A TECHNICAL LEMMA FOR MONOTONE LIKELIHOOD RATIO FAMILIES

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Let $\mathcal{A}$ be a σ-algebra on an abstract set X. Let $\mathcal{P}/\mathcal{A}$ be an ordered family of p-measures which is dominated by a σ-finite measure $\mu/\mathcal{A}$. Let p be a density of $P \varepsilon \mathcal{P}$ with respect to μ. $\mathcal{P}$ is said to have monotone likelihood ratios, if there exists an $\mathcal{A}$-measurable map $T: X \to R$, defined $\mathcal{P}$-a.e., such that for each pair P', $P'' \varepsilon \mathcal{P}$ with $P' < P''$ there exists a nondecreasing function $H_{P'P''} : R \to \bar{R}$ such that

$$(1) \qquad p''(x)/p'(x) = H_{P'P''}(T(x)) \qquad \mu\text{-a.e.},$$

whenever $p''(x)/p'(x)$ is defined.

In this formulation, the μ-null set on which (1) is violated depends on P', P''. The purpose of this note is to show that for MLR-families the densities can always be chosen such that (1) holds except on a fixed μ-null set. This leads to a simplification in many proofs connected with MLR-families, (e.g. those connected with the theory of uniformly most powerful tests).

Lemma. *If $\mathcal{P}$ is a* MLR-*family there exists a dominating σ-finite measure μ_0 and a coherent system of densities with respect to μ_0, such that for each pair P', $P'' \varepsilon \mathcal{P}$ with $P' < P''$ the ratio $p''(x)/p'(x)$ is a nondecreasing function of $T(x)$ with the exception of a fixed μ_0-null set. In other words: There exists a subset $X_0 \subset X$ such that $P(X_0) = 1$ for all $P \varepsilon \mathcal{P}$ and such that $p''(x)/p'(x)$ is a nondecreasing function of $T(x)$ for all $x \varepsilon X_0$ for which this ratio is defined.*

Proof. As $\mathcal{P}/\mathcal{A}$ is dominated by a σ-finite measure $\mu/\mathcal{A}$, according to the lemma of Halmos and Savage (see Lehmann, p. 354, Theorem 2), there exists a σ-finite measure $\mu_0 = \sum_{n=1} c_n \cdot P_n$, $P_n \varepsilon \mathcal{P}_0$, which is equivalent to $\mathcal{P}$. The densities of $P/\mathcal{A}$ with respect to $\mu_0/\mathcal{A}$ can be assumed to depend on x only through $T(x)$ (Lehmann, p. 48, Theorem 8). Hence for the following proof we might change the p-space: instead of $(X, \mathcal{A}, \mathcal{P})$ we will consider $(\mathcal{R}, \mathcal{B}, \mathcal{Q})$, where $\mathcal{B}$ is the Borel-algebra on $\mathcal{R}$ and $\mathcal{Q}/\mathcal{B}$ is the family of p-measures induced by $\mathcal{P}/\mathcal{A}$ through T (i. e., $Q(B) = P(T^{-1}B)$). Let $\nu_0/\mathcal{B}$ be the p-measure induced by $\mu_0/\mathcal{A}$. Then $\nu_0/\mathcal{B}$ is equivalent to $\mathcal{Q}/\mathcal{B}$. If $q(t)$ is a density of $Q/\mathcal{B}$ with respect to $\nu_0/\mathcal{B}$, $p(x) := q(T(x))$ is a density of $P/\mathcal{A}$ with respect to $\mu_0/\mathcal{A}$.

As $\mathcal{B}$ is separable, $\mathcal{Q}/\mathcal{B}$ is separable with respect to uniform convergence, i. e., there exists a countable subset $\mathcal{Q}_0 \subset \mathcal{Q}$ such that for each $Q \varepsilon \mathcal{Q}$ there exists a sequence $(Q_n)_{n=1,2,\cdots}$ with $Q_n \varepsilon \mathcal{Q}_0$ such that $Q_n(B) \to Q(B)$ uniformly in $B \varepsilon \mathcal{B}$. (See Lehmann, p. 352, Theorem 1.)

For each $Q \varepsilon \mathcal{Q}_0$, we fix a version, say q, of $dQ/d\nu_0$. For convenience we choose

Received 8 August 1966.

q finite everywhere. For $Q' < Q''$, Q', $Q'' \varepsilon \mathfrak{Q}_0$, we have

$$(2) \qquad q''(t)/q'(t) = H_{Q'Q''}(t) \qquad \nu_0\text{-a.e.}$$

Let $B_{Q',Q''} := \{t: q''(t)/q'(t) \neq H_{Q'Q''}(t)\}$. Then $\nu_0(B_{Q',Q''}) = 0$. Hence for $B_0 := \cup \{B_{Q'Q''} : Q', Q'' \varepsilon \mathfrak{Q}_0\}$ we have $\nu_0(B_0) = 0$. We have (2) for all $t \not\varepsilon B_0$.

For each $Q \varepsilon \mathfrak{Q}$ we choose a monotone sequence $(Q_n)_{n=1,2,\cdots}$, $Q_n \varepsilon \mathfrak{Q}_0$, converging uniformly to Q. Uniform convergence of measures implies ν_0-convergence in the mean of densities which furthermore implies ν_0-a.e. convergence of a subsequence. Taking this sequence instead of the original one we obtain: To each $Q \varepsilon \mathfrak{Q}$ we may choose a monotone sequence $(Q_n)_{n=1,2,\cdots}$, $Q_n \varepsilon \mathfrak{Q}_0$, such that $(q_n)_{n=1,2,\cdots}$ converges ν_0-a.e. We will show that $(q_n(t))_{n=1,2,\cdots}$ converges for all $t \not\varepsilon B_0$. As $(q_n)_{n=1,2,\cdots}$ converges ν_0-a.e., the limit is a density of Q with respect to ν_0. Hence there exists $t_0 \not\varepsilon B_0$ such that $\lim_{n\to\infty} q_n(t_0)$ is positive and finite. Let $t \not\varepsilon B_0$ be arbitrary. Without loss of generality we may assume $t_0 < t$. If $(Q_n)_{n=1,2,\cdots}$ is increasing, we have by definition of B_0:

$$q_{n+1}(t_0)/q_n(t_0) = H_{Q_nQ_{n+1}}(t_0) \leqq H_{Q_nQ_{n+1}}(t) = q_{n+1}(t)/q_n(t).$$

Hence $q_n(t)/q_n(t_0) \leqq q_{n+1}(t)/q_{n+1}(t_0)$. This relation also holds if $q_n(t) = q_{n+1}(t) = 0$. As $(q_n(t)/q_n(t_0))_{n=1,2,\cdots}$ is nondecreasing, it converges (possibly to $+\infty$). As $(q_n(t_0))_{n=1,2,\cdots}$ converges by assumption to a positive and finite value, $(q_n(t))_{n=1,2,\cdots}$ converges too. Hence $(q_n)_{n=1,2,\cdots}$ converges for all $t \not\varepsilon B_0$. We define for $t \not\varepsilon B_0 : q(t) := \lim_{n\to\infty} q_n(t)$. By this procedure we obtain for each $Q \varepsilon \mathfrak{Q}$ a fixed version of the density defined in $\bar{B}_0$.

It remains to show that $q''(t)/q'(t)$ is nondecreasing in $\bar{B}_0$ for all Q', $Q'' \varepsilon \mathfrak{Q}$ with $Q' < Q''$. Let $(Q_n')_{n=1,2,\cdots}$ and $(Q_n'')_{n=1,2,\cdots}$ be monotone sequences out of $\mathfrak{Q}_0$, converging uniformly to Q' and Q'', such that the corresponding sequences $(q_n')_{n=1,2,\cdots}$ and $(q_n'')_{n=1,2,\cdots}$ of densities converge to q' and q'', respectively, everywhere on $\bar{B}_0$. As $Q' < Q''$, there exists n_0 such that $Q_n' < Q_n''$ for all $n \geqq n_0$. If $Q_n' \uparrow Q'$, $Q_n'' \downarrow Q''$, this is trivial. If $Q_n'' \uparrow Q''$, there exists n' such that $Q' < Q_n''$ for all $n \geqq n'$, because $Q_n'' \leqq Q' < Q''$ implies that q''/q' as well as q'/q_n'' are nondecreasing; together with $q'' = \lim_{n\to\infty} q_n''$ this implies $q' = q''$, which contradicts $Q' < Q''$. If $Q_n' \uparrow Q'$ the assertion holds with $n_0 = n'$; if $Q_n' \downarrow Q'$, there exists n'' such that $Q_n' < Q_n''$ for all $n \geqq n''$. Then the assertion holds with $n_0 := \max(n', n'')$. For t_0, $t_1 \not\varepsilon B_0$ and $t_0 < t_1$ we have

$$\begin{aligned} q''(t_0)/q'(t_0) &= \lim_{n\to\infty} q_n''(t_0)/\lim_{n\to\infty} q_n'(t_0) = \lim_{n\to\infty} q_n''(t_0)/q_n'(t_0) \\ &\leqq \lim_{n\to\infty} q_n''(t_1)/q_n'(t_1) = \lim_{n\to\infty} q_n''(t_1)/\lim_{n\to\infty} q_n'(t_1) \\ &= q''(t_1)/q'(t_1), \end{aligned}$$

whenever $q''(t)/q'(t)$ is defined for $t = t_0$ and $t = t_1$. Therefore q''/q' is nondecreasing for $t \not\varepsilon B_0$. Hence $p''(x)/p'(x) = q''(T(x))/q'(T(x))$ is a nondecreasing function of $T(x)$ for $x \not\varepsilon T^{-1}(B_0)$.

REFERENCE

LEHMANN, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.