NOTES

GENERALIZATIONS OF THE MAXIMAL ERGODIC THEOREM[1]

By Glen Baxter

Purdue University

Let L_1 be the class of real-valued, integrable functions on the measure space $(X, \mathcal{B}, \mu)$, and let T be a norm-reducing, positive linear operator from L_1 into L_1. For any $f \varepsilon L_1$, let

$$z_n(f) = f + Tf + \cdots + T^{n-1}f, \qquad z_0(f) = 0, \tag{1}$$

$$M_n(f) = \max_{0 \leqq k \leqq n-1} z_k(f), \qquad n \geqq 1.$$

We use the abbreviation imk for the statement *for infinitely many values of* k and the abbreviation fmk for the statement *for only finitely many values of* k. Hopf's maximal ergodic theorem [1] states:

Theorem 1. *For any* $f \varepsilon L_1$ *and any* $n \geqq 1$,

$$\int_{\{M_n(f)>0\}} f \, d\mu \geqq 0. \tag{2}$$

We give here some generalizations of the maximal ergodic theorem. We state first a result which according to the established nomenclature would be called the limsup ergodic theorem.

Theorem 2. *For any* $f \varepsilon L_1$,

$$\int_{\{z_k(f)>0 \text{ imk}\}} f \, d\mu \geqq 0. \tag{3}$$

As a standard application of the maximal ergodic theorem one shows that the ratios $Q_n(f, p) = z_n(f)/z_n(p)$, where f and p are non-negative functions in L_1, satisfy

$$\sup_n Q_n(f, p) < \infty \tag{4}$$

almost everywhere on the set where $p > 0$. By an auxiliary argument, one extends (4) to the set where $\sum_0^\infty T^k p > 0$. We give a generalization of (2) which shows directly that (4) holds on the set where $\sum_0^\infty T^k p > 0$. Theorems 1 and 2 correspond in the following to the case $t = 1$.

Theorem 3. *For any* $f \varepsilon L_1$ *and any integers* n, t *with* $n \geqq t \geqq 1$,

$$\int_{\{M_n(f)>M_t(f)\}} z_t(f) \, d\mu \geqq -\int_{\{M_n(f)=M_t(f)\}} M_t(f) \, d\mu. \tag{5}$$

Theorem 4. *For any* $f \varepsilon L_1$ *and any integer* $t \geqq 1$,

$$\int_{\{z_k(f)>M_t(f) \text{ imk}\}} z_t(f) \, d\mu \geqq -\int_{\{z_k(f)>M_t(f) \text{ fmk}\}} M_t(f) \, d\mu. \tag{6}$$

Received 23 February 1965.

[1] This research was supported in part by the United States Air Force under grant number AF-AFOSR 722-65.

If one applies (5) to the function $f - ap$, one can deduce that $\mu\{(z_t(p) > 0) \cap (\limsup Q_n = \infty)\} = 0$. Since t is arbitrary, (4) must hold on the set where $z_\infty(p) > 0$.

We now sketch a proof of Theorem 4. For the moment t and n are fixed. We suppress dependence of z_k on f and employ the notation $a^+ = \max(a, 0)$. Let $s(0), \cdots, s(n)$ be a permutation of $0, \cdots, n$ such that $z_{s(0)} \geqq \cdots \geqq z_{s(n)}$ is an ordering of $z_0, \cdots, z_n$. If $z_j = z_k$, $j < k$, then $s^{-1}(j) < s^{-1}(k)$. Let $A_m = \{x : z_{s(m)} > 0\}$, let $L_{m,t} = \{x : s(i) \geqq t \text{ all } 0 \leqq i \leqq m\}$, and let $B_m = A_m \cap L_{m,t}$. It follows that $B_0 \supset B_1 \supset \cdots \supset B_n$. On B_m

$$\sum_{j=0}^{m} z_{s(j)}^+ \leqq \sum_{j=0}^{m} (z_t + T^t z_{s(j)}^+). \tag{7}$$

For, at each point $x \varepsilon B_m$, one has $z_{s(j)}^+ = z_{s(j)} = z_t + T^t z_{k_j}$ for some set of distinct integers k_j with $0 \leqq k_0, \cdots, k_m \leqq n - t$. Of course, the particular values of k_j depend on x. In any case, $\sum_{j=0}^{m} z_{k_j} \leqq \sum_{j=0}^{m} z_{s(j)}^+$ for all choices of k_j, so that (7) is established. Next, for any integer h, $0 \leqq h \leqq n$, let $C_m = B_m - B_{m+1}$, $0 \leqq m \leqq h - 1$, and let $C_h = B_h$. Then, by (7)

$$\begin{aligned} \sum_{j=0}^{h} \int_{B_j} z_t \, d\mu &= \sum_{m=0}^{h} \int_{C_m} \sum_{j=0}^{m} z_t \, d\mu \\ &\geqq \sum_{m=0}^{h} \int_{C_m} \sum_{j=0}^{m} (z_{s(j)}^+ - T^t z_{s(j)}^+) \, d\mu \\ &= \sum_{j=0}^{h} \int_{B_j} (z_{s(j)}^+ - T^t z_{s(j)}^+) \, d\mu. \end{aligned} \tag{8}$$

The last term in (8) would be non-negative if the integration were over X instead of B_j, so we deduce finally

$$\sum_{j=0}^{h} \int_{B_j} z_t \, d\mu \geqq -\sum_{j=0}^{h} \int_{\tilde{B}_j} z_{s(j)}^+ \, d\mu. \tag{9}$$

On $\tilde{B}_j$, one has $z_{s(j)}^+ \leqq M_t$. For, if $x \varepsilon \tilde{A}_j$, then $z_{s(j)}^+ = 0$, while if $x \varepsilon \tilde{L}_{j,t}$, then $s(i) < t$ for some integer $0 \leqq i \leqq j$. This means $M_t \geqq z_{s(i)}^+ \geqq z_{s(j)}^+$. We rewrite (9) accordingly,

$$\sum_{j=0}^{h} \int_{B_j} z_t \, d\mu \geqq -\sum_{j=0}^{h} \int_{\tilde{B}_j} M_t \, d\mu. \tag{10}$$

It is noted in passing that Theorem 3 follows from (10) by setting $h = 0$.

We now let n vary, keeping t fixed. Denoting dependence on n by a superscript, we have first that $B_j^{(n)} \subset B_j^{(n+1)}$. Letting $F_j = \bigcup_{n=0}^{\infty} B_j^{(n)}$, it follows that

$$\sum_{j=0}^{h} \int_{F_j} z_t \, d\mu \geqq -\sum_{j=0}^{h} \int_{\tilde{F}_j} M_t \, d\mu. \tag{11}$$

But, F_j is the set on which more than j of the sums $z_0, z_1, \cdots$ are greater than M_t. Thus, $E = \bigcap_{j=0}^{\infty} F_j = \{z_k > M_t \text{ imk}\}$, and (6) follows by dividing (11) by h and letting h tend to infinity.

Continuous flow analogues of the above inequalities have been found recently by Mr. Kenneth Berk.

REFERENCE

[1] Hopf, E. (1954). The general temporally discrete Markov process. *J. Rat. Mech. Anal.* **3** 13–45.