NOTES

ON MEASURES EQUIVALENT TO WIENER MEASURE[1]

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Recently Shepp [5] has given, among several other results, a simple condition for the equivalence of a Gaussian measure to Wiener measure. His proof is ingenious but long. The purpose of this note is to record a somewhat simpler proof of this result which we had obtained independently. Our proof uses a reproducing kernel Hilbert space (*rkhs*) criterion, due to Oodaira [2], for the equivalence of Gaussian measures.

We first state Shepp's theorem (in a terminology more familiar to engineers): Discrimination between a zero-mean Gaussian process with covariance function $R(t, s)$ and a Wiener process with covariance function $\min(t, s)$, $[0 \leqq t, s \leqq T]$ will be nonsingular[2] if and only if there exists a unique, symmetric, square-integrable function on $T \times T$,

$$K(t, s) = K(s, t), \qquad \int_0^T \int_0^T K^2(t, s)\, dt\, ds < \infty \tag{1}$$

such that

$$R(t, s) - \min(t, s) = \int_0^t \int_0^s K(u, v)\, du\, dv \tag{2}$$

and

$$1 \text{ is not an eigenvalue of } K(u, v) \tag{3}$$

We shall, as mentioned above, obtain this result by direct application of the following theorem[3] of Oodaira [2]: *Discrimination between zero-mean Gaussian processes with covariance functions* $R_i(t, s)$, $t, s \,\varepsilon\, T \times T$, $i = 1, 2$, *will be nonsingular if and only if* (i) $R_1 - R_2$ *belongs to the direct product space* $H(R_2) \otimes H(R_2)$, *where* $H(R_2)$ *is the reproducing kernel Hilbert space* (*rkhs*) *of* R_2 *and* (ii) *there exist positive* (>0) *constants* c_1 *and* c_2 *such that* $c_1 R_2 \leqq R_1 \leqq c_2 R_2$.

We shall apply this theorem when $R_1(t, s) = R(t, s)$ and $R_2(t, s) = \min(t, s)$, $0 \leqq t, s \leqq T$. We need to calculate $H(R_2) \otimes H(R_2)$. This is easy to do and ac-

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Received 31 May 1966; revised 23 September 1966.

[1] This work was supported by AFOSR Contract AF 49(638)1517 and Tri Service Contract Nonr 225(83).

[2] That is, the measures corresponding to these two processes will be equivalent.

[3] We refer for definitions to Oodaira [2] or Parzen [4]. A similar theorem was given by Capon [1]. Both these theorems are extensions of, and were suggested by, the work of Parzen [3] [4]. We note that other rkhs proofs are possible, see, e.g., J. I. Golosov, Soviet Math. Doklady 7 (1966), 48–51. This reference was also noted by Shepp [5].

tually the result is available in the literature on *rkhs* [see, e.g., Parzen [4], p. 164]: $g(t, s)$, $0 \leqq t, s \leqq T$, will belong to $H(R_2) \otimes H(R_2)$, where $R_2(t, s) = \min(t, s)$, $0 \leqq t, s \leqq T$, if and only if

$$\int_0^T \int_0^T \partial^2 g(t, s)/\partial t \partial s \, dt \, ds < \infty \tag{4}$$

It therefore follows easily that conditions (1) and (2) are equivalent to the condition (i) of Oodaira's theorem. To show the equivalence of (3) and (11) we proceed as follows. We first note that the inner product in $H(R_2)$ is

$$\langle f, g \rangle_{H(R_2)} = \int_0^T f'(t) g'(t) \, dt \tag{5}$$

where the primes denote differentiation with respect to t. Let the (orthonormalized) eigenfunctions and eigenvalues of the (Hilbert-Schmidt) kernel $K(t, s)$ be given by

$$\int_0^T K(t, s) \phi_j(s) \, ds = \lambda_j \phi_j(t), \qquad j = 1, 2, \cdots \tag{6}$$

and let us define

$$\Phi_j(t) = \int_0^t \phi_j(s) \, ds, \qquad j = 1, 2, \cdots \tag{7}$$

Then it is easy to verify that

$$\langle \min(t, s), \Phi_j(s) \rangle_{H(R_2)} = \int_0^t 1 \cdot \phi_j(s) \, ds = \Phi_j(t) \tag{8}$$

and

$$\langle \int_0^t \int_0^s K(u, v) \, du \, dv, \Phi_j(s) \rangle_{H(R_2)} = \lambda_j \Phi_j(t) \tag{9}$$

Taking inner products with Φ_j in $H(R_2)$, the condition

$$c_1[\min(t, s)] \leqq \min(t, s) - \int_0^t \int_0^s K(u, v) \, du \, dv \leqq c_2[\min(t, s)] \tag{10}$$

becomes

$$c_1 \Phi_j(t) \leqq (1 - \lambda_j) \Phi_j(t) \leqq c_2 \Phi_j(t), \qquad j = 1, 2, \cdots \tag{11}$$

Taking inner products with Φ_j again, we get

$$c_1 \leqq 1 - \lambda_j \leqq c_2$$

i.e.,

$$1 - c_2 \leqq \lambda_j \leqq 1 - c_1 < 1 \tag{12}$$

which is equivalent to condition (3).

This completes our proof of the theorem. We close with some additional remarks.

If the process with covariance R has a nonzero mean, $m(t)$, $0 \leqq t \leqq T$, then for nonsingular detection $m(t)$ must belong to $H(R_2)$, which in turn will only be true if $dm(t)/dt$ is square integrable on $(0, T)$ [see, e.g., Parzen [4], p. 159].

This condition, due to Parzen [3], has also been obtained in a different way by Shepp [5], Eq. 1.3. In his paper Shepp treats several questions related to Wiener measure. Some of these results [Eqs. 5.6, 5.14, 5.15, 6.8–6.10] can again be obtained by *rkhs* methods. The expansion theorems in Shepp [5], Theorem 2 and 3, can also be generalized by such methods. We shall give the details in a later paper.

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