ASYMPTOTIC EFFICIENCY OF CERTAIN RANK TESTS FOR COMPARATIVE EXPERIMENTS

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1. Introduction and summary. For comparative experiments with two or more treatments, rank methods possess, in their insensitivity to gross errors and extreme observations, a distinct advantage over the classical normal theory procedures (see [10]) (beside providing exact significance level when the form of the underlying distribution is unknown). First of the rank tests developed was the Wilcoxon-two-sample test, subsequently generalized to the K-sample problem by Kruskal and Wallis (see [12]). Both these tests have been shown to possess asymptotic (Pitman) efficiency equal to $3/\pi$ (against normal shift) relative to the classical t- and $\mathfrak{F}$-tests respectively (see Andrews [1]). However, in many comparative experiments, it is desirable in the interest of increased precision to stratify the population or divide the experimental subjects into homogeneous (randomized) blocks. For such experimental designs, the first attempt at providing a rank test was made as far back as 1937 by Friedman [5] (for the one observation per cell case), who proposed a test based on independent rankings of observations within each block. This procedure, which we shall refer to in the sequel as the *separate-ranking* procedure, was extended subsequently to more general designs by Durbin [4] and Benard and van Elteren [2]. Van Elteren and Noether [22] computed the asymptotic efficiency of the separate-ranking procedure and showed (for the one observation per cell case) that relative to the normal theory $\mathfrak{F}$-statistic its efficiency (against normal shift) is $3K/\pi(K + 1)$ (which takes the value $2/\pi$ for $K = 2$ and increases to $3/\pi$ as $K \to \infty$).

In 1962, however, Hodges and Lehmann [9] pointed out that the rather low efficiency of the separate-ranking procedure was due, presumably, to the absence of interblock comparisons and proposed a conditional test based on a combined ranking of all the observations after "alignment" (defined below) within each block (see also Mehra [17]). Subsequently, Lehmann in a series of papers [13], [14], [15] laid the foundations of an entirely new and remarkable approach to nonparametric inference parallel to the classical normal theory (parametric) analysis of variance. However the question of asymptotic efficiency of the test proposed in [9] was left essentially unanswered.

It is the purpose of the present paper to study the asymptotic efficiency of the conditional test proposed in [9]. In Section 2, the asymptotic version of this test is discussed. In Section 3, limit distributions under translation alternatives are obtained. Section 4 contains a discussion of the asymptotic efficiency and Section 5 consists of certain concluding remarks.

2. The conditional test. Consider K treatments in an experimental design, with n blocks and m_{ij} ($\geqq 1$) observations in the (i, j)th cell ($i = 1, 2, \cdots, n$;

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$j = 1, 2, \cdots, K$). Let X_{ijl} ($l = 1, 2, \cdots, m_{ij}$) be the lth observation in the cell (i, j), and assume that these m_{ij} (independent) observations have a common continuous cumulative distribution function (cdf) $F_{ij}(x)$ satisfying the condition

$$(2.1) \qquad \Omega: F_{ij}(x) = F_j(x + \xi_i)$$

where ξ_i are unknown constants (block-effects). The hypothesis of interest, namely, that there is no difference among the treatment effects can be expressed as $H_0: F_1 = F_2 = \cdots = F_K$. It may be observed that the model (2.1) corresponds to the usual assumption of additivity of block effects in the linear analysis of variance model.

The conditional test proposed by Hodges and Lehmann [9], based on the *joint-ranking* of all the observations after "alignment," is on the premise that the interblock comparisons introduced through joint-ranking would improve the relative efficiency. We shall show below that this is indeed true. By "alignment" is essentially meant removing the block effects ξ_i ($i = 1, \cdots, n$) from the observations by subtracting from each observation in a block, say the ith, some reasonable function μ of the observations in the block which satisfy the condition

$$(2.2) \quad \mu(X_{i11} + a, \cdots, X_{i1m_{i1}} + a, \cdots, X_{iK1} + a, \cdots, X_{iKm_{iK}} + a)$$
$$= \mu(X_{i11}, \cdots, X_{i1m_{i1}}, \cdots, X_{Ki1}, \cdots, X_{iKm_{iK}}) + a.$$

The object of alignment is obvious. For, on account of possibly unequal (and unknown) block effects, no worthwhile information would be contained in the ranks based on joint-ranking before alignment. However, if the block-effects ξ_i are removed before ranking, it is clear (at least when the marginal distributions of all the aligned observations, under H_0, are identical) that the ranks do contain relevant information. Also, it is desirable (although not necessary) that, under H_0, the aligned observations in each block have a symmetric joint distribution. Both these properties would hold if μ satisfies (2.2) and is a symmetric function of its arguments, (cf. condition B of [9]) as is evident from the following:

LEMMA 2.1. *Let $X_1, X_2, \cdots, X_c$ be independent identically distributed random variables and let $\mu = \mu(X_1, X_2, \cdots, X_c)$ be some symmetric function of these variables. Then the random variables Z_j, $j = 1, 2, \cdots, c$, where $Z_j = X_j - \mu(X_1, \cdots, X_c)$ have a symmetric joint distribution.*

PROOF. The proof is immediate, since on account of symmetry of $\mu(X_1, X_2, \cdots, X_c)$

$$\begin{aligned} F(t_2, t_1, \cdots, t_c) &= P[Z_1 \leqq t_2, Z_2 \leqq t_1, \cdots, Z_c \leqq t_c] \\ &= P[X_1 - \mu(X_1, X_2, \cdots, X_c) \leqq t_2, \\ &\qquad X_2 - \mu(X_1, X_2, \cdots, X_c) \leqq t_1 \cdots Z_c \leqq t_c] \\ &= P[X_2 - \mu(X_2, X_1, \cdots, X_c) \leqq t_2, \\ &\qquad X_1 - \mu(X_2, X_1, \cdots, X_c) \leqq t_1 \cdots Z_c \leqq t_c] \\ &= P[Z_2 \leqq t_2, Z_1 \leqq t_1, \cdots, Z_c \leqq t_c] \\ &= F(t_1, t_2, \cdots, t_c). \end{aligned}$$

Description of the test. The conditional test procedure for small sample sizes, along with the asymptotic theory for $K = 2$ is fully discussed in [9]. We will discuss here only the asymptotic version of the test for general K. Let the aligned observations be denoted by Z_{ijl} ($l = 1, 2, \cdots, m_{ij}$; $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, K$) and let r_{ijl} be the rank of Z_{ijl} in a combined ranking of all the $N = \sum_i \sum_j m_{ij}$ aligned observations. Consider now the conditional situation given the set of ranks for each block. Each conditional situation (an event in the original sample space) is referred to as a *configuration* in [9]. Given a configuration, the only randomness that remains is due to independent assignment of ranks to the treatments in each block. Clearly complete random assignment of the ranks to the treatments would be justifiable, under H_0, if the vector of aligned observations for each block has a symmetric joint distribution. For the discussion below we will assume (i) that the same method of alignment is used in each block and (ii) that the conditions A and B of [9] are satisfied where

(A) after alignment each block contains at least one positive and one negative observation;

(B) within each block, complete randomization is employed.

The condition (A) above is satisfied and the condition (B) is justifiable under all reasonable methods of alignment, such as on the mean, Winsorized or censored mean, or the median. Let $N_i = \sum_j m_{ij}$, $R_{ij} = \sum_l r_{ijl}$, $R_j = \sum_i R_{ij} = \sum_i \sum_l r_{ijl}$ = sum of the ranks for the jth treatment and let $\tilde{E}(\cdot)$, $\tilde{\text{V}}\text{ar}(\cdot)$ and $\tilde{\text{C}}\text{ov}(\cdot, \cdot)$ stand for the conditional expectation, variance and covariance respectively, under H_0, given a configuration. Then

$$\tilde{E}(R_j) = \sum_{i=1}^{n} m_{ij} \bar{r}_i;$$

$$\sigma_j^2 = \sigma_{jj} = \tilde{\text{V}}\text{ar}(R_j) = \sum_{i=1}^{n} (N_i - m_{ij})\, m_{ij}\, [\tau_i^2/(N_i - 1)]; \tag{2.3}$$

$$\sigma_{jj'} = \tilde{\text{C}}\text{ov}(R_j, R_{j'}) = -\sum_{i=1}^{n} m_{ij}\, m_{ij'} [\tau_i^2/(N_i - 1)];$$

where $\tau_i^2 = \sum_j \sum_l (r_{ijl} - \bar{r}_i)^2/N_i$ and $\bar{r}_i = \sum_j \sum_l r_{ijl}/N_i$. We will first write down the test statistic for the case when the m_{ij} do not vary with the blocks, viz.,

$$m_{1j} = m_{2j} = \cdots = m_{nj} = m_j \quad \text{for each} \quad j = 1, 2, \cdots, K. \tag{2.4}$$

This is the more meaningful case, since then the vectors of aligned observations, as well as the vectors $(R_{i1}, R_{i2}, \cdots, R_{iK})$, $i = 1, 2, \cdots, n$, of rank sums R_{ij} for individual treatments in block i, have identical distributions under H_0. Also, in this case, the conditional test-statistic takes a convenient form. Under (2.4), the expressions (2.3) reduce to

$$\tilde{E}(R_j) = m_j n(nN' + 1)/2;$$

$$\sigma_j^2 = \sigma_{jj} = \tilde{\text{V}}\text{ar}(R_j) = [(N' - m_j)m_j/(N' - 1)] \left(\sum_{i=1}^{n} \tau_i^2\right); \tag{2.5}$$

$$\sigma_{jj'} = \tilde{\text{C}}\text{ov}(R_j, R_{j'}) = -[m_j \cdot m_{j'}/(N' - 1)] \left(\sum_{i=1}^{n} \tau_i^2\right);$$

where $N' = \sum_{j=1}^{K} m_j$. The proposed conditional test statistic of [9], then, takes the form

$$(2.6) \quad W_n = [(N' - 1)/N'(\sum_{i=1}^{n} \tau_i^2)] \sum_{j=1}^{K} m_j^{-1}\{R_j - m_j n(nN' + 1)/2\}^2$$

with the test consisting in rejecting H_0 at level α, if W_n exceeds a predetermined number $W_{n,\alpha}$ such that the conditional probability $\tilde{P}_{H_0}[W_n > W_{n,\alpha}] = \alpha$. The large sample approximation of $W_{n,\alpha}$ is provided by the following:

THEOREM 2.1 *Let the hypothesis H_0 be true and the method of alignment be such that* (A) *and* (B) *hold. Then* (i) *the conditional statistic W_n, given a configuration, converges in distribution as $n \to \infty$, to the χ^2-variable with $K - 1$ degrees of freedom; and* (ii) *the convegrence is uniform in the configuration.*

The proof of the theorem will be accomplished in a number of steps by extending the method of proof of [9].

Consider an arbitrary linear combination $U_i = \sum_{j=1}^{K-1} c_j R_{ij}$ of R_{ij}, $j = 1, 2, \cdots, K - 1$. Let b_i^2 and β_i denote the conditional (given a configuration) variance and the conditional third absolute moment about the mean of U_i and let $S_n^2 = \sum_{i=1}^{n} b_i^2$. Assume further that

$$(2.7) \qquad N_i < \eta \quad \text{for all} \quad i.$$

Lemma 2.2 below is proved for arbitrary m_{ij} not necessarily satisfying the assumption (2.4), and consequently can also be used to prove a more general version of Theorem 2.1 viz., Theorem 2.1 A.

LEMMA 2.2. *Under assumptions* (2.7), (A) *and* (B), (i) *the conditional* (*given a configuration*) *distribution of $\sum_{i=1}^{n} [U_i - \tilde{E}(U_i)]/S_n$ converges, as $n \to \infty$, to the standard normal distribution; and* (ii) *the convergence is uniform in the configuration.*

PROOF. First note that by arguments of Lemma 1 of [9] it follows that there exists a constant $0 < a_1 < \infty$, independent of the configuration, such that

$$(2.8) \qquad \sum_{i=1}^{n} \beta_i \leqq \sum_{j=1}^{K-1} |c_j|^3 (K - 1)^2 \sum_{i=1}^{n} |R_{ij} - \tilde{E}R_{ij}|^3 \leqq a_1 n^4.$$

We now obtain for $S_n^2 = \sum_{i=1}^{n} b_i^2$ a lower bound which is independent of the configuration: We assume first that at least two of the c's (out of $c_1, c_2, \cdots, c_{K-1}$) are different from zero and not all non-zero c's are equal. For otherwise the desired lower bound is provided by the inequality (2.9) itself (cf. Lemma 2 of [9]). Now note that from (2.3)

$$\begin{aligned} S_n^2 = \sum_{i=1}^{n} b_i^2 &= \sum_{i=1}^{n} [\sum_{j=1}^{K-1} c_j^2(N_i - m_{ij})m_{ij} \\ &\quad - 2 \sum_{j<j'} c_j c_{j'} m_{ij} m_{ij'}]\tau_i^2/(N_i - 1) \\ (2.8a) \qquad &\geqq \sum_{i=1}^{n} [\sum_{j=1}^{K-1} c_j^2(m_{ij}/N_i^*) \\ &\quad - (\sum_{j=1}^{K-1} c_j(m_{ij}/N_i^*))^2][N_i^{*2}/(N_i - 1)]\tau_i^2 \\ &\geqq b \sum_{i=1}^{n} \tau_i^2, \end{aligned}$$

where $N_i^* = N_i - m_{iK}$ and $0 < b < \infty$ is a constant independent of the configuration. The last inequality follows using (2.7), $m_{ij} \geqq 1$ and the fact that the expression

$$\sum_{j=1}^{K-1} c_j^2(m_{ij}/N_i^*) - \{\sum_{j=1}^{K-1} c_j(m_{ij}/N_i^*)\}^2$$

is the variance of some non-degenerate random variable and is consequently bounded below by a positive constant independent of i and the configuration.

We shall now prove the existence of a positive constant b_1 independent of the configuration, such that[1]

$$\sum_{i=1}^{n} \tau_i^2 \geqq b_1 n^3. \tag{2.9}$$

For convenience we suppress the index j representing the treatment and denote by $Z_{i1}, Z_{i2}, \cdots, Z_{iN_i}$, $i = 1, 2, \cdots, n$, the observations after alignment. Let r_{ij} denote the rank of Z_{ij} in the total set and

$$r_i' = \min_j r_{ij}, \qquad r_i'' = \max_j r_{ij}.$$

Then on the account of the assumption (A), $\min_j Z_{ij} < 0 < \max_j Z_{ij}$, so that

$$\max_i r_i' < \min_i r_i''. \tag{2.10}$$

Now using (2.7),

$$\begin{aligned} \tau_i^2 &= N_i^{-2} \sum_{1 \leqq j < k \leqq N_i} (r_{ij} - r_{ik})^2 \\ &\geqq N_i^{-2} (r_i'' - r_i')^2 \\ &\geqq \eta\, (r_i'' - r_i')^2 \end{aligned}$$

and hence

$$\sum_{i=1}^{n} \tau_i^2 \geqq \eta^{-2} \sum_{i=1}^{n} (r_i'' - r_i')^2.$$

Let $a_1' < a_2' < \cdots < a_n'$ and $a_1'' < a_2'' < \cdots < a_n''$ denote the ordered values $r_1', r_2', \cdots, r_n'$ and $r_1'', r_2'', \cdots, r_n''$ respectively. By a well-known inequality (cf. [8] Theorem 368)

$$\sum_{i=1}^{n} r_i' r_i'' \leqq \sum_{i=1}^{n} a_i' a_i'',$$

which implies

$$\sum_{i=1}^{n} (r_i'' - r_i')^2 \geqq \sum_{i=1}^{n} (a_i'' - a_i')^2,$$

and hence

$$\sum_{i=1}^{n} \tau_i^2 \geqq \eta^{-2} \sum_{i=1}^{n} (a_i'' - a_i')^2. \tag{2.11}$$

Now from (2.10) we have $a_1' < a_2' < \cdots < a_n' < a_1'' < a_2'' < \cdots < a_n''$, so

[1] The above proof of the inequality (2.9) is due to Professor Wassily Hoeffding and provides a correct proof of Lemma 2 of [9].

that

$$a_i'' - a_i' \geqq n \quad \text{for} \quad i = 1, 2, \cdots, n. \tag{2.12}$$

From (2.11) and (2.12), the inequality (2.9) follows with $b_1 = \eta^{-2}$.

From (2.8a) and (2.9), it follows that there exists positive a constant a_2 independent of the configuration such that

$$S_n^2 \geqq a_2 n^3. \tag{2.12a}$$

The rest of the proof follows from (2.8a) and (2.12a) by using the Berry-Esseen theorem as in [9]; and the proof is complete.

For the proof of Theorem 2.1, we need the following lemmas:

LEMMA 2.3. *Let* $V^{(n)} = (V^{(n)} \cdots V_c^{(n)})$, $n = 1, 2, 3, \cdots$, *be a sequence of random vectors with* cdf. $F_\theta^{(n)}(v)$ (*in c-space* $R^{(c)}$) *depending on a parameter* θ, *and let* $G_\theta^{(n)}(u)$ *denote the* cdf *of an arbitrary linear function* $U^{(n)} = f(V^{(n)}) = \sum_{j=1}^{c} c_j V_j^{(n)}$ $(f: R^{(c)} \to R^{(1)})$ *of the components, such that, as* $n \to \infty$

(i) $F_\theta^{(n)}(v)$ *converges to* $F_\theta(v)$ *for each* θ (*and* v), *where* $F_\theta(v)$ *is the* cdf *of a* rv V,

(ii) *for every linear* f, $G_\theta^{(n)}(u)$ *converges to* $G_\theta(u)$ *uniformly in* θ (*and* u), *where* $G_\theta(u)$ *is the* cdf *of the* rv $f(V)$,

then $F_\theta^{(n)}(v)$ *also converges to* $F_\theta(v)$ *uniformly in* θ (*and* v).

PROOF. One can easily verify that the statement (ii) is equivalent to the statement: for every linear function f and every Borel set $B \varepsilon \mathcal{B}^{(1)}$ (Borel field in $R^{(1)}$), $\Delta_\theta^{(n)}(B) = P[f(V^{(n)}) \varepsilon B] - P[f(V) \varepsilon B]$ converges to zero, as $n \to \infty$, uniformly in θ and B. Now since $\Delta_\theta^{(n)}(B) = P[V^{(n)} \varepsilon f^{-1}(B)] - P[V \varepsilon f^{-1}(B)]$, and since the minimal σ-field containing the class of all sets in $R^{(c)}$ which are inverse images, under the family of linear functions $f: R^{(c)} \to R^{(1)}$, of Borel sets in $\mathcal{B}^{(1)}$ coincides with the Borel field $\mathcal{B}^{(c)}$ in $R^{(c)}$, it follows that $\Delta_\theta^{*(n)}(B^*) = P[V^{(n)} \varepsilon B^*] - P[V \varepsilon B^*]$ converges to zero, for every $B^* \varepsilon \mathcal{B}^{(c)}$, uniformly in B^* and θ; and the proof is complete.

LEMMA 2.4. *Let the sequences* $\{V^{(n)}\}$ *and* $\{F_\theta^{(n)}(v)\}$, $n = 1, 2, 3, \cdots$, *be as defined in Lemma* 2.3 *and suppose that the conclusion of Lemma* 2.3 *holds. Let* $f: R^{(c)} \to R^{(1)}$ *be a measurable function, with the set of discontinuity points of measure zero, then* (i) *the distribution* $G_\theta^{(n)}(x)$ *of* $f(V^{(n)}(x)$ *of* $f(V^{(n)})$ *converges to the distribution* $G_\theta(x)$ *of* $f(V)$ *and* (iii) *the convergence is uniform in* θ (*and* x).

PROOF. Part (i) is proved in Mann and Wald [16] and part (ii) can be proved by arguments similar to those of Lemma 2.3.

PROOF OF THEOREM 2.1. From part (i) of Lemma 2.2 it follows by arguments similar to those of Wald and Wolfowitz ([20], Section 7) that, under (2.4), the random variables $(V_j/d^{\frac{1}{2}})$, $j = 1, 2, \cdots, K$, where

$$V_j = [R_j - \tilde{E}(R_j)]/m_j^{\frac{1}{2}}, \qquad d = \{N'/(N' - 1)\} \sum_{i=1}^{n} \tau_i^2$$

are conditionally, given a configuration, distributed in the limit as $n \to \infty$, as multivariate normal distribution with zero means and covariance matrix

$\|\delta_{jj'} - (m_j/N')^{\frac{1}{2}}(m_{j'}/N')^{\frac{1}{2}}\|$. We now make an orthogonal transformation

$$V_0^* = \sum_{j'=1}^{K} (m_{j'}/N')^{\frac{1}{2}} (V_{j'}/d^{\frac{1}{2}});$$
$$V_j^* = \sum_{j'=1}^{K} a_{jj'} (V_{j'}/d^{\frac{1}{2}}), \qquad j = 1, 2, \cdots, K-1,$$

where a's are chosen to make the transformation orthogonal. Part (1) of the theorem now follows from part (i) of Lemma 2.4 and the fact that V_j^* ($j = 1, 2, \cdots, K-1$) are independent normal variates in the limit and V_0^* is degenerate at zero, and that $W_n = \sum_{j=1}^{K} (V_j^2/d) = \sum_{j'=0}^{K-1} V_{j'}^{*2}$. Part (ii) of the theorem follows from Lemma 2.3 and parts (ii) of Lemmas 2.2 and 2.4.

For the case where the sample sizes m_{ij} do not satisfy the condition (2.4), one can base the test on the statistic

$$\mathbf{V}' \mathbf{\Delta}^{-1} \mathbf{V}, \tag{2.13}$$

where $\mathbf{V} = (V_1, V_2, \cdots, V_{i-1}, V_{i+1}, \cdots, V_K)'$ and $\mathbf{\Delta}$ denotes the exact covariance matrix $\|\sigma_{jj}\|$ of $\mathbf{V}$ given by (2.3). It is easily verified that the quadratic form (2.13) does not depend on the choice of the omitted variable V_i. Benard and van Elteren [2] have given a convenient representation of the quadratic form (2.13). The following theorem extends Theorem 2.1 to the case when (2.4) is not satisfied; its proof can be accomplished by using Lemmas 2.2–2.4 and arguments similar to those of Theorem 2.1.

THEOREM 2.1A. *Let the hypothesis H_0 be true and the conditions* (A), (B) *and* (2.7) *hold. Then* (i) *conditionally, given a configuration, the statistic* (2.13) *is distributed in the limit, as $n \to \infty$, as a χ^2-variable with* $(K-1)$ *degrees of freedom*; *and* (ii) *the convergence is uniform in the configuration.*

3. Asymptotic distributions under translation alternatives. The results of the preceding section were proved under any method of alignment, provided (2.2) and the conditions (A) and (B) were satisfied. As pointed out in Section 2, most reasonable methods of alignment satisfy these conditions. We will, however, confine our attention henceforth to alignment on the mean only, and obtain in this section the (unconditional) limiting distribution of the statistic W_n, defined by (2.4) and (2.6), under a suitable sequence of translation alternatives approaching H_0. This will enable us in the next section to compare the asymptotic efficiency of this test relative to other competing test procedures, namely, the classical $\mathfrak{F}$-test, the Friedman-Benard-van Elteren separate ranking procedure, as well as the new test procedure proposed by Lehmann in [15].

Consider for each n, the alternative hypothesis

$$K_n: F_j(x) = F(x + \theta_j n^{-\frac{1}{2}}) \quad \text{for} \quad j = 1, 2, \cdots, K \quad (\text{not all } \theta_j \text{ equal}). \tag{3.1}$$

Let X_{jl} ($l = 1, 2, \cdots, m_j$; $j = 1, 2, \cdots, K$) be $N' = \sum_{j=1}^{K} m_j$ independent random variables distributed identically with cdf $F(x)$. Further let $G(u)$ denote the cdf of $Z_{11} = (X_{11} - \bar{X})$, where $\bar{X} = \sum_{j=1}^{K} \sum_{l=1}^{m_j} X_{jl}/N'$ and let $H(u, v)$ denote the joint cdf of (Z_{11}, Z_{21}). For our experimental design, then, $G(u)$ and $H(u, v)$ represent the marginal and joint cdf's of aligned observations Z_{111} and

Z_{121} under H_0. Let $\chi_r^2(\Delta)$ denote the non-central chi-square variable with r degrees of freedom and the noncentrality parameter Δ. We now state

THEOREM 3.1. *Assume for each n, the truth of the hypothesis K_n and suppose, for any real number η, that $\lim_{n\to\infty} \int n^{\frac{1}{2}}\{G(x + \eta n^{-\frac{1}{2}}) - G(x)\} dG(x)$ exists and is finite. Then the statistic W_n, defined by (2.4) and (2.6), converges in distribution as $n \to \infty$, to $\chi^2_{K-1}(\Delta_W)$ with $\Delta_W = [3/N'^2(1 - 3\lambda)] \sum_{j=1}^{K} (\mu_j^2/m_j)$, where*

$$(3.2) \quad \mu_j = \lim_{n\to\infty} m_j \sum_{j'=1}^{K} m_{j'} \int_{-\infty}^{\infty} n^{\frac{1}{2}}\{G(x + (\theta_{j'} - \theta_j)/n^{\frac{1}{2}}) - G(x)\} dG(x),$$

$$\lambda = \lambda(N',F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u) G(\nu) dH(u, \nu).$$

The proof of the theorem will be based on the following lemma, for which we set out some notation: Let $\mathbf{T}_n$ and $\boldsymbol{\mu}$ stand for the vectors $(T_{n,1}, T_{n,2}, \cdots, T_{n,K})'$ and $(\mu_1, \mu_2, \cdots, \mu_K)'$ respectively where μ_j ($j = 1, \cdots, K$) is defined by (3.2) and

$$(3.3) \qquad T_{n,j} = 3^{\frac{1}{2}} \{R_j - n m_j (n N' + 1)/2\}/m_j^{\frac{1}{2}} N (1 - 3\lambda)^{\frac{1}{2}} n^{\frac{3}{2}} \qquad (j = 1, 2, \cdots, K),$$

and $\mathbf{\Lambda}$ stand for the covariance matrix $\|\delta_{jj'} - (m_j/N')^{\frac{1}{2}}(m_{j'}/N')^{\frac{1}{2}}\|$.

LEMMA 3.1. *Under the assumptions of Theorem 3.1, the random vector $\mathbf{T}_n$ converges in distribution, as $n \to \infty$, to a multivariate normal vector $N(\boldsymbol{\mu}, \mathbf{\Lambda})$.*

PROOF. The proof of this lemma depends on the results of Hoeffding [9] on U-statistics and runs parallel to that of Andrews [1]:

For convenience, we first give a proof for the case $m_j = 1$ for each $j = 1, 2, \cdots, K$, from which, then, by a linear transformation the general result will follow. Define $\delta(y_j, y_{j'}) = 1$ if $y_j > y_{j'}$ and $\delta(y_j, y_{j'}) = 0$ otherwise and $h^j(y_1, y_2, \cdots, y_K) = \sum_{j'=1}^{K} \delta(y_j, y_{j'})$. Let $Z_i = (Z_{i1}, \cdots, Z_{iK})'$ denote the vector of aligned observations for the ith block ($i = 1, 2, \cdots, n$) and set

$$(3.4) \quad U'^j = n^{-K} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_K=1}^{n} h^j (Z_{i_1 1}, Z_{i_2 2}, \cdots, Z_{i_K K}).$$

Then right hand side of (3.4) is easily verified to be

$$(3.4a) \qquad = n^{-2} \sum_{j'=1}^{K} \sum_{i_j=1}^{n} \sum_{i_{j'}=1}^{n} \delta(Z_{i_j j}, Z_{i_{j'} j'}) = (R_j - n)/n^2.$$

Define now

$$\varphi^j(Z_1, \cdots, Z_K) = K!^{-1} \sum_{(j_1 \cdots j_K)} h^j(Z_{1j_1}, \cdots, Z_{Kj_K})$$

with $(j_1, \cdots, j_K)$ extending over all permutations of the set of integers $\{1, 2, \cdots, K\}$, and (for $n \geqq K$)

$$U^j = \sum_{(i_1 \cdots i_K)} \varphi^j (Z_{i_1}, \cdots, Z_{i_K})/\binom{n}{K}$$

with $(i_1, \cdots, i_K)$ extending over all sets such that $1 \leqq i_1, i_2, \cdots, i_K \leqq n$ and no two i's equal. Then U^j is the average of only those h^j terms for which arguments of a given j are each elements of a different vector, whereas U'^j is the average of all possible h^j terms (U^j, $j = 1, \cdots, K$, are U-statistics as defined in Hoeffding [11]). Further let $Y^j = (K/n^{\frac{1}{2}}) \sum_{i=1}^{n} \Psi_1^j(Z_i)$, where

$$(3.5) \qquad \Psi_1^j(Z_i) = E\{\varphi^j(Z_1, \cdots, Z_K) \mid Z_i\} - E\{\varphi^j(Z_1, \cdots, Z_K)\}.$$

Then, since the vectors Z_i ($i = 1, 2, \cdots, n$) are identically distributed, Theorem 7.1 of Hoeffding [11] applies. This, coupled with the arguments on page 727 of Andrews [1] (concerning U^j and U'^j), proves that the vectors $\{n^{\frac{1}{2}}(U'^j - EU'^j); j = 1, \cdots, K\}'$, $\{n^{\frac{1}{2}}(U^j - EU^j); j = 1, 2, \cdots, K\}'$ and $\{Y_j; j = 1, \cdots, K\}'$ are all identically distributed in the limit, as $n \to \infty$. Consequently, using (3.4), it follows that $n^{-\frac{3}{2}}(R_j - ER_j)$, $j = 1, \cdots, K$, have in the limit a multivariate normal distribution with mean vector zero and covariance matrix $\lim_{n\to\infty} K^2 E(\Psi_1\Psi_1')$ where $\Psi_1 = (\Psi_1^1, \cdots, \Psi_1^K)'$ is defined by (3.5). To find this covariance matrix, let $G_j(u)$ denote under (3.1) the cdf of Z_{ijl}, an aligned observation corresponding to the jth treatment, then one observes that

$$\begin{aligned} E(\varphi^k \mid Z_i) &= K!^{-1} \textstyle\sum_{(j,\cdots,j_K)} \sum_{k'=1}^{K} E(\delta(Z_{kj_k}, Z_{k'j_{k'}}) \mid Z_i\} \\ &= K^{-1} \textstyle\sum_{k'=1}^{K} \{G_{k'}(Z_{ik}) + 1 - G_k(Z_{ik'}) + \int_{-\infty}^{\infty} G_{k'}(u)\, dG_k(u)\} \end{aligned}$$

so that from (3.5),

$$\begin{aligned} \Psi_1^k(Z_i) = K^{-1} \textstyle\sum_{k'=1}^{K} \{G_{k'}(Z_{ik}) &- \int_{-\infty}^{\infty} G_{k'}(u)\, dG_k(u) \\ &- G_k(Z_{ik'}) + \int_{-\infty}^{\infty} G_k(u)\, dG_{k'}(u)\}. \end{aligned}$$

Using the fact that, as $n \to \infty$, the hypothesis $K_n \to H_0$ and the Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned} E(\Psi_1^j \cdot \Psi_1^{j'}) \to K^{-2} E_0\{[(K - 1)G(Z_{ij}) \\ - \textstyle\sum_{k\neq j} G(Z_{ik})][(K - 1)G(Z_{ij'}) - \sum_{k\neq j'} G(Z_{ik})]\} \\ = -(1 - 3\lambda)/3K \qquad \text{if } j \neq j' \\ = (K - 1)(1 - 3\lambda)/3K \quad \text{if } j = j'. \end{aligned}$$

Consequently,

$$(3.6) \qquad \lim_{n\to\infty} K^2 E(\Psi_1\Psi_1') = \| \{\delta_{jj'} - K^{-1}\} \{K^2(1 - 3\lambda)/3\} \|.$$

Further, from (3.4a), it easily follows that

$$\begin{aligned} (3.7) \quad \lim_{n\to\infty} n^{-\frac{3}{2}}\{E(R_j) - \tfrac{1}{2}n(nK + 1)\} \\ = \lim_{n\to\infty} \textstyle\sum_{j'=1}^{K} \int_{-\infty}^{\infty} n^{\frac{1}{2}}\{G(x + (\theta_{j'} - \theta_j)/n^{\frac{1}{2}}) - G(x)\}\, dG(x). \end{aligned}$$

From (3.6) and (3.7), the result follows immediately for the case: $m_j = 1$ for all $j = 1, 2, \cdots, K$.

For the case when not all m_j are equal to 1, the proof immediately follows from the above result by considering vectors Z_i ($i = 1, 2, \cdots, n$) with $N' = \sum_{j=1}^{K} m_j$ components each, (the first m_1 corresponding to the first treatment, the next m_2 corresponding to the second and so on) and using an appropriate linear transformation of the $T_{n,j}$'s and part (i) of Lemma 2.4. In this case, however,

$$\lim_{n\to\infty} n^{-\frac{3}{2}} \{E(R_j) - \tfrac{1}{2} m_j n(nN' + 1)\} = \mu_j \qquad (j = 1, \cdots, K) \tag{3.8}$$

where μ_j is defined by (3.2). The proof is complete.

LEMMA 3.2. *Under the assumptions of Theorem* 3.1,

$$n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{K} \sum_{l=1}^{m_j} (r_{ijl} - \bar{r}_i)^2 = N'n^{-3} \sum_{i=1}^{n} r_i^2$$
$$= (N'^2/3)(N' - 1)(1 - 3\lambda) + o_p(1)$$

as $n \to \infty$, *where* λ *is given by* (3.2).

PROOF. Clearly it suffices to prove the result for the case $m_j = 1$ for all $j = 1, \cdots, K$. Thus, if we prove that

$$\lim_{n\to\infty} n^{-3} E[\sum_{i=1}^{n} \sum_{j=1}^{K} (r_{ij} - \bar{r}_i)^2] = \tfrac{1}{3} K^2 (K - 1)(1 - 3\lambda) \tag{3.9}$$

and

$$\lim_{n\to\infty} \operatorname{Var} [n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{K} (r_{ij} - \bar{r}_i)^2] = 0, \tag{3.10}$$

the result would follow on account Chebyshev's inequality. To prove (3.9), note that

$$n^{-3} \sum_{i=1}^{n} \sum_{j=1}^{K} (r_{ij} - \bar{r}_i)^2$$
$$= n K(nK + 1)(2nK + 1)/6 n^3 - Kn^{-3} \sum_{i=1}^{n} \bar{r}_i^2$$
$$= [K(nK + 1)(2nK + 1)/6n^2](1 - K^{-1}) - (n^3K)^{-1} \sum_{i=1}^{n} (\sum_{j\neq j'} r_{ij} r_{ij'}).$$

Taking expectation and then limit, as $n \to \infty$, on both sides

$$\lim_{n\to\infty} n^{-3} E[\sum_{i=1}^{n} \sum_{j=1}^{K} (r_{ij} - \bar{r}_i)^2]$$
$$= K^2(K - 1)/3 - K^{-1} \lim_{n\to\infty} \{n^{-2} \sum_{j\neq j'} E(r_{1j} r_{1j'})\}. \tag{3.11}$$

But

$$E(r_{1j} r_{1j'}) = \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{r=1}^{K} \sum_{s=1}^{K} E \{\delta(Z_{1j}, Z_{ir}) \ \delta(Z_{1j'}, Z_{i's})\} + O(n)$$
$$= \sum_{i=2\neq i'}^{n} \sum_{i'=2}^{n} \sum_{r=1}^{K} \sum_{s=1}^{K} E \{\delta(Z_{1j}, Z_{ir}) \ \delta(Z_{1j'}, Z_{i's})\} + O(n) \tag{3.12}$$

and, if 1, i and i' are all different,

$$E \{\delta(Z_{1j}, Z_{ir}) \ \delta(Z_{1j'}, Z_{i's})\} = P[Z_{ir} < Z_{1j}, Z_{i's} < Z_{ij'}]$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u + (\theta_r - \theta_j)n^{-\frac{1}{2}}) G(v + (\theta_s - \theta_{j'})n^{-\frac{1}{2}}) \, dH(u, v)$$
$$= \lambda + o(1) \tag{3.13}$$

as $n \to \infty$. Using (3.12) and (3.13) in (3.11), we get (3.9). To prove (3.10), we note from above that

$$\begin{aligned}
&\operatorname{Var}\,[n^{-3} \textstyle\sum_{i=1}^{n} \sum_{j=1}^{K} (r_{ij} - \bar{r}_i)^2] \\
(3.14)\qquad &= (K^2 n^6)^{-1} \operatorname{Var}\,\{\textstyle\sum_{i=1}^{n} \sum_{j\neq j'} r_{ij}\, r_{ij'}\} \\
&= (K^2 n^6)^{-1}\{n \operatorname{Var}\,(\textstyle\sum_{j\neq j'} r_{1j}\, r_{1j'}) + n(n-1) \\
&\qquad \cdot \operatorname{Cov}\,(\textstyle\sum_{j\neq j'} r_{1j}\, r_{1j'}, \sum_{j\neq j'} r_{2j}\, r_{2j'})\}.
\end{aligned}$$

One observes that Var $(r_{1j}, r_{1j'})$ and Cov $(r_{1j}\, r_{1j'}, r_{1l}\, r_{1l'})$ where $j \neq j'$ and $l \neq l'$ are $O(n^4)$ and in the expansion of

$$\begin{aligned}
&\operatorname{Cov}\,(r_{1j}\, r_{1j'}, r_{2l}\, r_{2l'}) \\
&\quad = E[\{\textstyle\sum_{i=1}^{n} \sum_{s=1}^{K} \delta(Z_{1j}, Z_{is})\}\, \{\sum_{u=1}^{n} \sum_{v=1}^{K} \delta(Z_{1j'}, Z_{uv})\} \\
&\qquad \{\textstyle\sum_{i'=1}^{n} \sum_{s'=1}^{K} \delta(Z_{2l}, Z_{i's'})\}\, \{\sum_{u'=1}^{n} \sum_{v'=1}^{n} \delta(Z_{2l'}, Z_{u'v'})\}] \\
&\quad - E[\{\textstyle\sum_{i=1}^{n} \sum_{s=1}^{K} \delta(Z_{1j}, Z_{is})\}\, \{\sum_{u=1}^{n} \sum_{v=1}^{K} \delta(Z_{1j'}, Z_{uv})\}] \\
&\quad \cdot E[\{\textstyle\sum_{i'=1}^{n} \sum_{s'=1}^{K} \delta(Z_{2l}, Z_{i's'})\}\, \{\sum_{u=1}^{n} \sum_{v=1}^{K} \delta(Z_{1j}, Z_{u'v'})\}] \\
&\quad + O(n^3),
\end{aligned}$$

one finds that the terms corresponding to pairs of subscripts (i, u) and (i', u') with no element common cancel out and the remaining terms are of order n^3. Using these results in (3.14) one finds that the right hand side of (3.14) is $O(n^{-1})$; this completes the proof of the lemma.

Proof of Theorem 3.1. From Lemma 3.1, it follows that the vector $\mathbf{T}_n^* = (T_{n,1}, \cdots, T_{n,K-1})'$ is distributed in the limit, as $n \to \infty$, as a multivariate normal vector $N(\boldsymbol{\mu}^*, \boldsymbol{\Lambda}^*)$, where $\boldsymbol{\mu}^* = (\mu_1, \cdots, \mu_{K-1})$ and $\boldsymbol{\Lambda}^*$ is the corresponding submatrix of $\boldsymbol{\Lambda}$. Observing that $\sum_{j=1}^{K} m_j^{\frac{1}{2}}\, T_{n,j} = 0$ and that $\boldsymbol{\Lambda}^*$ is non-singular, it follows easily that the statistic

$$(3.15)\qquad [3/N'^2(1 - 3\lambda)\, n^3] \textstyle\sum_{j=1}^{K} m_j^{-1}\, \{R_j - \frac{1}{2}\, n\, m_j(n N' + 1)\}^2 = \sum_{j=1}^{K} T_{n,j}^2$$

is equivalent to the statistic $\mathbf{T}_n^{*\prime}\, \boldsymbol{\Lambda}^{*-1}\, \mathbf{T}_n^*$, which is distributed in the limit, as $n \to \infty$, as a $\chi_{K-1}^2\,(\Delta_W)$ using part (i) of Lemma 2.4, and a thoerem of Rao [19] p. 57). If we now replace $(1 - 3\lambda)$ in (3.15) by its consistent estimate

$$3 \textstyle\sum_{i=1}^{n} r_i^2/N' \cdot (N' - 1)n^3$$

given by Lemma 3.2, we obtain the statistic W_n. The proof of the theorem is complete in view of a well-known theorem (see Cramér [3], p. 254).

Lemma 3.5 below gives sufficient conditions for interchanging the limit and integration in (3.2). Its proof in turn is based on Lemmas 3.3 and 3.4[2]: Let μ denote the Lebesgue-measure on the real line.

Lemma 3.3. *Let $F_1(x)$ be an absolutely continuous distribution function with a*

[2] Lemma 3.4 provides an improvement on Lemma 3(a) of Hodges and Lehmann's "Comparison of the normal-scores and Wilcoxon tests," *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 307–317.

square μ-integrable derivative $f_1(x) = F_1'(x)$ (a.e. μ) and let $F = F_1 * F_2$ denote the convolution of F_1 with any distribution function F_2. Then $F(x)$ is also absolutely continuous with a square-integrable derivative given by $f(x) = \int_{-\infty}^{\infty} f_1(x-y)\, dF_2(y)$ (a.e. μ).

PROOF. It is well-known that the absolute continuity of $F_1(x)$ implies that of $F(x)$ with derivative $f(x)$ (a.e. μ) as given above. To see that $\int_{-\infty}^{\infty} f^2(x)\, dx < \infty$, observe that by Schwarz's inequality

$$|f(x)|^2 = \{\int_{-\infty}^{\infty} f_1(x-y)\, dF_2(y)\}^2$$
$$\leqq \int_{-\infty}^{\infty} f_1^2(x-y)\, dF_2(y) \qquad \text{(a.e. } \mu),$$

so that

$$\text{(3.15a)} \qquad \int_{-\infty}^{\infty} f^2(x)\, dx \leqq \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} f_1^2(x-y)\, dF_2(y)\}\, dx$$
$$= \int_{-\infty}^{\infty} f_1^2(x)\, dx.$$

Since $\int_{-\infty}^{\infty} f_1^2(x)\, dx < \infty$, the proof is complete.

LEMMA 3.4. *If the distribution function $G(x)$ possesses a square-integrable density* (i.e., $\int_{-\infty}^{\infty} \{G'(x)\}^2\, dx < \infty$), *then*

$$\lim_{h\to 0} \int_{-\infty}^{\infty} [G(x+h) - G(x)]/h\, dG(x) = \int_{-\infty}^{\infty} \{G'(x)\}^2\, dx.$$

PROOF. The proof follows immediately by using Lemma 4.3 of Hájek [7]. To see this, note that by Schwarz's inequality again

$$|\int_{-\infty}^{\infty} [G(x+h) - G(x)]/h\, dG(x) - \int_{-\infty}^{\infty} \{G'(x)\}^2\, dx|^2$$
$$= |\int_{-\infty}^{\infty} \{[G(x+h) - G(x)]/h - G'(x)\} G'(x)\, dx|^2$$
$$\leqq (\int_{-\infty}^{\infty} \{G'(x)\}^2\, dx) \int_{-\infty}^{\infty} \{[G(x+h) - G(x)]/h - G'(x)\}^2\, dx$$
$$\to 0,$$

as $h \to 0$, on account of Lemma 4.3 of [7] and the square-integrability of $G'(x)$; the proof is complete.

LEMMA 3.5. *If the distribution function $F(x)$ possesses a square-integrable density, then* (i) *the* (*marginal*) *distribution function $G(u)$ of the aligned observations also possess a square-integrable density and* (ii) *the non-centrality parameter Δ_w of the Theorem* 3.1 *takes the form*

$$\text{(3.16)} \qquad \Delta_w = [3/(1-3\lambda)](\int_{-\infty}^{\infty} \{G'(u)\}^2\, du)^2 \sum_{j=1}^{K} m_j(\theta_j - \bar{\theta})^2$$

where $\bar{\theta} = \sum_{j=1}^{k} m_j\theta_j/N'$.

PROOF. Since $G(u)$ is the (marginal) distribution of the aligned observations, i.e., of a random variable

$$z_{11} = X_{11} - \bar{X} = (N'-1)N'^{-1}X_{11} - N'^{-1}X_{12} \cdots - N'^{-1}X_{Km_k},$$

where X's are independently and identically distributed according to the distribution $F(x)$, it is a convolution and a repeated application of Lemma 3.3 proves part (i). Part (ii) then follows immediately on account of Lemma 3.4 and part (i); the proof is complete.

Consider now the separate-rankings statistic W_n^* under (2.4), namely

$$W_n^* = [12/N'(N'+1)n] \sum_{j=1}^{K} mj^{-1}\{R_j^* - mj\, n(N'+1)/2\}^2,$$

where R_j^* = sum of the ranks for the jth treatment based on independent rankings of X-observations in each block.

THEOREM 3.2. *Assume for each n the truth of K_n and that the conditions of Lemma 3.5 are satisfied. Then, the statistic W_n^* converges in distribution, as $n \to \infty$, to $\chi^2_{K-1}(\Delta_{W^*})$, where*

$$(3.17) \quad \Delta_{W^*} = [12N'/(N'+1)] \left(\int_{-\infty}^{\infty} F'(x)\, dF(x)\right)^2 \sum_{j=1}^{K} m_j(\theta_j - \bar{\theta})^2.$$

PROOF. The proof of this theorem can be accomplished using the central limit theorem for random vectors. (For the case $m_j = 1$ for all j see van Elteren and Noether [22]).

For the case where the sample sizes m_{ij} satisfy (2.4), the classical $\mathfrak{F}$-statistic is also distributed in the limit, as $n \to \infty$, as a $\chi^2_{K-1}(\Delta_{\mathfrak{F}})$ where

$$(3.18) \quad \Delta_{\mathfrak{F}} = \sum_{j=1}^{K} m_j\,(\theta_j - \bar{\theta})^2/\sigma^2$$

provided for the distribution $F(x)$ the variance σ^2 exists. (See Scheffé [21], p. 119. Case of proportional frequencies.)

We will use the results of this section to obtain the asymptotic (Pitman) efficiency of the statistic W_n relative to the statistics W_n^* and $\mathfrak{F}$.

4. Asymptotic efficiency. The asymptotic efficiency $e_{S,S^*} = e_{S,S^*}(\alpha, \beta, K_n)$ of a test S relative to a test S^*, for the alternatives K_n, is defined to be the limiting inverse ratio of the sample sizes needed to attain the same power β at the same level of significance α. The concept originally due to Pitman is discussed by Noether [18]. For the statistics under consideration it is given by the ratio of their respective noncentrality parameters. (see [1], [6]). Thus, from (3.16) to (3.18),

$$(4.1) \quad e_{W,W^*} = [(N'+1)/4N'(1-3\lambda)]\left[\int_{-\infty}^{\infty} \{G'(x)\}^2\, dx \Big/ \int_{-\infty}^{\infty} \{F'(x)\}^2\, dx\right]^2$$

and

$$(4.2) \quad e_{W,\mathfrak{F}} = [3\sigma^2/(1-3\lambda)]\left[\int_{-\infty}^{\infty} \{G'(x)\}^2\, dx\right]^2$$

where $\lambda = \lambda(N', F)$ is given by (3.2).

Although the conditions of Lemma 3.5 are satisfied for a large class of distributions—including normal, logistic, uniform, (double) exponential, Cauchy etc.,—the explicit evaluation of the efficiency expressions (4.1) and (4.2) is rather difficult in most cases, except for the normal distribution for which case it is given below.

Efficiency when $F(x)$ is $N(a, \sigma^2)$. It is easily seen that for this case,

$\int_{-\infty}^{\infty} \{F'(x)\}^2\, dx = (2\sigma\, \pi^{\frac{1}{2}})^{-1}$ and $\int_{-\infty}^{\infty} \{G'(x)\}^2\, dx = [N'^{\frac{1}{2}}/2\sigma\, \pi^{\frac{1}{2}}(N' - 1)^{\frac{1}{2}}]$ and

$$\lambda = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x)G(y)\, dH(x, y)$$
$$= P[U > 0, \quad V > 0]$$

where U, V have a bivariate normal distribution with zero means, common variance $\{2\sigma^2(N' - 1)/N'\}$ and covariance $\{-\sigma^2/N'\}$. From [3], p. 290, we have

$$\lambda = \tfrac{1}{4} + (2\pi)^{-1} \text{arc sin } [(1 + \rho)/(1 - \rho)]^{\frac{1}{2}}, \quad \text{with} \quad \rho = -[2(N' - 1)]^{-1}$$
$$= \pi^{-1} \text{arc tan } [(2N' - 3)/(2N' - 1)]^{\frac{1}{2}}.$$

Thus, (4.1) and (4.2) give

$$\begin{aligned} e_{W,W^*}(\text{normal}) &= [(N' + 1)/4(N' - 1)]\cdot \\ &\quad \cdot\{1 - (3/\pi) \text{ arc tan } [(2N' - 3)/(2N' - 1)]^{\frac{1}{2}}\}^{-1}; \\ e_{W,\mathfrak{F}}(\text{normal}) &= [3N'/4\pi(N' - 1)]\cdot \\ &\quad \cdot\{1 - (3/\pi) \text{ arc tan } [(2N' - 3)/(2N' - 1)]^{\frac{1}{2}}\}^{-1}. \end{aligned} \tag{4.3}$$

For the case of one-observation-per-cell, i.e., $m_j = 1$ for all j, (4.3) reduces to

$$\begin{aligned} e_{W,W^*}(\text{normal}) &= [(K + 1)/4(K - 1)] \\ &\quad \cdot\{1 - (3/\pi) \text{ arc tan } [(2K - 3)/(2K - 1)]^{\frac{1}{2}}\}^{-1}; \\ e_{W,\mathfrak{F}}(\text{normal}) &= [3K/4\pi(K - 1)] \\ &\quad \cdot\{1 - (3/\pi) \text{ arc tan } [(2K - 3)/(2K - 1)]^{\frac{1}{2}}\}^{-1}. \end{aligned} \tag{4.4}$$

One can easily verify that $e_{W,W^*}(\text{normal})$, given by (4.4) takes value $\frac{3}{2}$ for $K = 2$, and decreases monotonically to 1, as $K \to \infty$. It means that the efficiency-advantage of the joint-ranking procedure over the separate-ranking procedure decreases steadily with increase in the number of treatments. The corresponding efficiency expression $e_{W,\mathfrak{F}}(\text{normal})$ takes the value $3/\pi$ for $K = 2$, attains a maximum at $K = 3$ (considering only the integral values of $K = 2, 3, 4, \cdots$) and then decreases monotonically to $3/\pi$, as $K \to \infty$. Consequently,

$$e_{W,\mathfrak{F}}(\text{normal}) \geqq 3/\pi \quad \text{for all} \quad K.$$

The following table gives some values of efficiency expressions (4.4):

	K				
	2	3	4	5	∞
e_{W,W^*} (normal	1.5	1.355	1.263	1.210	1
$e_{W,\mathfrak{F}}$ (normal)	$3/\pi$ = .9549	.9662	.9647	.9632	$3/\pi$

The above remarks apply verbatim, with K replaced by N', to the efficiency expressions (4.3). (This includes the case when K is fixed and sample sizes m_j,

$j = 1, \cdots, K$, increase to ∞). Our results, thus, provide a verification of the statement by Hodges and Lehmann ([9], p. 495) that the gain in efficiency, in using the joint-ranking procedure in place of the separate-ranking one, tends to zero if "instead of a large number of small blocks we are dealing with a small number of large blocks."

Efficiency for large K. In view of the last remark, it seems interesting to see what happens to the general efficiency expressions (4.1) and (4.2), as the block-size N' increases to ∞ (this includes the case when $K \to \infty$). The result that e_{W,W^*}(normal) converges to 1, as $N' \to \infty$, in fact, remains true for any distribution $F(x)$ which has a finite first moment.

THEOREM 4.1 *If the distribution $F(x)$ possesses a square-integrable density and $|a| = |\int_{-\infty}^{\infty} x\, dF(x)| < \infty$, then the efficiency expression* (4.1) *satisfies* $\lim_{N' \to \infty} e_{W,W^*}(F) = 1$.

PROOF. Let X_{ij} $(i = 1, 2, 3, \cdots; j = 1, 2, \cdots, N')$ be independent rv's with the same distribution $F(x)$ and let $\bar{X}_i = \sum_{N'' j=1}^{N'} X_{ij}/N'$. Since

$$|a| = |\int_{-\infty}^{\infty} x\, dF(x)| < \infty,$$

the strong law of large numbers applies, so that from (3.2)

$$\begin{aligned} \lim_{N' \to \infty} \lambda(N', F) &= \lim_{N' \to \infty} P[(X_{11} - \bar{X}_1) > (X_{21} - \bar{X}_2), \\ &\qquad (X_{12} - \bar{X}_1) > (X_{31} - \bar{X}_3)] \\ &= P[X_{11} > X_{21},\ X_{12} > X_{31}] = \tfrac{1}{4}. \end{aligned} \tag{4.5}$$

We shall now prove

$$\lim_{N' \to \infty} \int_{-\infty}^{\infty} \{G'_{N'}(x)\}^2\, dx = \int_{-\infty}^{\infty} \{F'(x)\}^2\, dx, \tag{4.6}$$

where we have written $G'_{N'}(x)$ in place of $G'(x)$ to indicate dependence of $G(x)$ on N'. Let $f_{N'}(x)$ denote the density of $\{(N' - 1)X_{11}/N'\}$, then

$$f_{N'}(x) = \{N'/(N' - 1)\}F'(N'x/(N' - 1))$$

and by argument similar to those used for deriving (3.15a) we have

$$\begin{aligned} \int_{-\infty}^{\infty} \{G'_{N'}(x)\}^2\, dx &\leqq \int_{-\infty}^{\infty} f_{N'}^2(x)\, dx \\ &= \{N'/(N' - 1)\} \int_{-\infty}^{\infty} \{F'(x)\}^2\, dx \end{aligned}$$

which gives

$$\limsup_{N' \to \infty} \int_{-\infty}^{\infty} \{G'_{N'}(x)\}^2\, dx \leqq \int_{-\infty}^{\infty} \{F'(x)\}^2\, dx. \tag{4.7}$$

Further, since $F(x)$ is continuous,

$$\lim_{N' \to \infty} G_{N'}(x) = \lim_{N' \to \infty} P\{X_{11} - \bar{X}_1 \leqq x] = F(x + a),$$

so that

$$\begin{aligned} \lim_{N' \to \infty} G'_{N'}(x) &= \lim_{N' \to \infty} \lim_{h \to 0} [G_{N'}(x + h) - G_{N'}(x)]/h \\ &= \lim_{h \to \infty} [F(x + h + a) - F(x + a)]/h \\ &= F'(x + a) \qquad \text{a.e. } (\mu). \end{aligned} \tag{4.8}$$

From (4.7) and (4.8) it follows by Fatou's lemma that

$$\begin{aligned}\int_{-\infty}^{\infty} \{F'(x)\}^2\, dx &= \int_{-\infty}^{\infty} \liminf_{N'} \{G'_{N'}(x)\}^2\, dx \\ &\leqq \liminf_{N'} \int_{-\infty}^{\infty} \{G_{N'}(x)\}^2\, dx \\ &\leqq \limsup_{N'} \int_{-\infty}^{\infty} \{G_{N'}(x)\}^2\, dx \\ &\leqq \int_{-\infty}^{\infty} \{F'(x)\}^2\, dx,\end{aligned}$$

so that (4.6) is proved. From (4.5) and (4.6) the desired result follows forthwith; the proof is complete.

The Cauchy distribution provides a typical example for which the conclusion of Theorem 4.1 does not hold. For the Cauchy distribution, it is easily seen that, as $N' \to \infty$, $[\int_{-\infty}^{\infty} \{G'(x)\}^2\, dx / \int_{-\infty}^{\infty} \{F'(x)\}^2\, dx] \to \frac{1}{2}$ and $\lambda \to P[X > U, X > V]$, where X, U, V are independent Cauchy variables with scale parameters 1, 3, 3 respectively (and the same arbitrary location parameter). The last probability is

$$\tfrac{1}{4} + \pi^{-3} \int_{-\infty}^{\infty} \{(\arctan x/3)^2/(1 + x^2)\}\, dx$$

and consequently from (4.1),

$$(4.9)\quad \lim_{N' \to \infty} e_{W,W^*}(\text{Cauchy}) = \{4[1 - (12/\pi^3) \int_{-\infty}^{\infty} \{(\arctan x/3)^2/(1 + x^2)\}\, dx]\}^{-1},$$

which is approximately equal to .458. One also observes that for the case $N' = 2$, i.e., two treatments with one observation per cell, the asymptotic efficiency (4.1) reduces to

$$e_{W,W^*} = [3 \int_{-\infty}^{\infty} \{G'(x)\}^2\, dx / \{G'(0)\}^2],$$

which is the asymptotic efficiency of the Wilcoxon signed-rank test relative to the sign test (cf. [10]). This expression takes the value .75 for the Cauchy distribution. Thus for the Cauchy distribution the efficiency expression (4.1) decreases from .75 when $N' = 2$ to .458 as $N' \to \infty$. This makes the joint ranking procedure with alignment on the mean further less desirable in this case.

It is, however, important to observe that the above decrease in efficiency for the Cauchy distribution, as $N' \to \infty$, is due (presumably) not to the joint-ranking procedure but to the method of alignment, namely, the alignment on the mean. Clearly one would reasonably expect the conclusion of Theorem 4.1 to hold in case of Cauchy distribution also, if instead the alignment is done on the median, or for that matter, on any other consistent estimate of location. But we shall not enter into the investigation of this question here.

5. Concluding remarks. A comparison of the results of the preceding section with those of Lehmann [14] and [15] shows that, for the case of the normal distribution, the conditional test based on the joint-ranking statistic W is less efficient than the test procedure, say L, of [15] and is more efficient than the procedure, say L', of [14].[3] On the other hand, one may find other interesting distributions

[3] Lehmann has shown in [15] that the asymptotic efficiency $e_{L,\mathfrak{F}}(F)$ is uniformly higher than the efficiency $e_{L',\mathfrak{F}}(F)$ for all distributions F and all K.

F for which the W-test is more efficient than the test procedure L or L'. It would, therefore, be further illuminating to evaluate the efficiency expressions $e_{W,\mathfrak{F}}(F)$ or $e_{W,W^*}(F)$ for the uniform, logistic, (double) exponential and other distributions F. The results of the preceding section, particularly the remarks of the last paragraph, also point out the need for investigating the efficiency of the joint-ranking procedure for other methods of alignment, especially for the alignment on the median.

It is worth pointing out that *the conditional test W discussed in this paper provides exact significance level for all sample sizes and is consequently truly distribution-free*, a property which is not possessed by L or L' (see [14], p. 1495). The question of robustness of its power, however, (particularly against distributions with heavy tails) needs investigation. It may be possible to find for the efficiency $e_{W,\mathfrak{F}}(F)$ a lower bound comparable to the one obtained in [10] for the efficiency of the Wilcoxon (Kruskal-Wallis) test relative to the t-test ($\mathfrak{F}$-test). The authors hope to discuss these points in a subsequent paper.

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