EXPANSIONS OF t DENSITIES AND RELATED COMPLETE INTEGRALS[1]

By James M. Dickey

Yale University

1. Introduction and summary. A class of alternatives is here presented to Fisher's (1925) expansion of Student's t density function. These expansions involve Appell's polynomials; and hence, recurrence schemes are available for the coefficients.

Complete integrals of products of t densities are of interest as Behrens-Fisher densities (viewed as Bayesian posterior distributions: Jeffreys, 1940; Patil, 1964) and as moments of Bayesian posterior distributions (Anscombe, 1963; Tiao and Zellner, 1964). Asymptotic expansions of complete integrals, obtained by term-by-term integration of these expansions, are favorably compared with those obtained from Fisher's expansion. Although expansions of complete integrals of products of *multivariate* t densities can be developed from these expansions by the methods of Tiao and Zellner, the resulting coefficients are practically as complicated as the Tiao and Zellner coefficients; methods will be published soon (Dickey, 1965) for reducing the dimensionality of such integrals for quadrature.

The paper concludes with a numerical study of the integral expansions.

2. Expansions of t densities. In 1880, Appell obtained the power series representation,

$$e^{-x}(1 - \epsilon x)^{-1/\epsilon} = \textstyle\sum_0^\infty A_p(x)\epsilon^p, \qquad |\epsilon x| < 1, \tag{1}$$

considered as the exponential transformation of the power series for $-x - \epsilon^{-1} \log (1 - \epsilon x)$. Since the left-hand member of (1) satisfies the differential equation, $(1 - \epsilon x)\, dy/dx = \epsilon y$, with $y(0) = 1$, the polynomials A_p, sometimes called Appell's polynomials (Erdélyi, 1953, p. 256), satisfy the recurrence relation, $A_p'(x) = x[A_{p-1}(x) + A'_{p-1}(x)]$, with $A_0 \equiv 1$. Hence,

$$A_p(x) = x^p \textstyle\sum_{s=0}^p B_{p,s}x^s, \tag{2}$$

where

$$B_{0,0} = 1, \tag{3a}$$

and for $p > 0$,

$$(p + s)B_{p,s} = (p + s - 1)B_{p-1,s} + B_{p-1,s-1}. \tag{3b}$$

If $p > 0$, $B_{p,0} = 0$.

By expanding the right-hand side of $dy/dx = (1 - \epsilon x)^{-1}\epsilon xy$, obtain the recurrence relation, $A_p'(x) = xA_{p-1}(x) + x^2A_{p-2}(x) + \cdots$. Hence for $p > 0$,

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TABLE 1

Coefficients of the Appell polynomials
$A_p(x) = x^p \sum_0^p B_{p,s} x^s$

p	s = 0	1	2	3	4	5	6
0	1						
1	0	1/2					
2	0	1/3	1/8				
3	0	1/4	1/6	1/48			
4	0	1/5	13/72	1/24	1/384		
5	0	1/6	11/60	17/288	1/144	1/3840	
6	0	1/7	29/160	59/810	7/576	1/1152	1/46080

(3c) $$(p + s)B_{p,s} = -B_{p-1,s-1} + B_{p-2,s-1} - \cdots .$$

The coefficients $B_{p,\nu}$ for $p \leqq 6$ appear in Table 1.

We apply (1) to the kernel of the density of the t distribution with $\nu > 0$ degrees of freedom. With $\epsilon = 2/(\nu + 1)$, $h = (\nu + 1)/\nu$, and $x = \frac{1}{2}ht^2$. We have

(4) $$(1 + \nu^{-1}t^2)^{-(\nu+1)/2} = \exp(-\tfrac{1}{2}ht^2) \textstyle\sum_0^\infty A_p(-\tfrac{1}{2}ht^2)2^p(\nu + 1)^{-p}, \quad t^2 < \nu.$$

More generally, given $n > 0$, and $\rho > 0$, with $h = (n + \rho)/\nu$,

(5) $$(1 + \nu^{-1}t^2)^{-(\nu+1)/2} = \exp(-\tfrac{1}{2}ht^2) \textstyle\sum_0^\infty P_q[-\tfrac{1}{2}(n/\nu)t^2]n^{-q}, \quad t^2 < \nu,$$

where $\sum_q P_q(x)n^{-q}$ is the product of the expansion of $e^{-x}(1 - 2n^{-1}x)^{-n/2}$ according to equation (1), times the binomial series for $(1 - 2n^{-1}x)^{(n-\nu-1)/2}$, times the series for $e^{-\rho x/n}$.

Depending on the purpose of the expansion, n and ρ can be chosen so that the sum of the first few terms on the right-hand side in equation (5) (for example, the first term $\exp(-\frac{1}{2}ht^2)$) approximates $(1 + \nu^{-1}t^2)^{-(\nu+1)/2}$ in some sense. Fisher's (1925) expansion of Student's t distribution is based on the choice $n = \nu$, $\rho = 0$ $(h = 1)$.

For some applications it is desirable that, as in Fisher's expansion, the normalizing constant of the t density,

$$[B(\tfrac{1}{2}, \tfrac{1}{2}\nu + \tfrac{1}{2})\nu^{\frac{1}{2}}]^{-1} = 1/\textstyle\int_{-\infty}^{\infty} (1 + \nu^{-1}t^2)^{-(\nu+1)/2}\, dt,$$

where B denotes the complete beta function, be expanded and included as a factor to yield an approximating normal density with correction terms. Thus, by term-by-term integration of the right-hand member of (5), followed by formal division of series, we obtain an asymptotic expansion for the t density,

(6) $$[B(\tfrac{1}{2}, \tfrac{1}{2}\nu + \tfrac{1}{2})\nu^{\frac{1}{2}}]^{-1}(1 + \nu^{-1}t^2)^{-(\nu+1)/2} \sim (h/2\pi)^{\frac{1}{2}} \exp(-\tfrac{1}{2}ht^2) \textstyle\sum_0^\infty Q_p[-\tfrac{1}{2}(n/\nu)t^2]n^{-p},$$

where

TABLE 2

Coefficients of the Quotient polynomials

$Q_p(x) = \sum_0^{2p} V_{p,r}x^r$

p	r: 0 6 12	1 7	2 8	3 9	4 10	5 11
0	1.000000					
1	−.7500000	.000000	1.000000			
2	−.2187500	.000000	−.7500000	1.333333	.5000000	
3	.07031282	.000000	−.2187500	−1.000000	1.625000	1.333333
	.1666667					
4	.02880504	.000000	−.07031282	−.2916667	−1.609375	2.200000
	2.763889	.6666666	.04166667			
5	.05888299	.000000	.02880504	−.09375043	−.4726565	−2.691667
	3.130208	5.366666	1.857639	.2222222	.008333333	
6	−.03316466	.000000	.05888299	.03840671	−.1262231	−.7937505
	−4.643663	4.597024	10.17422	4.495062	.7715277	.05555555
	.00138889					

$$Q_q(x) = P_q(x) - \pi^{-\frac{1}{2}} \textstyle\sum_{p=0}^{q-1} Q_p(x)P_{q-p}(\Gamma).$$

$P_q(\Gamma)$ denotes the polynomial $P_q(x)$ with the powers x^r replaced by the quantities $[-(1 + \rho/n)]^{-r}\Gamma(r + \frac{1}{2})$. The coefficients of $Q_p(x)$, based on the Appell expansion (4) ($n = \nu + 1$, $\rho = 0$), are given in Table 2 for $p \leqq 6$.

Trivial modifications to equations (4), (5), and (6) yield expansions of a multivariate $-t$ density: a quadratic form replaces t^2; and $\nu + k$, where k is the dimensionality, replaces $\nu + 1$. The quotient series for the multivariate analogue of (6) is given by

$$Q_q(x) = P_q(x) - [\Gamma(\tfrac{1}{2}k)]^{-1} \textstyle\sum_{p=0}^{q-1} Q_p(x)P_{q-p}(\Gamma_k),$$

where Γ_k indicates substitution of $[-(1 + \rho/n)]^{-r}\Gamma(r + \frac{1}{2}k)$ for x^r. Tiao and Zellner (1964) make use of the multidimensional analogue of essentially Fisher's expansion.

3. Complete integrals. The expansions (4), (5), (6) interest us as tools to calculate integrals of the form,

$$(7) \qquad \int_{-\infty}^{\infty} g(t)\textstyle\prod_{k=1}^{K}[1 + \nu_k^{-1}l_k(t - x_k)^2]^{-(\nu_k+1)/2}\, dt, \qquad \nu_k > 0,$$

where $g(t)$ is a polynomial of low degree and the l_k's permit differences in scale.

With $K = 2$, (7) can be interpreted as the normalizing constant and moments of the posterior distribution for a normal mean μ, obtained as follows. One observes the sufficient statistics $\bar{y}$, s, $N - 1$ in the presence of an independent joint prior distribution for μ and σ^2, respectively Student-t (parameters ν_1, l_1, x_1) and $c/\chi_d^2(\nu_2 = d + N - 1, l_2 = \nu_2 N/(c + Ns - s), x_2 = \bar{y})$.

Again with $K = 2$, interpreting (7) as a convolution of two centered t densities, we obtain the density of the Behrens-Fisher random variable, $d = t_1 \sin \theta - t_2 \cos \theta$, where t_1 and t_2 are independent Student-t variables with degrees of freedom ν_1 and ν_2. The appropriate substitutions are

$$\begin{aligned} & x_1 - x_2 = d, \\ (8) \qquad & l_1 = 1/\sin^2 \theta, \qquad l_2 = 1/\cos^2 \theta, \\ & g \equiv [B(\tfrac{1}{2}, \tfrac{1}{2}\nu_1 + \tfrac{1}{2})(\nu_1/l_1)^{\frac{1}{2}} \cdot B(\tfrac{1}{2}, \tfrac{1}{2}\nu_2 + \tfrac{1}{2})(\nu_2/l_2)^{\frac{1}{2}}]^{-1}. \end{aligned}$$

With $g \equiv h^{K/2}$, $\nu_k = \nu$, and $l_k = l$, $k = 1, \cdots, K$, there is the obvious interpretation of the integrand in (7) as the likelihood function for a sample of size K from a t distribution. In practice, K is likely to be too large for one to easily make use of products of K many series.

By applying the expansion in equation (5) (say) to each of the K factors in the integrand of (7), completing the square in t in the exponent, and then integrating the resulting expansion term-by-term, we obtain an asymptotic expansion for (7),

$$(9) \qquad C\Sigma_{q_1, \cdots, q_K = 0}^{\infty} N_{g, q_1, \cdots, q_K} n_1^{-q_1} \cdots n_K^{-q_K},$$

where

$$C = (2\pi/\bar{h})^{\frac{1}{2}} \exp [-\tfrac{1}{2}\bar{h}^{-1}\Sigma_{k<j} h_k h_j (x_k - x_j)^2]$$

with $h_k = l_k(n_k + \rho_k)/\nu_k$ and $\bar{h} = \Sigma h_k$, and where $N_{g, q_1, \cdots, q_K}$ is the expectation,

$$E \; g(y) \pi_k P_{k, q_k}[-\tfrac{1}{2}(n_k/\nu_k) l_k (y - x_k)^2],$$

given that y is normally distributed with variance $1/\bar{h}$ and mean $\bar{x} = \Sigma h_k x_k/\bar{h}$. (Recall that if r is a positive integer, $E(y - \bar{x})^{2r} = \pi^{-\frac{1}{2}} 2r\Gamma(r + \tfrac{1}{2}) = 1 \cdot 3 \cdot 5 \cdots (2r - 3)(2r - 1)\bar{h}^{-r}$.)

Starkey (1938) has demonstrated the asymptotic property for expansions of the Behrens-Fisher densities based on Fisher's expansion. Her method of proof applies in the more general context. See also Wallace (1958).

4. Numerical results.

4.1. *One t-kernel factor.* A numerical study of the approximation to (7) by the first few terms of (9) began with consideration of the following simple example.

$$(10) \qquad \int_{-\infty}^{\infty} t^{2r}(1 + \nu^{-1}t^2)^{-(\nu+1)/2} \, dt \doteqdot S_p,$$

S_p a partial sum of the form, $S_p = \sum_0^p \delta_q$. The example (10) was chosen largely because of the availability of the closed form, $B(r + \tfrac{1}{2}, \tfrac{1}{2}\nu - r)\nu^{r+\frac{1}{2}}$, for the left-hand side, by which nominal true values T were calculated. The relative error $R_p = (S_p - T)/T$ was taken as a measure of accuracy.

Using the Appell expansion (4), we have

$$\delta_p = 2^p(\nu + 1)^{-p}((\nu + 1)/2\nu)^{(2r+1)/2} \sum_{s=0}^{p} B_{p,s}(-1)^{p+s}\Gamma(r + p + s + \tfrac{1}{2}),$$

TABLE 3

Accuracy of the Appell expansion of the moments of a t kernel
(d Ee denotes $d \times 10^e$)

ν	$r = 0$		$r = 1$		$r = 2$	
	p^*	R_{p^*}	p^*	R_{p^*}	p^*	R_{p^*}
1	7	−3 E-3				
2	7	−6 E-5				
3	·	−3 E-6	9	−.08		
4	·	3 E-7	9	−.02		
5	·	3 E-7	·	−5 E-3	10	−.17
6	$0 < R_{p^*} < 5$ E-7		·	−6 E-4	10	−.03
7	(round-off accuracy		·	−2 E-4	·	−8 E-3
8	attained)			−6 E-5	·	−2 E-3
9	·			−2 E-5	·	−7 E-4
10	·			−9 E-6		−2 E-4
11	·			−4 E-6		−8 E-5
12	7			−2 E-6		−3 E-5
13	9			−8 E-7		−1 E-5
14	9			−2 E-7		−4 E-6
15	·			−1 E-7	·	−1 E-6
16	·			2 E-7	·	−1 E-7
17	·		$0 < R_{p^*} < 5$ E-7 (round-		·	2 E-7
18			off accuracy attained)		10	5 E-7
19					11	−2 E-6
20					11	−7 E-7
21					·	−4 E-7
22					·	−4 E-8
23					·	−2 E-8
24						2 E-7
25					$0 < R_{p^*} < 5$ E-7 (round-	
26					off accuracy attained)	
·						
·						
·						·
·						·
65						·

by which the first 16 terms δ_q were calculated for the values $r = 0, 1, 2$ and $\nu = 2r + 1(1)35(5)65$. For each r and ν, the absolute value of δ_p was found to decrease monotonically in p until a term δ_{p^*}, $p^* = p^*(\nu, r)$, for which $|\delta_{p^*}|$ and $|R_{p^*}|$ were minima, and after which $|\delta_p|$ and $|R_p|$ increased astronomically. Hence, an appealing stopping rule is to retain terms up to and including the smallest term in absolute value. The terms do not alternate regularly in sign.

The values found for p^* and R_{p^*} appear in Table 3. Notice the anticipated improvement of accuracy with increasing ν and the surprising near independence of p^* from ν. The growth of p^* with r is accompanied by an increase in $|R_{p^*}|$ and an increase in $|R_p|$ for each fixed p.

TABLE 4

Accuracy of Fisher's expansion of the moments of a t Kernel (d Ee denotes d × 10^e)

ν	$r = 1$ p^*	$r = 1$ R_{p^*}	$r = 1$ R_5'
3	4	−.13	−.20
4	4	−.03	−.05
5	4	−.01	−.02
6	4	−4 E-3	−8 E-3
7	5	5 E-3	−3 E-3
8	5	6 E-3	−2 E-3
9	5	−1 E-3	−9 E-4
10	4	−1 E-3	−5 E-4
11	3	−1 E-3	−3 E-4
12	5	2 E-3	−2 E-4
13	3	−6 E-4	−1 E-4
14	3	−4 E-4	−8 E-5
15	4	−3 E-4	−6 E-5
16	4	−2 E-4	−4 E-5
17	4	−2 E-4	−3 E-5
18	4	−2 E-4	−2 E-5
19	4	−1 E-4	−1 E-5
20	4	−1 E-4	−1 E-5

When the right-hand side of (10) was based on the normalized form (6) of the Appell expansion (4), it was found to yield largely equivalent accuracy for the same numbers of terms with $r = 1$. Of course, when $r = 0$, the first term of any such normalized analogue is the true value T.

The approximation (10) based on Fisher's expansion was examined for $r = 1$ and $\nu = 3(1)20$. The values for p^* and R_{p^*} are displayed in Table 4, showing $p^* \leqq 5$ and $|R_{p^*}| \geqq 10^{-4}$. This contrasts with $p^* = 9$ and up to three more significant figures with (10) based on the Appell expansion. Table 4 displays also the six-term accuracy R_5' delivered by (10) based on the Appell expansion, showing $|R_{p^*}| \geqq |R_5'|$ for $\nu > 6$.

4.2 Two *t*-kernel factors. We consider also the asymptotic double-series expansion for the Behrens-Fisher densities given by (9) with the substitutions (8). The relative errors R_B of approximations of the form, $\sum_{(q_1,q_2)\varepsilon B} \delta_{q_1,q_2}$, were calculated for various parameter values ν_1, ν_2, θ, d, and using nominal true values obtained by Patil's (1964b) recurrence relations for Behrens-Fisher densities.

The following algorithm for choosing the index set B was found to give respectable accuracy in a modest amount of computer time. M_1 and M_2 are fixed upper bounds for q_1 and g_2. For successive values of $q_1 = 0, 1, \cdots$, let $q_2 = 0, 1, \cdots, Q(q_1)$, until $Q(q_1) < 0$ or $q_1 > M_1$. If $q_2 > Q(q_1 - 1)$, or $|\delta_{q_1,q_2}| > |\delta_{q_1-1,q_2}|$, or $|\delta_{q_1,q_2}| > |\delta_{q_1,q_2-1}|$, or $q_2 > M_2$, then choose $Q(q_1) = q_2 - 1$.

Values of R_B obtained with (9) based on the Appell expansion (4) ($P_{k,q_k} =$

TABLE 5

Accuracy of approximations to Behrens-Fisher densities (d Ee denotes $d \times 10^e$)

ν_1	ν_2	θ	d	Appell Expansion	Appell Normalized Expansion	Fisher's Expansion
3	3	45°	1	5 E-3	.15	−.02
3	3	7.5°	0	7 E-4	−3 E-4	−6 E-4
3	3	52.5°	1.4	.01	.05	.05
5	3	52.5°		6 E-3	.04	.02
5	5	30°		2 E-3	−4 E-3	.03
7	7	30°		8 E-4	−5 E-4	.01
9	7	60°		9 E-4	−4 E-4	.01
7	5	75°	4.2	−.51	−.48	−.70
7	7	15°	0	1 E-4	5 E-5	1 E-4
			1	1 E-4	4 E-4	−1 E-3
			2	8 E-3	−8 E-6	6 E-4
			3	.03	−.03	.13
			4	.19	.18	.36
			5	.76	.89	.97
		30°	0	−4 E-4	3 E-5	1 E-4
			1	2 E-4	7 E-3	−1 E-4
			2	2 E-3	.02	−5 E-3
			3	.05	.07	−8 E-3
			4	.03	.09	.06
			5	.21	.33	.44
		45°	0	−1 E-7	−1 E-4	−1 E-4
			1	7 E-4	.03	−6 E-4
			2	−2 E-3	1 E-3	.02
			3	.03	.03	.07
			4	−.34	−.28	−.19
			5	−.31	−.26	−.44

$2^{q_k}A_{q_k})$, $M_1 = M_2 = 15$, are displayed in column 5 of Table 5. The numbers of terms used averaged about 17. Notice the characteristic deterioration of accuracy for large $|d|$.

Comparable values of R_B, yielded by the analogue of (9), based on the normalized form (6) of the Appell expansion, $M_1 = M_2 = 15$, appear in column 6. Nearly identical values of R_B were obtained in this way with $M_1 = M_2 = 5$.

Column 7 contains the accuracies yielded by the analogue of (9), based on Fisher's expansion, $M_1 = M_2 = 5$.

The Appell expansions and Fisher's expansion appear to perform about as well with respect to numbers-of-terms used and accuracy, in the context of approximating Behrens-Fisher densities, with the Appell expansions seeming slightly more accurate.

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