AN INTEGRAL EQUATION IN AGE DEPENDENT BRANCHING PROCESSES

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1. Introduction. Let $Z(t)$ be the number of cells at time t of a branching process starting at $t = 0$ with one new cell. Let $N(t)$ be the total number of such cells born by time t. Each cell has lifetime distribution function $G(t)$ with $G(0) = 0$. At the end of its life the cell disappears and is replaced by k cells with probability p_k, $k = 0, 1, 2, \cdots$, where $p_k \geqq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Each cell has lifetime distribution function $G(t)$ and proceeds independently of the state of the system, and identically as the parent cell. Such a process, for this general G, is called an age-dependent branching process and is extensively treated in [3].

Define $h(s) = \sum_{k=0}^{\infty} p_k s^k$. For $h'(1) \equiv m > 1$, there is an increasing population with probability one. For this case, by use of Smith's key renewal theorem, it is shown ([3], Chap. 6) that for $\int_0^\infty u\, dG(u) < \infty$, as $t \to \infty$,

$$E[Z(t)] \sim K_1 \exp(\alpha t) \tag{1}$$

$$E[N(t)] \sim K_2 \exp(\alpha t) \tag{2}$$

where K_1, K_2 and α may be evaluated. Similarly, for $m < 1$ [5],

$$E[Z(t)] \sim K_3 \exp(-\beta t). \tag{3}$$

It is the purpose of this note to determine necessary and sufficient conditions under which the convergence, for $m > 1$, of $E[Z(t)] \exp(-\alpha t)$ and $E[N(t)] \exp(-\alpha t)$ and for $m < 1$, the convergence of $E[Z(t)] \exp(\beta t)$ is monotone, and to give an elementary proof of (1), (2) and (3) under these conditions. This will be accomplished by study of the asymptotic properties of certain monotonic solutions of an integral equation, special cases of which determine the behavior of $E[Z(t)]$ and $E[N(t)]$.

2. Solutions of an integral equation.

Theorem. *Let $Q(t)$ satisfy*

$$Q(t) = K(t) + \int_0^t Q(t-u)h(u)\, du \tag{4}$$

where

$$h(u) > 0 \quad \text{for} \quad u > 0, \qquad h(u) = 0, \qquad u \leqq 0,$$

$$\int_0^\infty h(u)\, du = 1 \quad \text{and} \quad \int_0^\infty uh(u)\, du < \infty.$$

Let

$$K(u) \geqq 0 \quad \text{for} \quad u > 0, \qquad K(u) = 0, \qquad u \leqq 0,$$

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and

$$\int_0^\infty K(u)\,du < \infty.$$

Let

$$k(u) = K'(u)$$

exist almost everywhere.

Then (a) *$Q(t)$ is increasing if and only if $K(0) + k(u)/h(u) > 0$ for $u > 0$, and then*

$$\lim_{t\to\infty} Q(t) = \int_0^\infty K(u)\,du \Big/ \int_0^\infty uh(u)\,du, \tag{5}$$

(b) *$Q(t)$ is decreasing if and only if $K(0) + k(u)/h(u) < 0$ for $u < 0$ and* (5) *holds.*

(c) *$Q(t) = \int_0^\infty K(u)\,du / \int_0^\infty uh(u)\,du$ for $t > 0$ if and only if $K(0) + k(u)/h(u) = 0$ for $u > 0$.*

PROOF. $\Leftarrow$ Let $R(t) = Q(t) - K(0)$. Then

$$R(t) = \int_0^t [R(t-u) + K(0) + k(u)/h(u)]h(u)\,du.$$

Let $\{X_i\}$ be independent, identically distributed random variables on the positive axis each with density $h(u)$. Let $Y_i = K(0) + k(u)/h(u)$ if $X_i = u$. Let $N(t) = \max\{n: \sum_{i=1}^n X_i < t\}$. Then

$$R(t) = E[\textstyle\sum_{i=1}^{N(t)} Y_i].$$

The desired monotonicity relations follow.

Define, for $A(t)$ such that $\int_0^\infty A(u)\,du < \infty$, the Laplace transform of $A(t)$ as

$$\psi_A(s) = \int_0^\infty \exp(-st)A(t)\,dt \qquad \text{for} \quad s > 0.$$

From (4), $\psi_Q = \psi_K[1 - \psi_h]^{-1}$ and observe that

$$\lim_{s\downarrow 0} s\psi_Q(s) = \int_0^\infty K(u)\,du \Big/ \int_0^\infty uh(u)\,du.$$

For (a), $Q(t)$ is increasing and positive, so that by a well known Tauberian theorem [2], we may conclude that

$$\lim_{t\to\infty} Q(t) = \int_0^\infty K(u)\,du \Big/ \int_0^\infty uh(u)\,du.$$

For (b), $Q(t)$ is decreasing. Then $K(0) - Q(t) = \Phi(t)$ is increasing and positive.

$$\begin{aligned}\psi_\Phi(s) &= K(0)s^{-1} - \psi_Q(s)\\ &= K(0)s^{-1} - \psi_K(s)[1 - \psi_h(s)]^{-1}.\end{aligned}$$

Hence,

$$\lim_{s\downarrow 0} s\psi_\Phi(s) = K(0) - \int_0^\infty K(u)\,du \Big/ \int_0^\infty uh(u)\,du$$

and the Tauberian theorem yields

$$\lim_{t\to\infty} \Phi(t) = K(0) - \int_0^\infty K(u)\,du / \int_0^\infty uh(u)\,du$$

or

$$\lim_{t\to\infty} Q(t) = \int_0^\infty K(u)\,du / \int_0^\infty uh(u)\,du.$$

Case (c) follows from (a) or (b).

$\Rightarrow$ Assume $R(t) = E[\sum_{i=1}^{N(t)} Y_i]$ is increasing. Suppose $K(0) + k(u)/h(u) < 0$ for $a \leqq u < b$. Then

$$R(b) - R(a) = E[\textstyle\sum_{\{i: a \leqq X_i < b\}} Y_i].$$

The right side is negative, but the left side is non-negative, a contradiction.

The other cases follow similarly.

3. Application to age dependent branching processes. Using the notation of the introduction, let $E[Z(t)] = m(t)$ and $E[N(t)] = n(t)$. We then obtain the following theorem.

THEOREM. Let $m > 1$, $G'(t) = g(t) > 0$ *for* $t > 0$ *exist almost everywhere. Let* $\alpha > 0$ *be the solution of* $1 = m \int_0^\infty \exp(-\alpha u) g(u)\,du$, *and let* $\int_0^\infty u \exp(-\alpha u) \cdot (u)\,du < \infty$.

Then (a) $m(t) \exp(-\alpha t)$ *is decreasing (increasing) if and only if*

$$g(t) < \alpha[1 - G(t)][m - 1]^{-1} \quad (g(t) > \alpha[1 - G(t)][1 - m]^{-1})$$

and then

$$m(t) \exp(-\alpha t) < 1 \quad (m(t) \exp(-\alpha t) > 1).$$

Further, in either case,

$$\lim_{t\to\infty} m(t) \exp(-\alpha t) = (m - 1)[\alpha m^2 \int_0^\infty u \exp(-\alpha u) g(u)\,du]^{-1}$$

(b) $m(t) \exp(-\alpha t) = (m - 1)[\alpha m^2 \int_0^\infty u \exp(-\alpha u) g(u)\,du]^{-1}$ *for* $t > 0$ *if and only if* $g(t)$ *is exponential with parameter* $\alpha[m - 1]^{-1}$.

(c) $n(t) \exp(-\alpha t)$ *is decreasing if and only if* $g(t) < \alpha/m$ *for* $t > 0$, *and then* $n(t) \exp(-\alpha t) < 1$ *and*

$$\lim_{t\to\infty} n(t) \exp(-\alpha t) = [\alpha m \int_0^\infty u \exp(-\alpha u) g(u)\,du]^{-1}.$$

(d) *Let* $m < 1$. *Assume that there exists a* $\beta > 0$ *such that*

$$1 = m \int_0^\infty \exp(\beta u) g(u)\,du,$$

and that

$$\int_0^\infty u \exp(\beta u) g(u)\,du < \infty.$$

Then $m(t) \exp(-\alpha t)$ *is decreasing (increasing) if and only if* $g(t) > \beta[1 - G(t)][1 - m]^{-1}$ $(g(t) < \beta[1 - G(t)][1 - m]^{-1})$ *for* $t > 0$, *and then*

$$m(t) \exp(\beta t) < 1 \quad (m(t) \exp(\beta t) > 1).$$

In either case,

$$\lim_{t\to\infty} m(t) \exp(\beta t) = (1 - m)[\beta m^2 \textstyle\int_0^\infty u \exp(\beta u) g(u)\, du]^{-1}.$$

(e) $m(t) \exp(\beta t) = (1 - m)[\beta m^2 \int_0^\infty u \exp(\beta u) g(u)\, du]^{-1}$ *if and only if* $g(t)$ *is exponential with parameter* $\beta(1 - m)^{-1}$.

(f) Let $m < 1$. *Then* $n(t)$ *is increasing and*

$$\lim_{t\to\infty} n(t) = (1 - m)^{-1}.$$

PROOF. Using the notation of the introduction, let

$$E[Z(t)] = m(t), \qquad E[N(t)] = n(t).$$

Then [3]

$$m(t) = 1 - G(t) + m \textstyle\int_0^t m(t - u) g(u)\, du. \tag{6}$$

Also, we have [6] that

$$n(t) = 1 + m \textstyle\int_0^t n(t - u) g(u)\, du. \tag{7}$$

Multiplying (6) and (7) by $\exp(-\alpha t)$, and applying the previous theorem yields (a) and (c) for $m > 1$. Multiplying (6) by $\exp(\beta t)$ and applying the theorem yields (d) for $m < 1$. (b) and (e) follow from limiting cases of (a) and (d), respectively, and a standard characterization of the exponential distribution. Applying Laplace transforms to (7) and re-inverting yields, for $m < 1$,

$$n(t) = \textstyle\sum_{n=0}^\infty m^n G^{(n)}(t), \tag{8}$$

where $G^{(n)}$ is the nth convolution of G, and (f) follows immediately from (8). The theorem is proved.

We note that the exponential law fits this characterization.

The case $m = 1$ yields $m(t) \equiv 1$ and

$$\lim_{t\to\infty} t^{-1} n(t) = [\textstyle\int_0^\infty u g(u)\, du]^{-1} = \mu^{-1}$$

If $\int_0^\infty u^2 g(u)\, du = \mu_2 < \infty$, then an application of Smith's key renewal theorem in [4] yields the partial result that

$$t^{-1} n(t) \geqq \mu^{-1} (t^{-1} n(t) \leqq \mu^{-1}) \qquad \text{if} \quad \mu_2 \geqq 2\mu^2 (\mu_2 \leqq 2\mu^2).$$

4. Alternative method for sufficiency. If we had allowed the continuity of $h(t)$, then the integral equation (4) for $Q(t)$ could have been differentiated to obtain

$$Q'(t) = k(t) + K(0) h(t) + \textstyle\int_0^t Q'(t - u) h(u)\, du. \tag{9}$$

Equation (9) can be solved for $Q'(t)$ by application of Laplace transforms and re-inversion from which the positivity properties of $Q'(t)$, and hence the monotonicity properties of $Q(t)$ could be immediately deduced. Then a straightforward use of a Hardy-Littlewood Tauberian theorem [1] on the Laplace transform of the equation for $Q'(t)$ would have yielded the limit.

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