L_1^*}$ and $e^{(n)}_{L_2,L_2^*}$ for the absolute-normal-score statistics defined in Section 3I, namely,

$$\begin{aligned}
(3.5)\quad e^{(n)}_{L,L^*}(\Pi) &= n^{-1}\textstyle\sum_{k=1}^n J^2_{n,k}\left(\int_0^1 \chi^{-1}(u)\psi(u)\,du\right)^2 \\
&\quad \cdot\left[\left(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-1}{k-1}\int_0^1 \psi(u)u^{k-1}(1-u)^{n-k}\,du\right)^2\right]^{-1}
\end{aligned}$$

which yields $e^{(n)}_{L_1,L_1^*}(\Pi)$ if the scores $J_{n,k} = J_n(k/(n+1))$, $k = 1, \cdots, n$, correspond to the Fisher-Yates type and $e^{(n)}_{L_2,L_2^*}(\Pi)$, if these scores correspond to the Van der Waerden type. From (3.3)II and (3.5)II we obtain

$$\begin{aligned}
e^{(n)}_{L,L^*}(\text{Normal}) &= \left(\left(\textstyle\sum_{k=1}^n J^2_{n,k}\right)/n\right)\left(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-1}{k-1}\right. \\
&\quad \left.\int_0^1 \Phi^{-1}((1+u)/2)u^{k-1}(1-u)^{n-k}\,du\right)^{-2}, \\
e^{(n)}_{L,L^*}(\text{Logistic}) &= n(n+1)^2\left(\textstyle\sum_{k=1}^n J^2_{n,k}\right) \\
&\quad \cdot\left(\pi\left(\textstyle\sum_{k=1}^n kJ_{n,k}\right)^2\right)^{-1}, \qquad (3.6)\\
e^{(n)}_{L,L^*}(\text{Double Exponential}) &= 2n\left(\textstyle\sum_{k=1}^n J^2_{n,k}\right)\left(\pi\left(\textstyle\sum_{k=1}^n J_{n,k}\right)^2\right)^{-1}, \\
e^{(n)}_{L,L^*}(\text{Cauchy}) &= \left(\textstyle\sum_{k=1}^n J^2_{n,k}/n\right)\left(\int_0^1 \Phi^{-1}((1+u)/2)\right. \\
&\quad \left.\cdot(\sin \pi u)\,du\right)^2\left(\textstyle\sum_{k=1}^n J_{n,k}\binom{n-k}{k-1}\right. \\
&\quad \left.\int_0^1 (\sin \pi u)u^{k-1}(1-u)^{n-k}\,du\right)^{-2}.
\end{aligned}$$

For evaluating $e^{*(n)}_{L,L^*}(\text{Uniform})$, we note from (3.5)II that

$$\begin{aligned}
e^{*(n)}_{L,L^*}(\text{Uniform}) &= \lim_{\sigma\to 0} e^{(n)}_{L,L^*}(\Pi_\sigma) \\
&\geqq \lim_{\sigma\to 0}\left(\int_0^1 \Phi^{-1}((1+u)/2)\psi_\sigma(u)\,du\right)^2 \qquad (3.7)\\
&\quad \cdot\left(n^2\left(\int_0^1 \psi_\sigma(u)\,du\right)^2\right)^{-1}
\end{aligned}$$

where, on account of Fatou's lemma,

TABLE 2

$e^{(n)}_{L_1,L_1^*}$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = \infty$
Normal	1.571	1.34	1.24	1.20	1.14	1
Logistic	1.273	1.152	1.102	1.075	1.058	1
Double exponential	.637	.746	.803	.839	.862	1
$e^{(n)}_{L_2,L_2^*}$						
Normal	1.571	1.34	1.24	1.20	1.14	1
Logistic	1.273	1.148	1.096	1.068	1.051	1
Double exponential	.637	.730	.781	.814	.835	1

$$\lim_{\sigma\to 0} \left(\int_0^1 \Phi^{-1}((1+u)/2)\psi_\sigma(u)\,du\right)$$

$$(3.8) \qquad \begin{aligned} &= \lim\inf_{\sigma\to 0} \int_0^1 (\pi_\sigma[\Pi_\sigma^{-1}(u)]/\phi[\Phi^{-1}(u)) \,du \\ &\geqq \int_0^1 \lim\inf_{\sigma\to 0} (\pi_\sigma[\Pi_\sigma^{-1}(u)]/\phi[\Phi^{-1}(u)]) \,du \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \{\phi[\Phi^{-1}(u+\tfrac{1}{2})]\}^{-1} \,du = \infty. \end{aligned}$$

Also, from (3.3)II we have

$$\int_0^1 \psi_\sigma(u)\,du = 2[\Phi(1/2\sigma) - \Phi(-1/2\sigma)] \to 2$$

as $\sigma \to \infty$. From (3.7), (3.8), and (3.9) of this section, it follows that $e^{*(n)}_{L,L^*}$(Uniform) $= \infty$ for both version $L_{N,1}$ and $L_{N,2}$ of the absolute-normal-score statistics.

The *approximate* numerical values of the local efficiency expressions (3.6) II are tabled in Table 2 for both versions of the absolute-normal-score statistics.

We observe that for the cases considered in Table 2, the numerical values of $e^{(n)}_{L,L^*}(\Pi)$ seem to converge monotonically to 1, as $n \to \infty$.

II.4. Conclusion. The local efficiency expressions and their numerical values obtained in the preceding section indicate the superiority of the "joint-ranking" procedure against shift alternatives with normal, uniform or logistic as the underlying distribution; whereas against a double exponential or Cauchy distribution the "separate-ranking" statistic L_N^* seems to have better local power. These observations however seem merely incidental to a presumably more basic pattern suggested by the following: (a) First, we note that for $n = 1$ the local efficiency $e^{(n)}_{L,L^*}$ reduces to $E_{L,D}$, the asymptotic efficiency of L_N relative to the Durbin statistic, and (b) secondly, that for the special cases considered above the local efficiency seems to converge monotonically to 1, as $n \to \infty$. Thus if we consider, for a given choice of function $J_N(u)$ and $J(u)$, the class of all distributions satisfying (b), it follows that $e^{(n)}_{L,L^*} > 1$ or $<$ for all n, according as $E_{L,D}(\Pi) > 1$ or <1. These considerations suggest the following heuristic conclusion (for the class of distributions satisfying the condition (b)): *For a given functions $J(u)$ and $J_N(u)$, the "joint-ranking statistic" $L_N(J, J_N)$ is preferable to its counterpart $L_N^*(J, J_N)$ based on "separate-rankings", except for alternative distributions for which the Durbin-statistic is relatively Pitman-efficient than the statistic $L_N(J, J_N)$* i.e., *for which $E_{L(J,JN),D}(\Pi) > 1$*. It would be of interest to characterize for a given function $J(u)$ the class of distributions satisfying the condition (b). For example, for the Wilcoxon-statistics W and W^*, a simple characterization of such a class would be: The class of all cdf's $\Pi(z)$ for which

$$t(\pi) = \pi(0)/\int_{-\infty}^{\infty} \pi^2(z)\, dz$$

is either $<(12/7)$ or $>(7/4)$. In fact, $e^{(n)}_{W,W^*}$ decreases monotonically to 1 if $t(\pi) < (12/7)$ (e.g., normal, logistic, uniform distributions) and increases monotonically to 1 if $t(\pi) > (7/4)$ (e.g. Cauchy and double exponential). Accordingly, since $E_{W,D} = e^{(1)}_{W,W^*}$, for this class of distributions the above heuristic conclusion clearly holds for the Wilcoxon-statistics.

A strong argument in favour of the "joint-ranking" procedure, however, is the following: Consider the problem of testing H_0 against the alternatives of shift in location and assume that the underlying distribution $\Pi(z)$ is specified. Then, one can select a most Pitman-efficient rank-order statistic by letting $J(u) = \psi(u)$ in L_N or L_N^*. However, since $E_{L,L^*} = 1$, the choice is still to be made between the "joint-ranking" and the "separate-ranking" procedures. Now one can easily show that, for the above choice of the function $J(u)$,

$$\begin{aligned} e^{(n)}_{L,L^*}(\pi) &= \left(\int_0^1 \psi^2(u)\, du\right)\left(n^{-1}\sum_{k=1}^n \psi^2_{n,k}\right) \\ &\quad \cdot \left(\sum_{k=1}^n \psi_{n,k}\binom{n-1}{k-1} \int_0^1 \psi(u) u^{k-1}(1-u)^{n-k}\, du\right)^{-2} \\ &\geqq \left(\left(\sum_{k=1}^n \psi^2_{n,k}\right)/n\right)\left(\int_0^1 \left[\sum_{k=1}^n \psi_{n,k}\binom{n-1}{k-1} u^{k-1}(1-u)^{n-k}\right]^2 du\right)^{-1} \geqq 1, \end{aligned}$$

with equality sign only if $J(u) = \psi(u) = \text{const.}$, $0 < u < 1$, in which case obviously the statistics L_N, L_N^* are identical. This leads us to the conclusion that, against a specified alternative distribution $\Pi(z)$, the "best" rank-order statistic (in the sense of local-power) is the one based on the "joint-ranking" procedure.

Finally, it seems worth mentioning that the form of the hypotheses H_0 favours the "joint-ranking" procedure. The "separate-ranking" statistic is essentially a test of symmetry about zero for each of the distributions $\Pi_{ij}(z)$ i.e., $\Pi_{ij}(z) + \Pi_{ij}(-z) = 1$, $(1 \leqq i < j \leqq K)$. It does not take into consideration the second part of the hypothesis H_0, namely, that $\Pi_{ij}(z) = \Pi_{i'j'}(z)$ for any two pairs (i, j) and (i', j'), whereas the "joint-ranking" statistic L_N does take this into consideration.

PART III

III.1. Summary. Let (ξ_{il}, η_{jl}), $l = 1, \cdots, N_{ij}$; $1 \leqq i < j \leqq K$ be independent samples from populations with absolutely continuous cdf's $D_{ij}(u, v)$. Denote $a^*_{ij,r} = +1$, if the rth smallest observation from the ordered absolute values $|Z_{ijl}|$ where $Z_{ijl} = \xi_{il} - \eta_{il}$, in the combined sample of size $N = \sum\sum_{i<j} N_{ij}$ is from a positive Z_{ij}, and otherwise let $a^*_{ij,r} = 0$. Denote $a^{**}_{ij,r} = -1$, if the rth smallest observation from the ordered absolute values $|Z_{ijl}|$ in the combined sample of size N is from a negative Z_{ij} and otherwise let $a^{**}_{ij,r} = 0$.

Denote

$$\tau_N^{(i,j)} = \tau_{ij}^{+} + \tau_{ij}^{-} \tag{1.1}$$

where

$$\tau_{ij}^{+} = m_{ij}T_{ij}^{+} = \sum_{r=1}^{N} E_{N,r}a_{ij,r}^{*} \tag{1.2}$$

and

$$\tau_{ij}^{-} = n_{ij}T_{ij}^{-} = \sum_{r=1}^{N} E_{N,r}\, a_{ij,r}^{**}\,. \tag{1.3}$$

The $E_{N,r}$ are given numbers satisfying certain restrictions to be stated below; and m_{ij} and n_{ij} are the number of positive and negative Z's among $Z_{ij1}, \cdots, Z_{ijN_{ij}}$. The purpose of this part is to find a set of sufficient conditions for the joint asymptotic normality of the statistics $\tau_N^{(i,j)}$. Various applications of these statistics are given in Part I where the problem of testing the hypothesis of no difference among several different treatments is considered for the case when the comparison between the treatment is possible only in pairs. (However, in Part I, the joint asymptotic normality of the statistics $\tau_N^{(i,j)}$'s which can be obtained as a special case of the more general theorem (Theorem 3.1 below), is obtained by following the methods of Hájek [9] so as to present a different approach to the reader).

III.2. Assumptions and notations. Let $c = \binom{K}{2}$ denote the number of all possible pairs and label them $\alpha = 1, \cdots, c$. Let m_α and n_α be the number of positive and negative Z's respectively for the αth pair. m_α and n_α are random but $m_\alpha + n_\alpha = N_\alpha$ is non-random. For given m_α, let $X_{\alpha 1}^{+}, \cdots, X_{\alpha m_\alpha}^{+}$ denote the positive Z's and $\bar{X}_{\alpha 1}, \cdots, \bar{X}_{\alpha n_\alpha}$ denote the absolute values of negative Z's among $Z_{\alpha 1}, \cdots, Z_{\alpha N_\alpha}$; $\alpha = 1, \cdots, c$. Let $F^{+(\alpha)}(x)$ and $F^{-(\alpha)}(x)$ denote the cdf's of X_α^{+}'s and X_α^{-}'s respectively. Let $F_{m_\alpha}^{+}(x)$ and $F_{n_\alpha}^{-}(x)$ denote the sample cdf's of X_α^{+}'s and X_α^{-}'s respectively. Define

$$H_N(x) = \sum_{\alpha=1}^{c} \rho_\alpha F_{n_\alpha}^{-}(x) + \sum_{\alpha=1}^{c} \rho_\alpha \nu_\alpha (F_{m_\alpha}^{+}(x) - F_{n_\alpha}^{-}(x)) \tag{2.1}$$

and

$$H(x) = \sum_{\alpha=1}^{c} \rho_\alpha F^{-(\alpha)}(x) + \sum_{\alpha=1}^{c} \rho_\alpha \nu_\alpha \Delta_\alpha(x) \tag{2.2}$$

where

$$\rho_\alpha = N_\alpha/N, \quad \nu_\alpha = m_\alpha/N_\alpha\,, \quad \Delta_\alpha(x) = F^{+(\alpha)}(x) - F^{-(\alpha)}(x). \tag{2.3}$$

Denote

$$H^*(\cdot x) = \sum_{\alpha=1}^{c} \rho_\alpha F^{-(\alpha)}(x) + \sum_{\alpha=1}^{c} \rho_\alpha p_\alpha \Delta_\alpha(x) \tag{2.4}$$

where

$$p_\alpha = E(\nu_\alpha) \quad \text{and} \quad E \quad \text{denotes the expectation.} \tag{2.5}$$

Let

(2.6) $$\mu_{2,\alpha} = E(\nu_\alpha - p_\alpha)^2; \qquad s_\alpha = (\nu_\alpha - p_\alpha)/\mu_{2,\alpha}^{\frac{1}{2}}.$$

Define

$$\tilde{E}(\cdot) = E[(\cdot)|\,|s_\alpha| \leqq \omega], \qquad \alpha = 1, \cdots, c,$$

where ω is a fixed positive constant, and similarly $\widetilde{\text{var}}(\cdot)$ and $\widetilde{\text{cov}}(\cdot)$. Note that ( )|[ ] stands for ( ) given [ ]. Denote

(2.7) $$a^*_{\alpha,r} = +1,$$

if the rth smallest observation from the ordered absolute values $|Z_{\alpha j}|$, $j = 1, \cdots, N_\alpha$; $\alpha = 1, \cdots, c$, is an X_α^+ observation and otherwise denote $a^*_{\alpha,r} = 0$. Denote $a^{**}_{\alpha,r} = -1$, if the rth smallest observation from the ordered absolute values $|Z_{\alpha j}|$, $j = 1, \cdots, N_\alpha$; $\alpha = 1, \cdots, c$, is an X_α^- observation and otherwise denote $a^{**}_{\alpha,r} = 0$. Then [cf. (1.1), (1.2), (1.3)], we can rewrite $\tau_N^{(\alpha)}, \tau_\alpha^+, \tau_\alpha^-, T_\alpha^+, T_\alpha^-$ as

(2.8) $$\tau_N^{(\alpha)} = \tau_\alpha^+ + \tau_\alpha^-$$

where

(2.9) $$\tau_\alpha^+ = m_\alpha T_\alpha^+ = \sum_{r=1}^N E_{N,r} a^*_{\alpha,r} = m_\alpha \int J_N[H_N(x)]\, dF^+_{m_\alpha}(x),$$

(2.10) $$\tau_\alpha^- = n_\alpha T_\alpha^- = \sum_{r=1}^N E_{N,r} a^{**}_{\alpha,r} = -n_\alpha \int J_N[H_N(x)]\, dF^-_{n_\alpha}(x),$$

and where

(2.11) $$E_{N,r} = J_N(r/N), \qquad r = 1, \cdots, N.$$

While J_N need be defined only at $1/N, \cdots, N/N$, it will be convenient to extend its domain of definition to (0, 1] by letting it have constant value on $(r/N, (r+1)/N]$. Let

$$J(H(x)) = \lim_{N\to\infty} J_N(H(x)).$$

Denote

(2.12) $$a_\alpha^+ = \int J[H(x)]\, dF^{+(\alpha)}(x), \qquad a_\alpha^- = -\int J[H(x)]\, dF^{-(\alpha)}(x);$$

(2.13) $$\tilde{a}_\alpha^+ = \tilde{E}(m_\alpha a_\alpha^+), \qquad \tilde{a}_\alpha^- = \tilde{E}(n_\alpha a_\alpha^-);$$

(2.14) $$L_0^{+(\alpha)} = \int J[H^*(x)]\, dF^{+(\alpha)}(x); \qquad L_0^{-(\alpha)} = \int J[H^*(x)]\, dF^{-(\alpha)}(x);$$

(2.15) $$L_{1,i}^{+(\alpha)} = \int \Delta_i(x) J'[H^*(x)]\, dF^{+(\alpha)}(x);$$

$$J'[H^*(x)] = dJ[H^*(x)]/dH^*(x);$$

(2.16) $$L_{1,i}^{-(\alpha)} = \int \Delta_i(x) J'[H^*(x)]\, dF^{-(\alpha)}(x);$$

(2.17) $$d_\alpha^+ = N_\alpha p_\alpha L_0^{+(\alpha)}, \qquad d_\alpha^- = -N_\alpha q_\alpha L_0^{-(\alpha)}, \qquad q_\alpha = 1 - p_\alpha;$$

(2.18) $$d_N^{(\alpha)} = d_\alpha^+ + d_\alpha^-;$$

(2.19) $$I_{(+\alpha;+i,+k)}(x, y) = F^{+(\alpha)}(x)[1 - F^{+(\alpha)}(y)] J'[H(x)] J'[H(y)];$$

$I^*_{(+\alpha;+i,+k)}(x, y)$ = the expression for $I_{(+\alpha;+i,+k)}(x, y)$ with H changed to H^*;

$$(2.20)\quad U_{(+\alpha;+i,+k)} = \int\int_{-\infty<x<y<\infty} I_{(+\alpha;+i,+k)}(x, y)\, dF^{+(i)}(x)\, dF^{+(k)}(y);$$
$$V_{(+\alpha;+i,+k)} = \int\int_{-\infty<y<x<\infty} I_{(+\alpha;+i,+k)}(y, x)\, dF^{+(i)}(x)\, dF^{+(k)}(y);$$

$U^*_{(+\alpha;+i,+k)}$ = the expression of $U_{(+\alpha;+i,+k)}$ with I changed to I^*, and $V^*_{(+\alpha;+i,+k)}$ = the expression for $V_{(+\alpha;+i,+k)}$ with I changed to I^*.

$$(2.21)\quad \begin{aligned} Nb^2_{\alpha^+} = {} & 2 \textstyle\sum_{i=1, i\neq\alpha}^{c} \lambda_i U_{(+i;+\alpha,+\alpha)} + 2 \sum_{i=1}^{c} \mu_i U_{(-i;+\alpha,+\alpha)} \\ & + (2/\lambda_\alpha) \textstyle\sum_{i=1,i\neq\alpha}^{c} \lambda_i^2 U_{(+\alpha;+i,+i)} + (2/\lambda_\alpha) \sum_{i=1}^{c} \mu_i^2 U_{(\alpha;-i,-i)} \\ & + (1/\lambda_\alpha) \textstyle\sum_{(3)} \lambda_i\lambda_k W_{(+\alpha;+i,+k)} + (1/\lambda_\alpha) \sum_{(2)} \mu_i\mu_k W_{(+\alpha;-i,-k)} \\ & + (2/\lambda_\alpha) \textstyle\sum_{(1)} \lambda_i\mu_k W_{(+\alpha;+i,-k)}\,; \qquad \lambda_i = m_i/N, \quad \mu_i = n_i/N, \end{aligned}$$

where (1) indicates the summation over all (i, k) with $i \neq \alpha$, the (2) over all (i, k) with $i \neq k$, and (3) over all (i, k) with $i \neq k, i \neq \alpha, k \neq \alpha$, and where $W = U + V$ with U and V having the same subscripts as W.

(2.22) $Nb^2_{\alpha^-}$ = the expression of $Nb^2_{\alpha^+}$ with λ's and μ's interchanged, and the subscripts of U's and V's written with opposite signs.

$$(2.23)\quad \begin{aligned} Nb^{++}_{\alpha,\alpha'} = {} & - \textstyle\sum_{i=1}^{c} \lambda_i[W_{(+\alpha;+i,+\alpha')} + W_{(+\alpha';+i,+\alpha)} - W_{(+i;+\alpha,+\alpha')}] \\ & - \textstyle\sum_{i=1}^{c} \mu_i[W_{(+\alpha;+i,+\alpha')} + W_{(+\alpha';+i,+\alpha)} - W_{(-i;+\alpha,+\alpha')}] \end{aligned}$$

and a similar expression for $Nb_{\alpha-,\alpha'-}$.

(2.24) Nb_{α^+,α'^-} = −(the right hand side of (2.23) with $+\alpha'$ changed to $-\alpha'$), and similar expressions for Nb_{α^+,α^-} and $Nb_{\alpha'^+,\alpha^-}$.

$$(2.25)\quad a_N^{(\alpha)} = m_\alpha a_\alpha^+ + n_\alpha a_\alpha^-$$

$$(2.26)\quad b_N^{(\alpha)2} = m_\alpha^2 b_\alpha^2 + n_\alpha^2 b_\alpha^2 + 2m_\alpha n_\alpha b_{\alpha^+,\alpha^-}$$

$$(2.27)\quad b_N^{(\alpha,\alpha')} = m_\alpha m_{\alpha'} b_{\alpha^+,\alpha'^+} + m_\alpha n_{\alpha'} b_{\alpha^+,\alpha'^-} + m_{\alpha'} n_\alpha b_{\alpha'^+,\alpha^-} + n_\alpha n_{\alpha'} b_{\alpha^-,\alpha'^-}\,.$$

$$(2.28)\quad \begin{aligned} \beta^2_{\alpha^+} = {} & 2N_\alpha\rho_\alpha p_\alpha^2 \textstyle\sum_{i=1,i\neq\alpha}^{c} \rho_i p_i U^*_{(+i;+\alpha,+\alpha)} \\ & + 2N_\alpha\rho_\alpha p_\alpha^2 \textstyle\sum_{i=1}^{c} \rho_i q_i U^*_{(-i;+\alpha,+\alpha)} \\ & + 2N_\alpha p_\alpha \textstyle\sum_{i=1,i\neq\alpha}^{c} \rho_i^2 p_i^2 U^*_{(+\alpha;+i,+i)} + 2N_\alpha p_\alpha \sum_{i=1}^{c} \rho_i^2 q_i^2 U^*_{(+\alpha;-i,-i)} \\ & + N_\alpha p_\alpha \textstyle\sum_{(3)} \rho_i\rho_k p_i p_k W^*_{(+\alpha;+i,+k)} + N_\alpha p_\alpha \sum_{(2)} \rho_i\rho_k q_i q_k W^*_{(+\alpha;-i,-k)} \\ & + 2N_\alpha p_\alpha \textstyle\sum_{(1)} \rho_i\rho_k p_i q_k W^*_{(+\alpha;+i,-k)} \\ & + N_\alpha^2[(L_0^{+(\alpha)})^2 \mu_{2,\alpha} + p_\alpha^2 \textstyle\sum_{i=1}^{c} \rho_i^2 \mu_{2,i} (L_{1,i}^{+(\alpha)})^2 + 2p_\alpha\rho_\alpha\mu_{2,\alpha} L_0^{+(\alpha)} L_{1,i}^{+(\alpha)}] \end{aligned}$$

where $W^* = U^* + V^*$ with U^* and V^* having the same subscripts as W^*.

(2.29) $\beta^2_{\alpha^-}$ = the expression for $\beta^2_{\alpha^+}$ with p's and q's interchanged; subscripts of U^* and V^* written with opposite signs; $L_0^{+(\alpha)}$ changed to $-L_0^{-(\alpha)}$; $L_{1,i}^{+(\alpha)}$ changed to $L_{1,i}^{-(\alpha)}$.

(2.30) $d^{xy}_{\alpha,\alpha'} = N_\alpha \rho_\alpha p_\alpha^x p^y_{\alpha'}$ (the expression for $b^{xy}_{\alpha,\alpha'}$ with λ_i, μ_i, U and V changed to $\rho_i p_i$, $\rho_i q_i$, U^* and V^* respectively), where $p_\alpha^x = p_\alpha$ or q_α as x is $+$ or $-$, and $p^y_{\alpha'} = p_{\alpha'}$ or $q_{\alpha'}$ as y is $+$ or $-$.

(2.31) $d_{\alpha^+,\alpha^-} = N_\alpha \rho_\alpha p_\alpha q_\alpha$ (the expression for b_{α^+,α^-} with λ_i, μ_i, U and V changed as in (2.30))

(2.32) $d_{\alpha'^+,\alpha^-} = N_\alpha \rho_\alpha p_{\alpha'} q_\alpha$ (the expression for $b_{\alpha'^+,\alpha^-}$ with λ_i, μ_i and U and V changed as in (2.30)).

$$\beta_{\alpha^+,\alpha^-} = d_{\alpha^+,\alpha^-} + N_\alpha^2 \mu_{2,\alpha} L_0^{+(\alpha)} L_0^{-(\alpha)} + N_\alpha^2 \mu_{2,\alpha} p_\alpha \rho_\alpha L_{1,\alpha}^{+(\alpha)} L_0^{-(\alpha)} - N_\alpha^2 \mu_{2,\alpha} q_\alpha \rho_\alpha L_{1,\alpha}^{-(\alpha)} L_0^{+(\alpha)} - N_\alpha^2 p_\alpha' q_\alpha \textstyle\sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha)} L_{1,i}^{-(\alpha)}. \tag{2.33}$$

$$\beta_{\alpha^+,\alpha'^-} = d_{\alpha^+,\alpha'^-} - N_\alpha N_{\alpha'} \mu_{2,\alpha} q_{\alpha'} \rho_\alpha L_{1,\alpha}^{-(\alpha')} L_0^{+(\alpha)} + N_\alpha N_{\alpha'} \mu_{2,\alpha'} p_\alpha \rho_{\alpha'} L_0^{-(\alpha')} L_{1,\alpha'}^{+(\alpha)} - N_\alpha N_{\alpha'} p_\alpha q_{\alpha'} \textstyle\sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha)} L_{1,i}^{-(\alpha')}. \tag{2.34}$$

$$\beta_{\alpha'^+,\alpha^-} = d_{\alpha'^+,\alpha^-} - N_\alpha N_{\alpha'} \mu_{2,\alpha'} q_\alpha \rho_{\alpha'} L_{1,\alpha'}^{-(\alpha)} L_0^{+(\alpha')} + N_\alpha N_{\alpha'} \mu_{2,\alpha} p_{\alpha'} \rho_\alpha L_{1,\alpha}^{+(\alpha')} L_0^{-(\alpha)} - N_\alpha N_{\alpha'} p_{\alpha'} q_\alpha \cdot \textstyle\sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha')} L_{1,i}^{-(\alpha)}. \tag{2.35}$$

$$\beta_{\alpha^-,\alpha'^-} = d_{\alpha^-,\alpha'^-} - N_\alpha N_{\alpha'} \mu_{2,\alpha} q_{\alpha'} \rho_\alpha L_{1,\alpha}^{-(\alpha')} L_0^{-(\alpha)} - N_\alpha N_{\alpha'} \mu_{2,\alpha'} q_\alpha \rho_{\alpha'} L_{1,\alpha'}^{-(\alpha)} L_0^{-(\alpha')} + N_\alpha N_{\alpha'} q_\alpha q_{\alpha'} \textstyle\sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{-(\alpha)} L_{1,i}^{-(\alpha')}. \tag{2.36}$$

$$\beta_{\alpha^+,\alpha'^+} = d_{\alpha^+,\alpha'^+} + N_\alpha N_{\alpha'} \mu_{2,\alpha} p_{\alpha'} \rho_\alpha L_{1,\alpha}^{+(\alpha')} L_0^{+(\alpha)} + N_\alpha N_{\alpha'} \mu_{2,\alpha'} p_\alpha \rho_{\alpha'} L_{1,\alpha'}^{+(\alpha)} L_0^{+(\alpha')} + N_\alpha N_{\alpha'} p_\alpha p_{\alpha'} \textstyle\sum_{i=1}^c \rho_i^2 \mu_{2,i} L_{1,i}^{+(\alpha)} L_{1,i}^{+(\alpha')}. \tag{2.37}$$

The methods used in the proofs for the asymptotic normality of $\tau_N^{(\alpha)}$'s are mainly adaptations of the methods of [18] and [7]. It is assumed that the sample sizes N_α tend to infinity in such a way that $N_\alpha = \rho_\alpha \cdot N$, $N \to \infty$.

III.3. Joint asymptotic normality.

THEOREM 3.1. *If*

(i) $E(\nu_\alpha) = p_\alpha \to p_{\alpha_0}$ *such that* $0 < p_{\alpha_0} < 1$,

(ii) $\mu_{2,\alpha} = E(\nu_\alpha - p_\alpha)^2 = O(1/N)$,

(iii) *for* m_α *such that* $|s_\alpha| \leqq \omega$ *for some fixed* $\omega > 0$,

$$\Pr(\underline{m_\alpha} = m_\alpha) = p(m_\alpha) = (N_\alpha(\mu_{2,\alpha})^{\frac{1}{2}})^{-1}\phi(s_\alpha) + o(1/N^{\frac{1}{2}})$$

where ϕ is the standard normal density function, and $s_\alpha = (\nu_\alpha - p_\alpha)/\mu_{2,\alpha}^{\frac{1}{2}}$ and, if for given $F^{+(\alpha)}(x)$, $F^{-(\alpha)}(x)$; λ_α, μ_α bounded away from zero and one,

(iv) *the conditions Ω_3 of Section 2, Part I, are satisfied then the random vector $(\tau_N^{(1)} - d_N^{(1)}, \cdots, \tau_N^{(c)} - d_N^{(c)})$ has a limiting normal distribution with zero mean vector and covariance matrix.*

$$\begin{aligned} &\operatorname{var}(\tau_N^{(\alpha)} - d_N^{(\alpha)}) \\ (3.1) \qquad &= \beta_N^{(\alpha)2} = \beta_{\alpha^+}^2 + \beta_{\alpha^-}^2 + 2\, d_{\alpha^+,\alpha^-} + 2N_\alpha^{\ 2}\mu_{2,\alpha}L_0^{+(\alpha)}L_0^{-(\alpha)} \\ &\quad - 2N_\alpha^{\ 2}q_\alpha\mu_{2,\alpha}p_\alpha L_{1,\alpha}^{-(\alpha)}L_0^{+(\alpha)} - 2N_\alpha^{\ 2}p_\alpha q_\alpha \textstyle\sum_{i=1}^c \rho_i^2\mu_{2,i}L_{1,i}^{+(\alpha)}L_{1,i}^{-(\alpha)} \end{aligned}$$

where $\beta_{\alpha^+}^2$, $\beta_{\alpha^-}^2$, d_{α^+,α^-}, $\mu_{2,i}$, $L_0^{+(\alpha)}$ and $L_0^{-(\alpha)}$, $L_{1,i}^{+(\alpha)}$ and $L_{1,i}^{-(\alpha)}$ are given by (2.28), (2.29), (2.33), (2.6), (2.14), (2.15) *and* (2.16) *respectively.*

$$\begin{aligned} &\operatorname{cov}(\tau_N^{(\alpha)} - d_N^{(\alpha)}, \tau_N^{(\alpha')} - d_N^{(\alpha')}) = \beta_N^{(\alpha,\alpha')} \\ &= d_{\alpha^+,\alpha'^+} + d_{\alpha^+,\alpha'^-} + d_{\alpha'^+,\alpha^-} + d_{\alpha^-,\alpha'^-} \\ &\quad + N_\alpha N_{\alpha'}p_{\alpha'}\rho_\alpha\mu_{2,\alpha}L_0^{+(\alpha)}L_{1,\alpha}^{+(\alpha)'} + N_\alpha N_{\alpha'}p_\alpha\rho_{\alpha'}\mu_{2,\alpha'}L_0^{+(\alpha')}L_{1,\alpha'}^{(+\alpha)} \\ &\quad + N_\alpha N_{\alpha'}p_\alpha p_{\alpha'} \textstyle\sum_{i=1}^c \rho_i^2\mu_{2,i}L_{1,i}^{+(\alpha)}L_{1,i}^{+(\alpha')} - N_\alpha N_{\alpha'}q_{\alpha'}\rho_\alpha\mu_{2,\alpha}L_0^{+(\alpha)}L_{1,\alpha}^{-(\alpha')} \\ (3.2) \qquad &\quad + N_{\alpha'}N_\alpha p_\alpha\rho_{\alpha'}\mu_{2,\alpha'}L_0^{-(\alpha')}L_{1,\alpha'}^{+(\alpha)} - N_\alpha N_{\alpha'}p_{\alpha'}q_{\alpha'} \textstyle\sum_{i=1}^c \rho_i^2\mu_{2,i}L_{1,i}^{+(\alpha)}L_{1,i}^{-(\alpha')} \\ &\quad - N_\alpha N_{\alpha'}q_\alpha\rho_{\alpha'}L_{1,\alpha'}^{-(\alpha)}L_0^{+(\alpha')}\mu_{2,\alpha'} + N_\alpha N_{\alpha'}p_{\alpha'}\rho_\alpha\mu_{2,\alpha}L_0^{-(\alpha)}L_{1,\alpha}^{+(\alpha')} \\ &\quad - N_\alpha N_{\alpha'}p_{\alpha'}q_\alpha \textstyle\sum_{i=1}^c \rho_i^2\mu_{2,i}L_{1,i}^{+(\alpha')}L_{1,i}^{-(\alpha)} \\ &\quad - N_\alpha N_{\alpha'}q_{\alpha'}\rho_\alpha L_0^{-(\alpha)}L_{1,\alpha}^{-(\alpha')}\mu_{2,\alpha} - N_\alpha q_\alpha N_{\alpha'}\rho_{\alpha'}\mu_{2,\alpha'}L_0^{-(\alpha')}L_{1,\alpha'}^{-(\alpha)} \\ &\quad + N_\alpha N_{\alpha'}q_\alpha q_{\alpha'} \textstyle\sum_{i=1}^c \rho_i^2\mu_{2,i}L_{1,i}^{-(\alpha)}L_{1,i}^{-(\alpha')}. \end{aligned}$$

REMARKS. (a) The Theorem 3.1 remains valid if the assumption (iii) is replaced by the assumption

(iii)′ $p(m_\alpha) = (1/N_\alpha\mu_{2,\alpha}^{\frac{1}{2}})[\phi(s_\alpha) + h(\phi(s_\alpha))] + o(1/N)$, where ϕ is the standard normal, density, $h(\phi)$ is a polynomial in ϕ whose coefficients involve inverse powers of N_α, and $s_\alpha = (\nu_\alpha - p_\alpha)/\mu_{2,\alpha}^{\frac{1}{2}}$.

(b) The assumptions (ii) and (iii) of Theorem 3.1 are satisfied if the random variable m_α has a binomial distribution with parameters N_α and p_α such that $p_\alpha \to p_{\alpha_0}$, $0 < p_{\alpha_0} < 1$.

(c) The assumptions (ii) and (iii) of Theorem 6.1 are also satisfied if m_α has a hypergeometric distribution, and the size of the population N_α^* and the size of the sample N_α, are such that $N^* = O(N^{k+\delta})$ for $k \geqq 2$ and some $\delta > 0$, for then (cf. [7]),

$$p(m_\alpha) = \binom{N_\alpha}{m_\alpha}p_\alpha^{m_\alpha}q_\alpha^{N_\alpha - m_\alpha} + o(1/N_\alpha^{\ k-2}).$$

To prove this theorem, we first consider the case when the sample sizes m_α, n_α; $\alpha = 1, \cdots, c$, are non-random instead of random. In such a case the random variables $(X_{\alpha 1}^+, \cdots, X_{\alpha m_\alpha}^+)$ and $(X_{\alpha 1}^-, \cdots, X_{\alpha n_\alpha}^-)$ can be regarded as constituting $2c$ independent samples from the distribution functions $F^{+(\alpha)}(x)$ and

$F^{-(\alpha)}(x)$ respectively, $\alpha = 1, \cdots, c$; and we have the following specializations of the conditional analogues of Theorem 3.1, the proofs of which follow by proceeding exactly as in Theorem 6.1 of Puri (1964), and are therefore omitted.

(3A) NON-RANDOM CASE.

LEMMA 3A.1. *If assumption* (iv) *of Theorem* 3.1 *is satisfied, then the random vector* $N^{\frac{1}{2}}(T_1^+ - a_1^+, \cdots, T_c^+ - a_c^+)$ *where* T^+*'s and* a^+*'s are defined by* (2.9) *and* (2.12) *respectively, has a limiting normal distribution with zero mean vector and variance-covariances given by* $Nb_{\alpha^+}^2$ *and* Nb_{α^+,α'^+} *where* $b_{\alpha^+}^2$ *and* b_{α^+,α'^+} *are defined in* (2.21) *and* (2.23) *respectively.*

LEMMA 3A.2. *If assumption* (iv) *of Theorem* 3.1 *is satisfied, then the random vector* $N^{\frac{1}{2}}(T_1^- - a_1^-, \cdots, T_c^- - a_c^-)$ *where* T^-*'s and* a^-*'s are defined by* (2.10) *and* (2.12) *respectively, has a limiting normal distribution with zero mean vector and variance-covariances given by* $Nb_{\alpha^-}^2$ *and* Nb_{α^-,α'^-} *where* $b_{\alpha^-}^2$ *and* b_{α^-,α'^-} *are defined in* (2.22) *and* (2 .23) *respectively.*

THEOREM 3A.2. *Under the assumptions of Lemma* 3A.1, *the random vector* $W = (W^{(1)}, \cdots, W^{(c)})$ *where*

$$W^{(\alpha)} = N^{-\frac{1}{2}}(m_\alpha T_\alpha^+ + n_\alpha T_\alpha^- - m_\alpha a_\alpha^+ - n_\alpha a_\alpha^-) \tag{3.3}$$

has a limiting normal distribution with zero mean vector and variance-covariances given by $N^{-1}b_N^{(\alpha)2}$ *and* $N^{-1}b_N^{(\alpha,\alpha')}$ *where* $b_N^{(\alpha)2}$ *and* $b_N^{(\alpha,\alpha')}$ *are defined in* (2.26) *and* (2.27) *respectively.*

We have thus established the joint asymptotic normality of the random variables $\tau_N^{(\alpha)}$'s when the sample sizes m_α, n_α ($\alpha = 1, \cdots, c$) are non-random. We now drop the assumption that m_α and n_α are non-random. We assume that m_α, n_α are random variables which satisfy the assumptions (i) to (iii) of Theorem 3.1.

(3B) RANDOM CASE. We shall need the following lemmas:

LEMMA 3B.1. *Under the assumptions* (ii) *and* (iii) *of Theorem* 3.1

$$\tilde{\mu}_{1,\alpha} = E\{(\nu_\alpha - p_\alpha) \mid |s_\alpha| \leqq \omega\} = o(N^{-\frac{1}{2}}) \tag{3.4}$$

$$|\mu_{2,\alpha} - \tilde{\mu}_{2,\alpha}| = O(\omega e^{-\omega^2/2}/N) + o(N^{-1}), \tag{3.5}$$

where $\tilde{\mu}_{2,\alpha} = E\{(\nu_\alpha - p_\alpha)^2 \mid |s_\alpha| \leqq \omega\}$.

The proof of this lemma is the same as in ([7], p. 37) and is therefore omitted

LEMMA 3B.2. *Let* $\{X_N\}$ *be a sequence of random variables and* $\{r_N\}$ *a sequence of numbers. If* $X_N = r_N + O_p(t_N)$ *where* $t_N \to 0$ *and* $r_N \to r$ *as* $N \to \infty$, *and* $h(x)$ *is a function admitting continuous* $(j + 1)$st *derivative in some interval containing* r, *then*

$$\begin{aligned} h(X_N) &= h(r_N) + \textstyle\sum_{i=1}^{j} h^{(i)}(r_N)(X_N - r_N)^i/i! \\ &\quad + [(X_N - r_N)^{j+1}/(j+1)!]\, h^{(j+1)}(eX_n + (1-e)r_N), \qquad 0 < e < 1, \end{aligned} \tag{3.6}$$

$$h(X_n) = h(r_N) + \textstyle\sum_{i=1}^{j} h^{(i)}(r_N)(X_N - r_N)^i/i! + o_p(t_N^{\,j}). \tag{3.7}$$

PROOF. (3.6) is just the Taylor expansion of $h(X_N)$ and (3.7) follows as a special case of the Corollary 3 of Mann and Wald [15].

LEMMA 3B.3. *Under the assumption* (ii) *of Theorem* 3.1

$$(3.8)\qquad J(H) = J(H^*) + J'(H^*)\sum_{i=1}^{c} \rho_i\Delta_i(x)(\nu_i - p_i) + o_p(N^{-\frac{1}{2}}),$$

$$(3.9)\qquad J'(H(x))J'(H(y)) = J'(H^*(x)J'(H^*(y)) + o_p(1).$$

PROOF. The proof follows by noting that $H(x) = H^*(x) + O_p(N^{-\frac{1}{2}})$, and applying Lemma 3B.2.

LEMMA 3B.4. *If the assumptions* (ii), (iii) *and* (iv) *of Theorem* 3.1 *are satisfied, then for large N*

$$(3.10)\qquad a_\alpha{}^*X = N_\alpha d_\alpha L_0^{X(\alpha)} + N_\alpha(\nu_\alpha - p_\alpha)L_0^{X(\alpha)} + N_\alpha e_\alpha \sum_{i=1}^{c} \rho_i(\nu_i - p_i)L_{1,i}^{X(\alpha)} + O(N^{\frac{1}{2}}),$$

$$(3.11)\qquad b_\alpha{}^*X = \beta_\alpha^2 X - N_\alpha^2[(L_0^{X(\alpha)})^2\mu_{2,\alpha} + d_\alpha^2 \sum_{i=1}^{c} \rho_i^2\mu_{2,i}(L_{1,i}^{X(\alpha)})^2 + 2\rho_\alpha e_\alpha\mu_{2,\alpha}L_0^{X(\alpha)}L_{1,i}^{X(\alpha)}] + O(N)$$

where

$$a_\alpha{}^*X = m_\alpha a_{\alpha^+},\quad b_\alpha{}^*X = m_\alpha^2 b_{\alpha^+}^2,\quad d_\alpha = e_\alpha = p_\alpha \quad \textit{if}\quad X \textit{ is } +;$$

$$a_\alpha{}^*X = n_\alpha a_\alpha^-,\quad b_\alpha{}^*X = n_\alpha^2 b_{\alpha^-}^2,\quad d_\alpha = e_\alpha = q_\alpha \quad \textit{if}\quad X \textit{ is } -.$$

(3.12) $m_\alpha m_{\alpha'} b_{\alpha^+,\alpha'^+} = N_\alpha \rho_{\alpha'} p_\alpha p_{\alpha'}$ [*the expression for* Nb_{α^+,α'^+} (cf. (2.23) *with* λ_i, μ_i, U *and* V *changed to* $\rho_i p_i$, $\rho_i q_i$, U^* *and* V^* *respectively*] $+ o(N)$;

(3.13) $n_\alpha n_{\alpha'} b_{\alpha^-,\alpha'^-} = N_\alpha \rho_{\alpha'} q_\alpha q_{\alpha'}$ [*the expression for* Nb_{α^-,α'^-} (cf. (2.23)) *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$;

(3.14) $m_{\alpha'} n_\alpha b_{\alpha'^+,\alpha^-} = N_\alpha \rho_{\alpha'} p_{\alpha'} q_\alpha$ [*the expression for* $Nb_{\alpha'^+,\alpha^-}$ *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$;

(3.15) $m_\alpha n_{\alpha'} b_{\alpha^+,\alpha'^-} = N_\alpha \rho_{\alpha'} p_\alpha q_{\alpha'}$ [*the expression for* Nb_{α^+,α'^-} *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$;

(3.16) $m_\alpha n_\alpha b_{\alpha^+,\alpha^-} = N_\alpha \rho_\alpha p_\alpha q_\alpha$ [*the expression for* Nb_{α^+,α^-} *with* λ_i, μ_i, U *and* V *changed as in* (3.12)] $+ o(N)$.

PROOF. Apply Lemma 3B.3 and make use of the facts that $\nu_\alpha^2\nu_i = p_\alpha^2 p_i + o(1)$; $\nu_\alpha^2(1 - \nu_i) = p_\alpha^2 q_i + o(1)$ and similar expressions for $\nu_\alpha\nu_i^2$, $\nu_\alpha(1 - \nu_i)^2$, $\nu_\alpha\nu_i\nu_k, \nu_\alpha(1 - \nu_i)(1 - \nu_k)$ and $\nu_\alpha\nu_i(1 - \nu_k)$.

LEMMA 3B.5. *If the hypothesis of Lemma* 3B.4 *hold, then for large* N_α, $\alpha = 1, \cdots, c$,

$$(3.17)\qquad (d_\alpha - m_\alpha a_{\alpha^+})/m_\alpha b_{\alpha^+} = -\sum_{i=1}^{c} s_i v_i/I_1 + o(1);$$

$$(3.18)\qquad \beta_{\alpha^+}/m_\alpha b_{\alpha^+} = I_2/I_1 + o(1);$$

where

$$N_\alpha\mu_{2,\alpha} = p_\alpha q_\alpha c_\alpha^{\;2}, \qquad c_\alpha = O(1);$$
$$v_i = p_\alpha(\rho_i\rho_\alpha p_i q_i)^{\frac{1}{2}} c_i L_{1,i}^{+(\alpha)}), \qquad i = 1, \cdots, c; \quad i \neq \alpha;$$
$$v_\alpha = (p_\alpha q_\alpha)^{\frac{1}{2}}(c_\alpha L_0^{\;+(\alpha)} + p_\alpha\rho_\alpha c_\alpha L_{1,\alpha}^{+(\alpha)};$$
$$\begin{aligned} I_1^{\;2} = {} & 2\rho_\alpha p_\alpha^{\;2} \textstyle\sum_{i=1,i\neq\alpha}^{c} \rho_i p_i U^*_{(+i;+\alpha,+\alpha)} + 2\rho_\alpha p_\alpha^{\;2} \sum_{i=1}^{c} \rho_i q_i U^*_{(-i;+\alpha,+\alpha)} \\ & + 2p_\alpha \textstyle\sum_{i=1,i\neq\alpha}^{c} \rho_i^{\;2} p_i^{\;2} U^*_{(+\alpha;+i,+i)} + 2p_\alpha \sum_{i=1}^{c} \rho_i^{\;2} q_i^{\;2} U^*_{(+\alpha;-i,-i)} \\ & + p_\alpha \textstyle\sum_{(3)} \rho_i\rho_k p_i p_k W^*_{(+\alpha;+i,-k)} + p_\alpha \sum_{(2)} \rho_i\rho_k q_i q_k W^*_{(+\alpha;-i,-k)} \\ & + 2p_\alpha \textstyle\sum_{(1)} \rho_i\rho_k p_i q_k W^*_{(+\alpha;+i,-k)} \end{aligned}$$

where W^* *is defined in* (2.28). *and* $I_2^{\;2} = I_1^{\;2} + \sum_{i=1}^{c} v_i^{\;2}$.

The proof of this lemma involves straightforward algebraic computations and is therefore omitted.

LEMMA 3B.6. *If*

(i) $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$ *for some* $\lambda_0 \leqq 1/2c$,
$0 < \mu_0 \leqq \mu_1, \cdots, \mu_c \leqq 1 - \mu_0 < 1$ *for some* $\mu_0 \leqq 1/2c$,

(ii) *the assumptions* (ii), (iii), *and* (iv) *of Theorem* 3.1 *hold*,

(iii) $E(T_{\alpha^+} \mid \lambda_1, \cdots, \lambda_c)$, $E(T_{\alpha^-} \mid \lambda_1, \cdots, \lambda_c)$, $\operatorname{var}(T_{\alpha^+} \mid \lambda_1, \cdots, \lambda_c)$, $\operatorname{var}(T_{\alpha^-} \mid \lambda_1, \cdots, \lambda_c)$ *exist, then for large* N_α *such that* $\omega(\mu_{2,\alpha})^{\frac{1}{2}} < p_\alpha q_\alpha$,

$$\tilde{E}(\tau_{\alpha\mp}) = d_{\alpha\mp} + o(N^{\frac{1}{2}}); \tag{3.19}$$
$$\tilde{E}(\tau_N^{\;(\alpha)}) = d_N^{\;(\alpha)} + o(N^{\frac{1}{2}}); \tag{3.20}$$
$$\tilde{\operatorname{var}}(\tau_{\alpha\mp}) = B^2_{\alpha\mp} + O(N\omega e^{-\omega^2/2}) + o(N); \tag{3.21}$$
$$\tilde{\operatorname{cov}}(\tau_{\alpha^+}, \tau_{\alpha^-}) = \beta_{\alpha^+,\alpha^-} + O(N\omega e^{-\omega^2/2}) + o(N); \tag{3.22}$$
$$\tilde{\operatorname{var}}(\tau_N^{\;(\alpha)}) = \beta^2_{\alpha^+} + \beta^2_{\alpha^-} + 2\beta_{\alpha^+,\alpha^-} + O(N\omega e^{-\omega^2/2}) + o(N); \tag{3.23}$$
$$\begin{aligned} \tilde{\operatorname{cov}}(\tau_N^{\;(\alpha)}, \tau_N^{\;(\alpha')}) = {} & \beta_{\alpha^+,\alpha'^+} + \beta_{\alpha^+,\alpha'^-} + \beta_{\alpha'^+,\alpha^-} + \beta_{\alpha^-,\alpha'^-} \\ & + O(N\omega e^{-\omega^2/2}) + o(N). \end{aligned} \tag{3.24}$$

NOTE. The quantities d_{α^+}, d_{α^-}, $d_N^{\;(\alpha)}$, $\beta^2_{\alpha^+}$, $\beta^2_{\alpha^-}$, $\beta_{\alpha^+,\alpha^-}$, $\beta_{\alpha^+,\alpha'^+}$, $\beta_{\alpha^+,\alpha'^-}$, $\beta_{\alpha'^+,\alpha^-}$, $\beta_{\alpha^-,\alpha'^-}$ are all defined in Section 2.

The proof of the lemma follows by straightforward computations.

LEMMA 3B.7. *Under the assumptions of Theorem* 3.1, *the random vector* $(\tau_1^{\;+} - d_1^{\;+}, \cdots, \tau_c^{\;+} - d_c^{\;+})$ *has a limiting normal distribution with zero mean vector and covariance matrix*

$$\operatorname{var}(\tau_{\alpha^+} - d_{\alpha^+}) = \beta^2_{\alpha^+}, \qquad \operatorname{cov}(\tau_{\alpha^+} - d_{\alpha^+}, \tau_{\alpha'^+} - d_{\alpha'^+}) = \beta_{\alpha^+,\alpha'^+} \tag{3.25}$$

where $\beta^2_{\alpha^+}$ *and* $\beta_{\alpha^+,\alpha'^+}$ *are given by* (2.28) and (2.39) *respectively.*

The proof of this lemma follows from Theorem 3.1 of [7] as does Lemma 3A.1 (or Theorem 6.1 of [18]) from Theorem 1 of [3].

Now proceeding as in Lemma 3B.7, we get the following lemma and the main Theorem 3.1.

LEMMA 3B.8. *Under the assumptions of Theorem* 3.1, *the random vector* $(\tau_1^- - d_1^-, \cdots, \tau_c^- - d_c^-)$ *has a limiting normal distribution with zero mean vector and covariance matrix*

$$(3.26) \quad \text{var}\,(\tau_{\alpha}^- - d_{\alpha}^-) = \beta_{\alpha}^{2}{}^-, \qquad \text{cov}\,(\tau_{\alpha}^- - d_{\alpha}^-, \tau_{\alpha'}^- - d_{\alpha'}^-) = \beta_{\alpha^-,\alpha'^-}$$

where $\beta_{\alpha}^{2}{}^-$ *and* $\beta_{\alpha^-,\alpha'^-}$ *are given by* (2.29) *and* (2.38) *respectively.*

Now as a special case, assume that the random variables $Z_{\alpha r}$, $r = 1, \cdots, N_\alpha$; $\alpha = 1, \cdots, c$, are distributed with distributions $\Pi_\alpha(z) = \Pi(z + \mu_\alpha N^{-\frac{1}{2}})$ where Π is symmetric about zero. Then

$$F^{+(\alpha)}(x) = [\Pi(x + \mu_\alpha N^{-\frac{1}{2}}) - \Pi(\mu_\alpha N^{-\frac{1}{2}})][1 - \Pi(\mu_\alpha N^{-\frac{1}{2}})]^{-1} \qquad \text{if } x \geqq 0$$
$$= 0 \qquad \text{otherwise;}$$
$$F^{-(\alpha)}(x) = [\Pi(\mu_\alpha N^{-\frac{1}{2}}) - \Pi(-x + \mu_\alpha N^{-\frac{1}{2}})][\Pi(\mu_\alpha N^{-\frac{1}{2}})]^{-1} \qquad \text{if } x \geqq 0$$
$$= 0 \qquad \text{otherwise,}$$

where $\mu_\alpha = \mu_{ij} = \theta_i - \theta_j$.

Then the following corollary is an immediate consequence of Theorem 3.1.

COROLLARY 3.1. *If*

(i) *the conditions of Theorem* 3.1 *and the condition* Ω_4 *of Section* 2, *Part* I, *are satisfied,*

(ii) $\Pi_\alpha(z) = \Pi(z + \mu_\alpha N^{-\frac{1}{2}})$, $\alpha = 1, \cdots, c$, *and* Π *is symmetric about zero, then the* $c = K(K-1)/2$ *random variables* $N_\alpha^{-\frac{1}{2}}\tau_N^{(\alpha)}$, $\alpha = 1, \cdots, c$, *are distributed in the limit, as* $N \to \infty$, *as independent* $N(\eta^{(\alpha)}, A^2)$ *where*

$$(3.27) \qquad \eta^{(\alpha)} = 4\rho_\alpha^{\frac{1}{2}}\mu_\alpha\left(\int_0^\infty J'[2\Pi(x) - 1]\pi(x)\, d\Pi(x)\right)$$

and

$$(3.28) \qquad A^2 = \int_0^1 J^2(u)\, du.$$

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