SOME APPLICATIONS OF MONOTONE OPERATORS IN MARKOV PROCESSES

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1. Introduction. This paper establishes uniqueness, stability, and methods of error estimation for a broad class of integro-differential equations that arise in the study of Markov processes. In Section 2 we consider the equation $Tu = 0$, where T is defined for real-valued differentiable functions u on an interval S of the real line by

$$(1) \qquad (Tu)(x) = u(x) - g(x, u'(x)) - \alpha(x) \int_S u(y)\, dF_x(y).$$

Here $g(x, y)$ is a real-valued function defined for all $x \varepsilon S$ and all real y, $\alpha(x)$ satisfies $0 \leqq \alpha(x) \leqq 1$, and for each $x \varepsilon S$, F_x is a distribution function on S. In Section 3 the treatment is extended to an arbitrary space S. Here the variable u' is suppressed and T is defined for real-valued functions u on S by

$$(2) \qquad (Tu)(x) = u(x) - g(x) - \alpha(x) \int_S u(y)\, dP_x(y),$$

where for each $x \varepsilon S$, P_x is a probability measure on a fixed σ-algebra in S.

Using very elementary methods, it is shown that these operators are monotone in the sense of Collatz [3], *viz.*, $Tu \leqq Tv \Rightarrow u \leqq v$. The uniqueness, stability, and error estimation mentioned above are easily obtained from this property.

Equations of the type $Tu = 0$ are frequently satisfied by absorption probabilities, mean passage times and various other expectations associated with a Markov process in the space S. Some examples illustrating how the operator T arises are described in Section 4. The same methods have been extended to the functional equations encountered in Markovian decision problems [1], [2]. These applications are considered elsewhere.

2. S an interval of the real line. Let S be a finite or infinite interval of the real line, and let S^* be its open interior (a, b). We say '$u \leqq v$ on the boundary' if $\limsup_{x \to a+, x \to b-} [u(x) - v(x)] \leqq 0$, and '$u = v$ on the boundary' if $\lim_{x \to a+, x \to b-} [u(x) - v(x)] = 0$.

We consider now the operator T defined by (1). The distribution function F_x is taken continuous from the right.

Theorem 1. *Suppose $u \leqq v$ on the boundary. If $\alpha(x)F_x(x) < 1$ at each $x \varepsilon S^*$, then $Tu \leqq Tv$ on S^* implies $u \leqq v$.*

Proof. Suppose $\sup_x [u(x) - v(x)] = m > 0$, with m necessarily finite. Then there is a largest value of x, say x_0, such that $u - v = m$ at x_0; moreover, $x_0 < b$, since $u \leqq v$ on the boundary. To the right of x_0 we have $u - v < m$, at

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x_0 have $u' = v'$ and hence

(3) $(Tu - Tv)(x_0) \geqq m - \alpha(x_0)[mF_{x_0}(x_0) + \int_{x_0}^{b} (u - v)\, dF_{x_0}] > 0.$

This contradiction completes the proof.

COROLLARY 1. *Let f be a given function. If $\alpha(x)F_x(x) < 1$ for every $x \varepsilon S^*$, there is at most one solution u to the equation $Tu = 0$, satifying $u = f$ on the boundary.*

PROOF. If $Tu = Tv = 0$, then $u \leqq v$ and $v \leqq u$ by Theorem 1, hence $u = v$,

COROLLARY 2. *In the definition of T, suppose $\alpha(x) \leqq c$ where c is a constant less than 1. Then $|Tu - Tv| \leqq \epsilon$ for $x \varepsilon S^*$ and $|u - v| \leqq \delta$ on the boundary implies*

(4) $$|u - v| \leqq \max\,[\epsilon/(1 - c),\, \delta],$$

for all non-negative constants ϵ and δ.

PROOF. Observe that $T(u - m) = Tu - m + \alpha m$ where m is any constant. If we take $m \geqq \epsilon/(1 - c)$, then $-m + \alpha m \leqq -m + cm \leqq -\epsilon$. Thus $T(u - m) \leqq Tu - \epsilon \leqq Tv$ on S^*. If in addition, $m \geqq \delta$, then $u - m \leqq v$ on the boundary, so that $u - m \leqq v$ by Theorem 1. Similarly, $v - m \leqq u$.

3. General state-space. Let $(S, \mathfrak{F})$ be a probability space, and for each $x \varepsilon S$, let P_x be a probability measure on $\mathfrak{F}$, with $P_x(A)$ itself being a measurable function of x for each measurable set A. Attention is restricted to measurable sets, and to real-valued measurable functions f on S such that $\int_S |f(y)|\, dP_x(y) < \infty$ for each x. The complement of a set A relative to S is denoted by A', and I_A is the characteristic function of A. Define the operator Π by

(5) $$(\Pi f)(x) = \alpha(x) \int_S f(y)\, dP_x(y),$$

so that T as defined by (2) is of the form $Tf = f - g - \Pi f$.

THEOREM 2. *Let u and v be given functions with $u - v$ bounded above on S. Assume there is a monotone sequence of sets $S_1 \subset S_2 \subset \cdots \subset S$ such that*

(6) $$\lim_{n\to\infty} \sup_{x \varepsilon S_n'} [u(x) - v(x)] \leqq 0,$$

and

(7) $$\sup_{x \varepsilon S_n} \inf_N (\Pi^N I_{S_n})(x) < 1.$$

Let $S^* = \lim_{n\to\infty} S_n$. Then $Tu \leqq Tv$ for $x \varepsilon S^*$ implies $u \leqq v$.

(For proof of a related result by means of martingale theory, see [4], p. 437.)

PROOF. Let δ be an arbitrary positive number. From (6) there is a set $A = S_n \subset S^*$ such that $u - v \leqq \delta$ on A'. It suffices to show that $\sup_{x\varepsilon A} (u - v) = \delta + m$ for $m > 0$ is impossible. Let $\tilde{v} = v + \delta$ so that $\sup_{x \varepsilon A} (u - \tilde{v}) = m$. Since $Tv \leqq T\tilde{v}$, we have $Tu \leqq T\tilde{v}$ on A and hence $I_A(u - \tilde{v}) \leqq I_A\Pi(u - \tilde{v}) \leqq I_A \Pi I_A(u - \tilde{v})$, with the latter inequality following from the fact that $u - \tilde{v} \leqq 0$ on A'. Applying the positive operator $I_A\Pi$ to $I_A(u - \tilde{v}) \leqq I_A\Pi I_A(u - \tilde{v})$ N

times, we see that

$$I_A(u - \tilde{v}) \leqq (I_A\Pi)^N I_A(u - \tilde{v}) \leqq m\Pi^N I_A .$$

In view of (7), a contradiction is now apparent, and the proof is complete.

For ease in application, it is sometimes desirable to separate Conditions (6) and (7). One natural way to do this occurs when S is a topological space, and $\mathfrak{F}$ consists of the Borel sets in S. Assume this is the case. Then we say '$u \leqq v$ on the boundary' if for a given non-compact set $S^* \subset S$, there is a monotone sequence of compact sets $S_1 \subset S_2 \subset \cdots, S_n \to S^*$, such that (6) holds for this sequence. We say '$u = v$ on the boundary' if $u \leqq v$ and $v \leqq u$ on the boundary. (Since S^* is non-compact, boundary conditions are transitive; e.g., $u = f$ and $f = v$ on the boundary implies $u = v$ on the boundary.) Theorem 2 thus takes the following convenient form:

THEOREM 2′. *If $u - v$ is bounded above, $u \leqq v$ on the boundary, and*

$$\sup_{x \varepsilon A} \inf_N (\Pi^N I_A)(x) < 1$$

for every compact set $A \subset S^*$, then $Tu \leqq Tv$ on S^* implies $u \leqq v$.

Straightforward analogs of Corollaries 1 and 2 of Theorem 1 are now possible. Also, we get a somewhat different stability result, valid for $\alpha = 1$.

COROLLARY 3. *Suppose $u - v$ is bounded, $|u - v| \leqq \delta$ on the boundary, and Π satisfies the conditions of Theorem 2′. Then $|Tu - Tv| \leqq \epsilon$ on S^* implies $|u - v| \leqq \epsilon t + \delta$, where t is any positive function satisfying $t \geqq 1 + \Pi t$ on S^*.*

The proof follows that of Corollary 2.

If $u - v$ is continuous, the condition on Π in Theorem 2′ can be weakened to $\inf_N (\Pi^N I_A)(x) < 1$ for $x \varepsilon A$. With reference to the proof of Theorem 2, the contradiction is now obtained at a point at which $I_A(u - \tilde{v})$ is maximized. Also, with continuity of $u - v$, and the stronger hypothesis $Tu < Tv$ on S^*, the contradiction is obtained without any condition on Π.

Another type of condition encountered in practice involves having $\sup_x \alpha(x) < 1$. In this case, if $Tu \leqq Tv$ on S and $u - v$ is bounded, we obtain directly $u - v \leqq \sup_x \alpha(x)m$ where $m = \sup_x (u - v)$, so that m cannot be positive.

A version of Theorem 2 can be reached by another route: Notice that if $f_1 \leqq f_2$ implies $T^{-1}f_1 \leqq T^{-1}f_2$, that is, if T^{-1} is monotone in the usual sense, then application of T^{-1} to $Tu \leqq Tv$ gives $u \leqq v$ directly. Define T^{-1} by $T^{-1}f = \lim_{N\to\infty} [(g + f) + \Pi(g + f) + \cdots + \Pi^N(g + f)]$. Since $(g + f) + \Pi(g + f) + \cdots \Pi^N(g + f)$ is monotone for each finite N, T^{-1} will be monotone when the defining limit exists. In particular, if $\lim_{N\to\infty} \Pi^N u = \lim_{N\to\infty} \Pi^N v = h$, say, we see that $T^{-1}Tu = u - h$, $T^{-1}Tv = v - h$, and hence $Tu \leqq Tv$ implies $u \leqq v$.

4. Some examples. (a) An example of the operator (1) is provided by the storage system model described by Gaver and Miller [7]. In their notation ([7], p. 116), the storage level $W(t)$ decreases with deterministic rate $W'(t) = -r(W(t))$ between non-negative, independent, and identically distributed inputs, $S_1, S_2, \cdots$, which occur at Poisson times $t_1, t_2, \cdots$, with intensity parameter λ. If $r(x) \geqq \epsilon > 0$, and $\lim_{x\to\infty} r(x) < \lambda ES$, then $u(x)$, the probability that $W(t)$ is never zero given $W(0) = x$, satisfies

$$u(x) = -r(x)u'(x)\lambda^{-1} + \int_0^\infty u(x+z)\, dG(z), \qquad x \geqq 0,$$

where G is the common distribution function of the S_n. Also, $u(x) \to 0$ as $x \to 0$, and $u(x) \to 1$ as $x \to \infty$. The derivation is quite simple, starting with $W(0) = x + h$ and decomposing at a time when $W(t)$ has fallen to x. Uniqueness for this equation, which is evidently of the form $Tu = 0$, is provided by Corollary 1.

A variation of this model provides an example with $\alpha(x) < 1$. Let τ be a random time which is independent of $W(t)$ and exponentially distributed on $[0, \infty)$ with mean $1/k$, and let $u(x) = E[\min(t_0, \tau) \mid W(0) = x]$, where t_0 is the first time at which $W(t) = 0$. We take $r(x) \geqq \epsilon > 0$ as above, but require only $\limsup_{x\to\infty} r(x) < \infty$ in place of $\lim_{x\to\infty} r(x) < \lambda ES$. Instead of assuming that the intensity parameter is constant we assume that it is given by $\lambda(W(t))$ where $\lambda \geqq 0$ is continuous and bounded. Then

$$u(x) = [1 - r(x)u'(x) + \lambda(x) \int_0^\infty u(x+z)\, dG(z)](\lambda(x) + k)^{-1}.$$

The boundary conditions are $u(x) \to 0$, as $x \to 0$, and $u(x) \to 1/k$ as $x \to \infty$.

Corollary 2 as well as Corollary 1 can be applied to this equation. Notice that specification of u and then r, say, determines λ explicitly. Thus it is possible to generate a variety of specific conditions for which u is known exactly, and thereby obtain a good quantitative understanding of the model. Our understanding is enhanced by the fact that $p(x) = P[t_0 \geqq \tau \mid W(0) = x] = ku(x)$.

With this same model, let $u(x) = E[\int_0^{t'} \varphi(W(t))\, dt \mid W(0) = x]$, where $t' = \min(t_0, \tau)$ and φ is continuous and bounded with $\lim_{x\to\infty} \varphi(x) = \varphi^* < \infty$. Then $u(x)$ satisfies the above equation, except that $\varphi(x)$ appears in place of the constant 1 on the right, and $u(x) \to \varphi^*/k$ as $x \to \infty$.

(b) To illustrate the application of Theorem 2′, and at the same time show how time dependent processes can be treated, we obtain a uniqueness condition for the Kolmogorov backward differential equations. Using Feller's notation and assumptions ([6], p. 423 ff.), the backward equation for the case of discrete state space $\{0, 1, \cdots\}$ is

$$(8) \qquad \partial P_{ik}(\tau, t)/\partial\tau = c_i(\tau)P_{ik}(\tau, t) - c_i(\tau) \textstyle\sum_v p_{iv}(\tau)P_{vk}(\tau, t).$$

Multiplying both sides by $\exp(-\int_s^\tau c_i(\xi)\, d\xi) = 1 - F(\tau; i, s)$ and integrating on τ from s to t gives

$$(9) \quad P_{ik}(s, t) = P_{ik}(t, t)(1 - F(t; i, s)) + \int_s^t \textstyle\sum_v p_{iv}(\tau)P_{vk}(\tau, t)F(d\tau; i, s).$$

Regarding $P_{ik}(s, t)$ as a function u of i and s alone, that is, a function on $S = B \times \Lambda$ where $B = \{0, 1, \cdots\}$ and $\Lambda = [0, \infty)$, we see that (9) is of the form $Tu = u - \Pi u = 0$, where

$$(\Pi u)(i, s) = \int_s^\infty \textstyle\sum_v u(v, \tau)p_{iv}(\tau)F(d\tau; i, s),$$

with the side condition $u(i, \tau) = P_{ik}(t, t)$ for $\tau \geqq t$. For the boundary conditions required by Theorem 2′, let $B_n = \{0, 1, \cdots, n\}$, let $\Lambda_n = [0, \tau_n]$ where $\tau_n < t$, $\tau_n \to t$, and let $S_n = B_n \times \Lambda_n$, so that $S^* = B \times [0, t)$. The continuity of the functions c_i implies $(\Pi I_{A'})(i, s) \geqq \epsilon_A > 0$ for every compact $A \subset S^*$, hence $(\Pi I_A)(i, s) \leqq 1 - \epsilon_A$. Now suppose return from the infinite boundary is forbidden, by imposing appropriate conditions on the $c_i(t)$ and the $p_{ij}(t)$, so that $\lim_{n\to\infty} \sup_{i \varepsilon B_n'} u(i, \tau) = 0$ with the passage to the limit being uniform in $\tau \varepsilon [0, t]$. Then $u = \delta$ on the boundary where $\delta(i, \tau) = P_{ik}(t, t)$ is 1 if $i = k$ and 0 otherwise. As in the proof of Corollary 1, we conclude from Theorem 2′ that there is only one solution to (9), and hence to (8), which satisfies this boundary condition. This method easily extends to cover the general case treated by Feller ([5], p. 502).

(c) In conclusion, we note that Spitzer's minimum principle can be obtained from Theorem 2. Let R, A, $P(x, y)$, and the function f be defined as in Theorem 31.1, p. 373, of [8], and identify S with R, and S^* with $R - A$. For $x \varepsilon S^*$ and $B \subset S$, let $P_x(B) = \sum_{y \varepsilon B} P(x, y)$, and for $x \varepsilon S - S^*$, let $P_x(S - S^*) = 1$. Also let $\alpha = 1$. The main hypothesis of the minimum principle then takes the form $Tf \geqq 0$ on S^*. Let $c = \inf_{t \varepsilon S - S^*} f(t)$, which may be assumed finite. We have $Tc = 0 \leqq Tf$ on S^*. Also $c - f$ is bounded since f is non-negative. Since $P(x, y)$ is the transition function of an aperiodic and recurrent random walk, and we have made $S - S^*$ absorbing under P_x, $\inf_N (\Pi^N I_{S^*})(x) = 0$ for $x \varepsilon S^*$, by a well known result, and (7) is satisfied. With $S_n = S^*$, $n = 1, 2, \cdots$, (6) is satisfied and we conclude from Theorem 2 that $f \geqq c$ on S.

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