INFINITELY DIFFERENTIABLE POSITIVE DEFINITE FUNCTIONS[1]

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The class of functions which are continuous and positive definite on the real line will be denoted by P. This paper presents two types of results concerning the derivatives of functions in P. The first type of result is that if a function in P agrees with a comparison function having certain properties on a sequence tending to the origin then the positive definite function either is identical to the comparison function or at least shares some of its properties. In 1960 R. G. Laha [6] proved the following theorem of this type:

A. Suppose $f \varepsilon P$ and g is analytic on the line (that is, g is the restriction to the real line of a function analytic in a strip of the complex plane along the line). If f and g agree on a double sequence x_k, $k = \pm 1, \pm 2, \cdots$, where $x_{-k} = -x_k$ and $x_k \rightarrow 0$, then f is analytic. Therefore $f = g$ on the line. Earlier, in a mimeographed note A. Devinatz [4] gave a similar result:

B. Suppose $f \varepsilon P$, $g \varepsilon C^\infty \cap P$, and the Hamburger moment sequence $(-i)^n g^{(n)}(0)$ is determined. If f and g agree on a sequence tending to the origin then $f = g$.

The second type of result is that certain properties possessed by a product of functions in P are shared by the factors. For instance, if the product of functions in P is infinitely differentiable or analytic then so are the factors. In 1959 A. Devinatz [5] proved the following result of this type:

C. Suppose $g \varepsilon P$, $h \varepsilon P$ and $f = gh$ is $2n$ times differentiable. Then g and h are also $2n$ times differentiable. For real r put $F_r(x) = e^{irx}f(x)$. Then for some real r, $|g^{(2k)}(0)| \leqq 2\,|F_r^{(2k)}(0)|$, $k = 0, \cdots, n$. A similar inequality holds for h.

We shall state our results in terms of certain classes of infinitely differentiable functions which were introduced by T. Carleman and S. Mandelbrojt. A positive sequence m_n is said to be *logarithmically convex* when the sequence $\log m_n$ is convex. A more useful equivalent definition of logarithmic convexity is that

$$m_0/m_1 \geqq m_1/m_2 \geqq \cdots \geqq m_n/m_{n+1} \geqq \cdots. \tag{1}$$

For a logarithmically convex sequence m_n we denote by $C(m_n)$ the class of functions, infinitely differentiable on the line, for which $f \varepsilon C(m_n)$ means that there is a finite $q = q(f)$ such that

$$\sup_x |f^{(n)}(x)| < q^n m_n, \qquad n = 0, 1, \cdots.$$

The purpose of introducing these classes of functions is to generalize certain

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properties of analytic functions which actually depend on the sequence of suprema of the derivatives. In particular, a class $C(m_n)$ is called *quasianalytic* provided that any two members, f and g, are identical whenever for some real x_0, $f^{(n)}(x_0) = g^{(n)}(x_0)$ for $n = 0, 1, \cdots$. Denjoy [3] and Carleman [2] proved that a class $C(m_n)$ is quasianalytic if and only if the decreasing sequence (1) sums to a divergent series.

The following two theorems are results of the first type.

THEOREM 1. *Suppose $f \varepsilon P$ and $g \varepsilon C^\infty$. If f and g agree on a sequence converging to the origin through both positive and negative values then $f \varepsilon C^\infty$.*

An improvement to be noted over Laha's result is that the comparison sequence of Theorem 1 need not be located symmetrically about the origin. On the other hand, the requirement that the comparison sequence tends to the origin both from the right and from the left is shown to be necessary by the positive definite function $e^{-|x|}$ which agrees with the analytic function e^{-x} for $x \geqq 0$. However, $e^{-|x|}$ is not differentiable at the origin.

THEOREM 2. *Suppose $f \varepsilon P$ and $g \varepsilon C(l_n)$. If f and g agree on a sequence converging to the origin through both positive and negative values then $f \varepsilon C(l_{n+1})$. In particular, if the class $C(l_n)$ is quasianalytic then $f = g$.*

The next two theorems are results of the second type mentioned above, although Theorem 3 is stated in terms of Borel measures whose Fourier-Stieltjes transforms are in P.

THEOREM 3. *Let μ and ν be non-negative, finite Borel measures on the line. Suppose the following absolute moments are finite;*

$$k_n = \int_{-\infty}^{\infty} |t|^n \, d\mu(t), \qquad l_n = \int_{-\infty}^{\infty} |t|^n \, d\nu(t), \qquad m_n = \int_{-\infty}^{\infty} |t|^n \, d\mu*\nu(t)$$

for $n = 0, 1, \cdots$. Put $a_n = \max(k_n, l_n)$, then $C(a_n) = C(m_n)$.

THEOREM 4. *Suppose g and h are in P, and $f = gh \varepsilon C(a_n)$. Then g and h are in $C(a_{n+1})$. In particular, if the product $f = gh$ is quasianalytic then so are the factors.*

These results apply to all infinitely differentiable positive definite functions. In fact, if $f(x) = \int_{-\infty}^{\infty} e^{itx} \, d\mu(t)$, where μ is a non-negative, finite Borel measure, then terms of the logarithmically convex sequence $m_n = \int_{-\infty}^{\infty} |t|^n \, d\mu(t)$ are all finite and $f \varepsilon C(m_n)$.

PROOF OF THEOREM 1. Let the comparison sequence in the hypotheses be written as a positive sequence $x_k \downarrow 0$ and a negative sequence $y_k \uparrow 0$. Put $F = \operatorname{Re} f$ and $G = \operatorname{Re} g$, then F and G also agree on x_k and y_k. Since $f \varepsilon P$,

$$(2) \qquad F(x) = \operatorname{Re} \int_{-\infty}^{\infty} e^{itx} \, d\mu(t) = \int_{-\infty}^{\infty} \cos(tx) \, d\mu(t)$$

where μ is a non-negative, finite Borel measure.

We shall show by an induction on the even order derivatives of G that the even moments of μ are finite, and therefore the even order derivatives of F and f exist. The agreement of F and G on both positive and negative sequences is used only to show that the odd order derivatives of G vanish at the origin.

The first step of the induction will illustrate the general step. By continuity $F(0) = G(0)$. Since $F \varepsilon P$, $F(0) \geqq F(x)$ for all real x. Therefore we can show that $G'(0) = 0$ as follows,

$$[G(x_k) - G(0)]/x_k = [F(x_k) - F(0)]/x_k \leqq 0 \quad \text{since} \quad x_k > 0$$

$$[G(y_k) - G(0)]/y_k = [F(y_k) - F(0)]/y_k \geqq 0 \quad \text{since} \quad y_k < 0.$$

Thus

$$0 \leqq \lim_{k\to\infty} [G(y_k) - G(0)]/y_k = G'(0) = \lim_{k\to\infty} [G(x_k) - G(0)]/x_k \leqq 0.$$

Since $G'(0) = 0$ we obtain from Taylor's formula that

$$\lim_{k\to\infty} [F(x_k) - F(0)]/x_k^2 = \lim_{k\to\infty} [G(x_k) - G(0)]/x_k^2 = \tfrac{1}{2}G''(0).$$

On the other hand, from (2),

$$[F(0) - F(x_k)]/x_k^2 = \int_{-\infty}^{\infty} [(1 - \cos(tx_k))/x_k^2]\, d\mu(t)$$
$$= 2 \int_{-\infty}^{\infty} [\sin^2(tx_k/2)/x_k^2]\, d\mu(t).$$

Now a standard argument using Fatou's lemma shows that the second moment of μ, $\int_{-\infty}^{\infty} t^2\, d\mu(t)$, is finite. Applying Lebesgue's convergence theorem we obtain $F''(x) = -\int_{-\infty}^{\infty} t^2 \cos(tx)\, d\mu(t)$. Thus $-F'' \varepsilon P$.

To complete the first step of the induction we use Rolle's theorem to show that there are positive and negative sequences converging to the origin on which F'' and G'' agree. Since F and G agree on $x_k \downarrow 0$ and on $y_k \uparrow 0$, there are sequences x_k' and y_k', where $y_k < y_k' < y_{k+1} < 0 < x_{k+1} < x_k' < x_k$, on which F' and G' agree. Therefore there are also sequences x_k'' and y_k'', where $y_k' < y_k'' < y_{k+1}' < 0 < x_{k+1}' < x_k'' < x_k'$, on which F'' and G'' agree.

The complete induction shows that the even moments $\int_{-\infty}^{\infty} t^{2n}\, d\mu(t)$ are finite for $n = 0, 1, \cdots$, hence $f \varepsilon C^\infty$.

Proof of Theorem 2. By Theorem 1, $f \varepsilon C^\infty$. Since f and g agree on a sequence converging to the origin we have $f^{(n)}(0) = g^{(n)}(0)$ for $n = 0, 1, \cdots$. If $f(x) = \int_{-\infty}^{\infty} e^{itx}\, d\mu(t)$ then the absolute moments, $m_n = \int_{-\infty}^{\infty} |t|^n\, d\mu(t)$, which form a logarithmically convex sequence, are all finite and $f \varepsilon C(m_n)$. Since $g \varepsilon C(l_n)$ there is a finite $q = q(g)$ such that $\sup_x |g^{(n)}(x)| < q^n l_n$. Also $m_{2n} = \int_{-\infty}^{\infty} t^{2n}\, d\mu(t) = |f^{(2n)}(0)| = |g^{(2n)}(0)| < q^{2n} l_{2n}$. Then by the logarithmic convexity of m_n and l_n we have $m_{2n} < q^{2n} l_{2n} \leqq (l_0/l_1) q^{2n} l_{2n+1}$ and $m_{2n-1} \leqq (m_0/m_1)/m_{2n} < (m_0/m_1) q^{2n} l_{2n}$. Hence $f \varepsilon C(l_{n+1})$.

Proof of Theorem 3. We first note that the assumption that the sequences k_n and l_n are finite is unnecessary. In the notation of the theorem, if m_n is finite, then an easy application of Fubini's theorem shows that k_n and l_n must be finite.

Straightforward use of inequalities (1) shows that the sequence a_n is logarithmically convex since it is the maximum of two such sequences. Then we obtain the following estimate for m_n from inequalities satisfied by logarithmically convex sequences.

$$m_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |s + t|^n\, d\mu(t)\, d\nu(s) \leqq \sum_{j=0}^{n} \binom{n}{j} \int_{-\infty}^{\infty} |t|^{n-j}\, d\mu(t) \int_{-\infty}^{\infty} |s|^j\, d\nu(s)$$
$$= \sum_{j=0}^{n} \binom{n}{j} k_{n-j} l_j \leqq \sum_{j=0}^{n} \binom{n}{j} a_{n-j} a_j \leqq a_0 a_n 2^n.$$

Hence any function in $C(m_n)$ is already in $C(a_n)$, so $C(m_n) \subset C(a_n)$.

To prove the opposite inclusion it will be enough to show that both $C(k_n)$ and $C(l_n)$ are contained in $C(m_n)$. A sufficient (and necessary) condition that $C(k_n) \subset C(m_n)$ is that $\limsup_{n\to\infty} (k_n/m_n)^{1/n} < \infty$. In our case, there is a more useful form of this condition,

$$(3) \qquad K(r) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (r^n |t|^n/m_n)\, d\mu(t) = \sum_{n=0}^{\infty} (k_n/m_n) r^n < \infty$$

for some $r > 0$. To show that (3) holds note that for $0 < r < 1$,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (r^n |t + s|^n/m_n)\, d\mu(t)\, d\nu(s) = \sum_{n=0}^{\infty} r^n < \infty.$$

Thus by Fubini's theorem, for $0 < r < 1$,

$$(4) \qquad \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (r^n |t + s|^n/m_n)\, d\mu(t) < \infty \text{ a.e. } (\nu).$$

We use the inequality $|t|^n \leqq 2^n(|t + s|^n + |s|^n)$ for some fixed real s to estimate $K(r)$.

$$K(r) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (r^n |t|^n/m_n)\, d\mu(t)$$
$$\leqq \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} [(2r\,|t + s|)^n/m_n]\, d\mu(t) + \sum_{n=0}^{\infty} [(2r\,|s|)^n/m_n] \int_{-\infty}^{\infty} d\mu(t).$$

The first term on the right is finite for $0 < r < \frac{1}{2}$ by (4). An estimate for the second term follows from an inequality implied by (1), $m_0/m_n \leqq (m_0/m_1)^n$. Thus

$$\sum_{n=0}^{\infty} [(2r\,|s|)^n/m_n] \int_{-\infty}^{\infty} d\mu(t) \leqq (k_0/m_0) \cdot \sum_{n=0}^{\infty} (2r\,|s|(m_0/m_1))^n.$$

The right side is finite for $0 < r < m_1/2\,|s|\,m_0$. Our estimate for $K(r)$ shows that it is finite for $0 < r < \frac{1}{2} \min(1, m_1/|s|\,m_0)$ where s is some fixed real number. Hence by (3), $C(k_n) \subset C(m_n)$.

The argument showing that $C(l_n) \subset C(m_n)$ is similar, so Theorem 3 is established.

Proof of Theorem 4. Suppose $g(x) = \int_{-\infty}^{\infty} e^{itx}\, d\mu(t)$ and $h(x) = \int_{-\infty}^{\infty} e^{itx}\, d\nu(t)$. Then $f(x) = \int_{-\infty}^{\infty} e^{itx}\, d\mu*\nu(t)$. Let the sequences k_n, l_n, and m_n be defined as in Theorem 3. Then $g \in C(k_n)$ and $h \in C(l_n)$. Let the sequence m_n' be the greatest logarithmically convex minorant of the sequence $\sup_x |f^{(n)}(x)|$. Since $f \in C(a_n)$ and since $m_n' \leqq \sup_x |f^{(n)}(x)|$ we have $C(m_n') \subset C(a_n)$. Furthermore, it is well known that $f \in C(m_n')$ [1], that is there is a finite $r = r(f)$ such that $\sup_x |f^{(n)}(x)| < r^n m_n'$. We use this inequality with the equality, $\sup_x |f^{(2n)}(x)| = \int_{-\infty}^{\infty} t^{2n}\, d\mu*\nu(t) = m_{2n}$, to show that $C(m_n) \subset C(m_{n+1}')$ as follows,

$$m_{2n} = \sup_x |f^{(2n)}(x)| < r^{2n} m_{2n}' \leqq (m_0'/m_1') r^{2n} m_{2n+1}'$$

and

$$m_{2n-1} \leqq (m_0/m_1) m_{2n} = (m_0/m_1) \sup_x |f^{(2n)}(x)| < (m_0/m_1) r^{2n} m_{2n}'.$$

We have obtained the inclusions $C(m_n) \subset C(m_{n+1}') \subset C(a_{n+1})$. By Theorem 3, since $g \in C(k_n)$ and $h \in C(l_n)$, both g and h are in $C(m_n)$. Therefore they are also in $C(a_{n+1})$.

REFERENCES

[1] BANG, T. (1960). *Om Quasi-analytiske Funktioner*. A. Busck, Copenhagen.
[2] CARLEMAN, T. (1926). *Fonctions Quasi Analytiques*. Gauthier-Villars, Paris.
[3] DENJOY, A. (1921). Sur les fonctions quasi-analytiques de variable reelle. *C. R. Acad. Sci. Paris* **173** 1329.
[4] DEVINATZ, A. (1958). On a lemma of U. V. Linnik. Unpublished.
[5] DEVINATZ, A. (1959). On a theorem of Lévy-Raikov. *Ann. Math. Statist.* **30** 583–586.
[6] LAHA, R. G. (1960). On a property of positive definite functions. *Bull. Amer. Math. Soc.* **66** 388–391.
[7] LOÈVE, M. (1960). *Probability Theory* (2nd ed.). Van Nostrand, Princeton, New Jersey.