A THEOREM ON THE GALTON-WATSON PROCESS[1]

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In this note we will prove a theorem concerning a limiting distribution associated with the Galton-Watson process. Specifically, we consider a stochastic process, $\{Z_n ; n = 0, 1, \cdots\}$, with the following properties:

(1) $Z_0 = 1$;

(2) if P denotes the probability measure associated with the process, then $P(Z_1 = i) = p_i, i = 0, 1, \cdots$. Moreover the process is a Markoff process with transition probabilities,

$$P_{ij} = P(Z_{n+1} = j \mid Z_n = i) = \sum_{k_1+k_2+\cdots+k_i=j} p_{k_1} \cdot p_{k_2} \cdots p_{k_i},$$

$$i = 1, 2, \cdots, j = 0, 1, \cdots, P_{0j} = 0, j = 1, 2, \cdots, \quad \text{and} \quad P_{00} = 1;$$

(3) $p_i \neq 1$ for all i; and

(4) $E(Z_1) = m > 1$.

We will show that the random variables, $(Z_n/m^n), n = 0, 1, \cdots$, converge a.e. to a random variable, W, whose probability distribution has a jump at the origin and a continuous density function on the set of positive real numbers. Levinson and Harris have proved similar theorems but under more restrictive assumptions and by using quite different arguments. Specifically, Levinson [4] by assuming that $E(Z_1 \log Z_1) < \infty$ and Harris [3] by assuming that $E(Z_1^2) < \infty$ have established our result. Harris has also proved that his assumptions imply convergence in the mean of the (Z_n/m^n)'s, and both Harris and Levinson have proved that their assumptions imply that $E(W) = 1$ and that $P(W = 0) = q < 1$, where q is a number to be defined later. In contrast under our assumptions we can only prove that $E(W) \leqq 1$ and that $P(W = 0) = q$ or 1. However we will show in a forthcoming paper with Harry Kesten that if Assumptions 1 through 4 hold and if $P(W = 0) = q$, then $E(W) = 1$. Moreover $E(W) = 1$ only if $E(Z_1 \log Z_1) < \infty$.

The probability generating function of Z_1 will be denoted $f(s)$ and is defined by the equation, $f(s) = \sum_{k=0}^{\infty} p_k s^k$, on the set of all complex numbers s such that $|s| \leqq 1$. The probability generating function of Z_n will be denoted by $f_n(\cdot)$. We will make repeated use of a few facts about the $f_n(\cdot)$'s that are stated briefly below.

(5) $f_{n+k}(s) = f_n(f_k(s)) = f_k(f_n(s))$.

(6) There exists a unique real number q such that $0 \leqq q < 1, f(q) = q$.

(7) For all $s \in [q, 1)$ we have $1 > s \geqq f(s) \geqq f_2(s) \geqq \cdots \geqq f_n(s) \geqq q$ with

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$\lim_n f_n(s) = q$. Similarly for all $s \varepsilon [0, q]$ we have $0 \leqq s \leqq f(s) \leqq f_2(s) \leqq \cdots \leqq f_n(s) \leqq q$ with $\lim_n f_n(s) = q$.

(8) The $f_n(\cdot)$'s are all differentiable and convex on the unit interval. Moreover for all $s \varepsilon [q, 1)$, $f_n'(s) \leqq \{f'(s)\}^n$, and for all $s \varepsilon [0, q]$, $f_n'(s) \geqq \{f'(s)\}^n$.

Doob has pointed out that the random variables, (Z_n/m^n), $n = 0, 1, \cdots$ constitute a martingale that converges a.e. to a random variable W with mean less than or equal to one (Harris [2], p. 13). We will denote the characteristic function of W by $\varphi(it)$ where t varies over the interval $(-\infty, \infty)$. It is easy to show that $\varphi(\cdot)$ must satisfy the functional equation, $\varphi(mit) = f(\varphi(it))$. In particular, $\varphi(m^n it) = f_n(\varphi(it))$ for all n. We will next show that either $\varphi(it) = 1$ identically in t or $|\varphi(it)| < 1$ for all $t \neq 0$.

LEMMA 1. *Either* $\varphi(it) = 1$ *for all* t *or* $|\varphi(it)| < 1$ *for all* $t \neq 0$.

PROOF. Throughout this proof we assume that $\varphi(it)$ is not identically equal to one. We will first show that the equality $|\varphi(it)| = 1$ for all t is impossible. Suppose the contrary to be true. Then $\varphi(it) = e^{+ita}$ for some a (Loève [5], p. 202). Moreover, when using the functional equation, $\varphi(mit) = f(\varphi(it))$, we find that $e^{\pm imta} = \sum_{k=0}^{\infty} p_k e^{\pm ikta}$. Hence $\sum_{k=0}^{\infty} p_k(1 - \cos (k - m)ta = 0$ for all t. This implies that m is an integer and that $p_m = 1$ which is contrary to Assumption 3 above.

We will next show that there exists a $\delta > 0$ such that for all $t \varepsilon (-\delta, \delta) - 0$, $|\varphi(it)| < 1$. Suppose not. Then there is a sequence of numbers t_n that converge to zero with the property that $|\varphi(it_n)| = 1$ for all n. We may assume without loss in generality that the t_n's are all positive. If $|\varphi(it_n)| = 1$, then there exists a number a_n in the interval $[0, 2\pi)$ such that $\varphi(it_n) = e^{+ia_n}$. Moreover, if x is a point of increase of the distribution associated with W, then there exists an integer k_n such that $t_n x = a_n + k_n 2\pi$. Since the t_n's converge to zero, we conclude that there is a large number N such that $x = (a_n/t_n)$ for all $n \geqq N$. This implies that $\varphi(\cdot)$ is degenerate which contradicts our previous result. (The argument used above is due to Levinson [4].)

Finally we will show that $|\varphi(it)| < 1$ for all $t \neq 0$. Let $\delta > 0$ be so small that for all $t \varepsilon (-\delta, \delta) - 0$, $|\varphi(it)| < 1$, and let M be an integer so large that $m^{-M} < \delta$. Moreover let $A = [m^{-(M+1)}, m^{-M})$. If t_0 is any positive number greater than δ, then there exists a number $u \varepsilon A$ and an integer n such that $t_0 = m^n u$. Hence $|\varphi(it_0)| = |\varphi(im^n u)| = |f_n(\varphi(iu))| \leqq f_n(|\varphi(iu)|) < 1$. This proves our assertion for all positive t's. Our arguments with only obvious modifications work equally well for negative numbers. Q.E.D.

LEMMA 2. *If* $\varphi(it)$ *is not identically equal to one and* $q = 0$, *then* $\lim_{t \to \pm\infty} |\varphi(it)| = 0$.

PROOF. To prove this lemma we let δ be a positive number and let A be defined as in the proof of the preceding lemma. Moreover we let $\alpha = \max_{t \varepsilon \bar{A}} |\varphi(it)|$, and let N be an integer so large that $f_N(\alpha) < \epsilon$, where ϵ is an arbitrarily chosen positive real number. If t_0 is any number greater than m^{N-M-1}, then there is a number u in A such that $t_0 = m^n u$ for some n. Hence $|\varphi(it_0)| = |\varphi(im^n u)| \leqq f_n(|\varphi(iu)|) \leqq f_N(|\varphi(iu)|) \leqq f_N(\alpha) < \epsilon$. The proof that $\lim_{t \to -\infty} |\varphi(it)| = 0$ can be obtained by essentially the same argument. Q.E.D.

We will next show that if $q = 0$, then $\varphi'(\cdot)$ is absolutely integrable.

LEMMA 3. *If $\varphi(it)$ is not identically equal to one, then $\varphi(it)$ is differentiable; and if in addition $q = 0$, then $\varphi'(it)$ is absolutely integrable.*

PROOF. Since $E|W| \leqq 1$, $\varphi(\cdot)$ is obviously differentiable (Doob [1], p. 38). Hence the only thing to prove is that $\varphi'(\cdot)$ is absolutely integrable whenever $q = 0$. To do this we proceed as follows: Since $\lim_{t\to\pm\infty} |\varphi(it)| = 0$, we can find a large constant K such that for all t with absolute value greater than or equal to K, $f'(|\varphi(it)|) < 1$. Let $\beta = \max_{|t|\varepsilon[K,mK]} f'(|\varphi(it)|) < 1$. Then $\int_{m^nK}^{m^{n+1}K} |\varphi'(it)|\, dt = m^n \int_K^{mK} |\varphi'(im^nt)|\, dt \leqq \int_K^{mK} f_n'(|\varphi(it)|)|\varphi'(it)|\, dt \leqq \int_K^{mK} f'(|\varphi(it)|)^n |\varphi'(it)|\, dt \leqq \beta^n \cdot \int_K^{mK} |\varphi'(it)|\, dt$. Similarly $\int_{-m^{n+1}K}^{-m^nK} |\varphi'(it)|\, dt \leqq \beta^n \int_{-mK}^{-K} |\varphi'(it)|\, dt$. Hence for all $T \geqq K$ there exists a constant Q independent of T such that $\int_{-T}^{T} |\varphi'(it)|\, dt \leqq \int_{-K}^{K} |\varphi'(it)|\, dt + 2Q(m-1)(1/(1-\beta))K < \infty$. Q.E.D.

When using the preceding lemmas we can give an exceedingly simple proof of the following theorem:

THEOREM 1. *If $\varphi(it)$ is not identically equal to one, then the distribution of W has a jump of magnitude equal to q at the origin and a continuous density function on the set of positive real numbers.*

PROOF. We will first prove that the distribution of W is differentiable on the set, $(0, \infty)$ under the additional assumption that $q = 0$. Let $g_T(x) = (1/2\pi) \int_{-T}^{T} e^{-itx}\varphi(it)\, dt$ for $T = 1, 2, \cdots$, and $x > 0$. Clearly, $g_T(\cdot)$ is a continuous function on the set, $(0, \infty)$. Moreover by integrating by parts we find that $g_T(x) = (-1/2\pi ix)\{e^{-iTx}\varphi(iT) - e^{iTx}\varphi(-iT)\} + (1/2\pi ix) \int_{-T}^{T} e^{-itx} \cdot (d\varphi(it)/dt)\, dt$. Hence if $0 < x_1 < x_2 < \infty$, we can use Lemmas 2 and 3 to deduce that as T tends to infinity $g_T(\cdot)$ converges uniformly on $[x_1, x_2]$ to a continuous function $g(x)$. Finally if $K(x) = P(W \leqq x)$, then by using the uniform boundedness of $g_T(\cdot)$ on closed bounded intervals we find that $K(x_2) - K(x_1) = \lim_{T\to\infty} \int_{-T}^{T} ((e^{-itx_2} - e^{-itx_1})/-2\pi it)\varphi(it)\, dt = \lim_{T\to\infty} \int_{x_1}^{x_2} g_T(x)\, dx = \int_{x_1}^{x_2} g(x)\, dx$, which establishes the required differentiability of $K(\cdot)$ for the case, $q = 0$ (the last argument is the same as that used by Harris [3]). To show that $K(0+) = 0$ we need only observe that

$$\lim_{t\to\pm\infty} |\varphi(it)| = \lim_{t\to\pm\infty} |K(0+) + \int_{0+}^{\infty} e^{+itx}K'(x)\, dx| = |K(0+) + \lim_{t\to\pm\infty} \int_{0+}^{\infty} e^{+itx}K'(x)\, dx| = K(0+).$$

To prove the theorem in the case $q > 0$ we proceed as follows: Let $\Psi(it) = (\varphi((1-q)it) - q)/(1-q)$ and let $h(s) = (f((1-q)s + q) - q)/(1-q)$. Then it is easily shown that on the set, $\{0 \leqq s \leqq 1\}$, $h(\cdot)$ is a convex, differentiable probability generating function such that $h(0) = 0$, $h(s) \neq s^m$, and $h'(1) = m$. Moreover, if for all n we let $h_n(s) = h(h_{n-1}(s))$, then for all $s \,\varepsilon\, [0, 1)$ we have $\lim_n h_n(s) = \lim_n (f_n((1-q)s + q) - q)/(1-q) = 0$. Finally, $h_n'(s) = f_n'((1-q)s + q) \leqq f'((1-q)s + q)^n = h'(s)^n$. Since $P(W = 0) \geqq \lim_n f_n(0) = q$, it is also easy to show that $\Psi(it)$ is a characteristic function and that it satisfies the condition, $\Psi(imt) = h(\Psi(it))$. From these observations it follows that Lemmas 1, 2, and 3 apply to $\Psi(it)$. Hence $|\Psi(it)| < 1$ for all $t \neq 0$, $\lim_{t\to\pm\infty} |\Psi(it)| = 0$, and $\Psi'(it)$ is absolutely integrable. When using the in-equality, $(1 - q)|\Psi(it)| = |\varphi((1 - q)it) - q| \geqq ||\varphi((1 - q)it)| - q|$, these results imply that $\lim_{t\to\pm\infty} |\varphi(it)| = q$ and that $\varphi'(it)$ is absolutely integrable. Similarly if $G(\cdot)$ is the probability distribution associated with $\Psi(\cdot)$, then the result obtained in the preceding paragraph applies to $G(\cdot)$. Hence $G(\cdot)$ is differentiable on the set $(0, \infty)$ and $G(0+) = 0$. Finally, when using the definition of $\Psi(\cdot)$ and the unique correspondence between characteristic functions and right-continuous probability distributions we find that

$$G(x) = (K(x/(1 - q)) - qE^*(x))/(1 - q)$$

for all $x \geqq 0$, where $E^*(x) \equiv 1$ for $x \geqq 0$ and 0 for $x < 0$. This implies that $K(\cdot)$ is differentiable on $(0, \infty)$ and that $K(0+) = q$. Q.E.D.

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