ON THE GALTON-WATSON BRANCHING PROCESS WITH MEAN LESS THAN ONE[1]

By A. Joffe

Université de Montréal

0. Introduction. Let Z_n denote a Galton-Watson branching process with $Z_0 = 1$, generating function

$$f(s) = \sum_{k=0}^{\infty} P[Z_1 = k]s^k, \qquad f(1) = 1, \quad 0 \leqq s \leqq 1.$$

The generating function of Z_n is given by the iterate

$$f_n(s) = f(f_{n-1}(s)) = \sum_{k=0}^{\infty} P[Z_n = k]s^k, \qquad f_0(s) = s \quad f_1(s) = f(s).$$

It is assumed that the mean m of Z_1 is less than 1:

$$0 < m = f'(1) = \sum_{k=1}^{\infty} kP[Z_1 = k].$$

The purpose of this note is to give very elementary proofs of results on the asymptotic behaviour of f_n and related quantities for large n. Most of the results are not new, but they hold under more general hypothesis than are to be found in the literature; for instance Yaglom's theorem (see Harris p. 18) will be proved without the existence of second moment.

It is a pleasure to acknowledge the numerous conversations with Professor F. Spitzer who stimulated this work.

1. Two Lemmas. We will use the following limited expansion of the function $f(s)$:

$$f(s) = 1 - m(1 - s) + (1 - s)\epsilon(s) \tag{1}$$

or

$$(1 - f(s))/(1 - s) = m - \epsilon(s). \tag{2}$$

Lemma 1. *The function* $\epsilon(s)$ *defined by* (1) *is monotone decreasing, tends to zero when* s *goes to one and is bounded above by* m.

Proof. The monotonicity of $\epsilon(s)$ is a consequence of the convexity of f. The right hand side of (2) is positive and converges to $f'(1) = m$ when s goes to one.

Replacing s by $f_{k-1}(s)$ in (2) we obtain

$$(1 - f_k(s))/(1 - f_{k-1}(s)) = m[1 - \epsilon(f_{k-1}(s))/m]. \tag{3}$$

Taking the product on both sides of (3) for $k = 1, 2, \cdots n$ we obtain

$$(1 - f_n(s))/(1 - s) = m^n \prod_{k=0}^{n-1} [1 - \epsilon(f_k(s))/m]. \tag{4}$$

Received 20 April 1966.

[1] This paper was supported by the N.S.F. while the author was visiting Cornell University.

Clearly the sequence $(1 - f_n(s))/(m^n(1 - s))$ is monotone decreasing in n and thus converges to a nondecreasing function $\varphi(s)$. Letting $s = 0$ we obtain

$$\varphi(0) = \lim_{n\to\infty} P[Z_n > 0]/m^n \geqq 0. \tag{5}$$

REMARK. From (4) it follows that $\varphi(0) > 0$ if and only if the series $\sum_{k=0}^{\infty} \epsilon(f_k(0))$ converges. In the following lemma we give sufficient conditions describing the behaviour of this series.

LEMMA 2. *If $\epsilon(s)$ goes to zero in such a way that the series $\sum_k \epsilon(1 - m^k)$ converges then $\varphi(0) > 0$. This will happen for instance if there is an $\alpha > 0$ and a constant C such that $\epsilon(s) < C(1 - s)^{\alpha}$, which is the case if $EZ_1^{1+\alpha} < \infty$. On the other hand if for $1 > s > s_0$ there is a constant C such that $\epsilon(s) > C/-\log(1 - s)$ then $\varphi(0) = 0$.*

PROOF. The first part follows from Lemma 1. We have $f(s) \geqq 1 - m(1 - s)$, therefore $f_k(s) \geqq 1 - m^k(1 - s)$. Since ϵ is monotone, $\epsilon(f_k(s)) \leqq \epsilon(1 - m^k(1 - s))$ and the proof is an easy consequence of the above remark.

The second part follows from the following observation: $f(s)$ being a convex function one sees easily that for any $m' < m$ there is an $s_0 < 1$ such that $f(s) < 1 - m'(1 - s)$ for $s \geqq s_0$. Therefore there is a k such that $f_k(0) > s_0$ and it follows that

$$\epsilon(f_{n+k}(0)) \geqq \epsilon(1 - m'^n(1 - f_k(0))) \geqq C[-n \log m - \log(1 - f_k(0))]^{-1}$$

which is the general term of a divergent series.

2. Yaglom's theorem.

THEOREM. *Let $b_{jn} = P[Z_n = j \mid Z_n > 0]$, $j = 1, 2, \cdots$, then as $n \to \infty$ the b_{jn} converge to quantities b_j such that $\sum_{j=1}^{\infty} b_j = 1$. Moreover $\sum_1^{\infty} nb_n = 1/\varphi(0) \leqq \infty$.*

PROOF. Let $g_n(s)$ be the generating function of the b_{jn}. It is easy to see that

$$g_n(s) = (f_n(s) - f_n(0))/(1 - f_n(0)) = 1 - (1 - f_n(s))/(1 - f_n(0)). \tag{6}$$

From (4) we obtain

$$g_n(s) = 1 - (1 - s) \prod_{k=0}^{n-1} (1 - \epsilon m^{-1}(f_k(s)))/(1 - \epsilon m^{-1}(f_k(0))). \tag{7}$$

Since $f_k(s) \geqq f_k(0)$ and by Lemma 1, $\epsilon(f_k(s)) \leqq \epsilon(f_k(0))$ we have that the general term in the product of (7) is larger than 1. It follows that the sequence g_n is monotone decreasing and has a limit $g(s)$. Following a suggestion of Professor Spitzer we show that g is a generating function.

Let $s_k = f_k(0)$,

$$\begin{aligned} g(s_k) &= \lim_{n\to\infty} (1 - (1 - f_{n+k}(0))/(1 - f_n(0))) \\ &= \lim_{s\to 1} (1 - (1 - f_k(s))/(1 - s)) = 1 - m^k. \end{aligned}$$

It follows that $g(s_k) \to 1$ as $k \to \infty$. Since g is monotone this shows that g is continuous and using the continuity theorem for generating function the theorem follows. The remark concerning $\sum nb_n$ follows from the above computation since

$$\sum nb_n = g'(1) = \lim_{k\to\infty} m^k/(1 - f_k(0)) = 1/\varphi(0).$$

3. Another theorem. Let us now consider the Markov process whose n-step transition matrix is given by

$$(8) \quad Q^{(n)}(i, j) = \lim_{k\to\infty} P[Z_n = j \mid Z_{n+k} > 0, Z_0 = i], \qquad i, j = 1, 2 \cdots .$$

It is easy to see that

$$Q^{(n)}(i, j) = P[Z_n = j \mid Z_0 = i]j/im^n.$$

From Yaglom's theorem we obtain

THEOREM. *The Q process is positive recurrent if and only if* $\varphi(0) > 0$. *The Q process is recurrent if and only if* $\sum_1^\infty (1 - f_n(0))/m^n$ *diverges.*

PROOF. Since $P[Z_n = j \mid Z_0 = i] = P[Z_n > 0 \mid Z_0 = i] \cdot P[Z_n = j \mid Z_n > 0, Z_0 = i]$ we obtain

$$Q^{(n)}(1, j) = [(1 - f_n(0))/m^n]b_{j,n}j.$$

Therefore from Yaglom's theorem

$$\lim_{n\to\infty} Q^{(n)}(1, j) = \varphi(0)jb_j = \pi_j \quad \text{(say)},$$

with $\sum \pi_j = 1$. This proves the first part of the theorem. The second part follows from the fact that $b_{j,n} \to b_j$. The series $\sum_{n=0}^\infty Q^{(n)}(1, j)$ converges if and only if $\sum (1 - f_n(0))/m^n$ converges. Proceeding as in the second part of Lemma 2 one can give examples where the last series diverges.

REFERENCE

HARRIS, T. (1963). *The Theory of Branching Processes.* Springer, Berlin.