A LOCAL LIMIT THEOREM FOR NONLATTICE MULTI-DIMENSIONAL DISTRIBUTION FUNCTIONS[1]

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1. Introduction and statement of results. Local limit theorems for asymptotically stable lattice distribution functions have been obtained by Gnedenko [2], [3] for the one-dimensional case and by Rvačeva [6] for the multi-dimensional case. We shall here obtain analogous results for nonlattice distribution functions.

Let F be a stable distribution function in d-dimensional space R^d which has a density p. Let F_1 be a distribution function in the domain of attraction of F, let F_n denote the n-fold convolution of F_1 with itself, and let B_n and A_n be constants in R and R^d respectively such that

$$\lim_{n\to\infty} F_n(B_n(x + A_n)) = F(x), \qquad x \in R^d. \tag{1}$$

Let f and f_1 denote the characteristic functions of F and F_1 respectively. We say that F_1 is *nonlattice* if

$$|f_1(\theta)| < 1, \qquad \theta \in R^d - (0). \tag{2}$$

We say that F_1 is *strongly nonlattice* if

$$e^{-c_1 d} = \limsup_{|\theta|\to\infty} |f_1(\theta)| < 1. \tag{3}$$

It is clear that F_1 is nonlattice if it is strongly nonlattice.

Note that lattice distribution functions and nonlattice (as defined here) distribution functions do not exhaust all possibilities unless $d = 1$, since a distribution function can be lattice in some directions, but not in others. The last possibility will not be considered in this paper.

For $x = (x^1, \cdots, x^d) \in R^d$ and $h > 0$, let $P(x, h)$ and $P_n(x, h)$ denote the measures assigned by F and F_n respectively to the set

$$\{y = (y^1, \cdots, y^d) | x^k \leqq y^k < x^k + h \quad \text{for} \quad 1 \leqq k \leqq d\}.$$

If $d = 1$, for example, then $P(x, h) = F(x + h) - F(x)$; while if $d = 2$, then

$$P(x, h) = F(x^1 + h, x^2 + h) - F(x^1 + h, x^2) - F(x^1, x^2 + h) + F(x^1, x^2).$$

It follows from (1) that

$$\lim_{n\to\infty} P_n(B_n(x + A_n), B_n h) = P(x, h), \quad x \in R^d \quad \text{and} \quad h > 0. \tag{4}$$

The purpose of this paper is to prove the following

THEOREM. *If F_1 is nonlattice, then*[2]

$$P_n(B_n(x + A_n), B_n h) = P(x, h) + o_n(1)(h^d + B_n^{-d}). \tag{5}$$

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[2] We use in this paper the convention that the behavior of any "o" term is uniform in all variables not listed in the term or previously fixed.

If F_1 is strongly nonlattice, then for any fixed $c < c_1$

$$P_n(B_n(x + A_n), B_nh) = P(x, h) + o_n(1)(h^d + e^{-cn}). \tag{6}$$

That this is really a "local" theorem becomes apparent on letting $h \to 0$. As we shall observe later, p is continuous and vanishes at ∞ and is therefore uniformly continuous. It now follows easily from (1) that the statements of the theorem are equivalent to those of

COROLLARY 1. *If F_1 is nonlattice, then*

$$P_n(B_n(x + A_n), B_nh) = h^d p(x) + o_{n,h}(1)(h^d + B_n^{-d}). \tag{7}$$

If F_1 is strongly nonlattice, then for any fixed $c < c_1$

$$P_n(B_n(x + A_n), B_nh) = h^d p(x) + o_{n,h}(1)(h^d + e^{-cn}). \tag{8}$$

By setting $h = B_n^{-1}$ we obtain

COROLLARY 2. *If F_1 is nonlattice, then*

$$P_n(B_n(x + A_n), 1) = B_n^{-d}p(x) + o(B_n^{-d}). \tag{9}$$

Similarly we have

COROLLARY 3. *If F_1 is nonlattice and $A_n \equiv 0$, then for fixed $x \in R^d$*

$$P_n(x, 1) = B_n^{-d}p(0) + o(B_n^{-d}). \tag{10}$$

Results such as Corollary 3 are useful in obtaining limit theorems for occupation times (see e.g. Kallianpur and Robbins [4]).

A further specialization is

COROLLARY 4. *If F_1 is nonlattice and has mean 0 and positive-definite covariance matrix Σ, then for fixed $x \in R^d$*

$$P_n(x, 1) = [(2\pi n)^{d/2}|\Sigma|^{\frac{1}{2}}]^{-1} + o(n^{-d/2}). \tag{11}$$

For the case $d = 1$, Corollary 4 can be stated as follows: *if F_1 is nonlattice and has mean 0 and finite variance $\sigma^2 > 0$, then for fixed $x \in R$*

$$F_n(x + 1) - F_n(x) = \sigma^{-1}(2\pi n)^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}). \tag{12}$$

This one-dimensional result has been obtained by Shepp [7] and, to the best of my knowledge, is essentially the only special case of the above theorem to have appeared previously in the literature.

In the proof of the theorem we use some of the methods of Gnedenko and Rvačeva from the papers quoted above. The idea of using the convolution method to eliminate the tail of the characteristic function f_1 was suggested by the work of Esseen ([1], pp. 30–36).

2. Proof. For $x = (x^1, \cdots, x^d) \in R^d$ and $\theta = (\theta_1, \cdots, \theta_d) \in R^d$ set $|\theta| = (\theta_1^2 + \cdots + \theta_d^2)^{\frac{1}{2}}$, $\|\theta\| = \max_{1 \leq k \leq d} |\theta_k|$, and $x \cdot \theta = x^1\theta_1 + \cdots + x^d\theta_d$. Define $K(x)$, $x \in R^d$, and $k(\theta)$, $\theta \in R^d$, by

$$K(x) = \frac{1}{(2\pi)^d}\left(\frac{\sin\frac{x^1}{2}}{\frac{x^1}{2}}, \cdots, \frac{\sin\frac{x^d}{2}}{\frac{x^d}{2}}\right)^2, \qquad x \in R^d,$$

and

$$k(\theta) = (1 - |\theta_1|) \cdot \cdots \cdot (1 - |\theta_d|), \qquad \|\theta\| < 1,$$
$$= 0, \qquad \|\theta\| \geqq 1.$$

Then $\int_{R^d} K(x)\,dx = 1$ and

$$\int_{R^d} e^{ix\cdot\theta} K(x)\,dx = k(\theta), \qquad \theta \in R^d.$$

For $a > 0$ set $K_a(x) = a^{-d}K(a^{-1}x)$, $x \in R^d$, and $k_a(\theta) = k(a\theta)$, $\theta \in R^d$. Then

$$\int_{R^d} K_a(x)\,dx = 1$$

and

$$\int_{R^d} e^{ix\cdot\theta} K_a(x)\,dx = k_a(\theta), \qquad \theta \in R^d.$$

Now $P_1(\cdot, h)$ is integrable and

$$\int_{R^d} e^{ix\cdot\theta} P_1(x, h)\,dx = h^d \prod_{k=1}^d [(1 - e^{-ih\theta_k})(ih\theta_k)^{-1}] f_1(\theta).$$

Similarly

$$\int_{R^d} e^{ix\cdot\theta} P_n(B_n(x + A_n), B_n h)\,dx = h^d \prod_{k=1}^d (1 - e^{-ih\theta_k})(ih\theta_k)^{-1} e^{-iA_n\cdot t} f_1{}^n(B_n{}^{-1}\theta).$$

Set

$$V_n(x, h, a) = \int_{R^d} K_a(x - y) P_n(B_n(y + A_n), B_n h)\,dy.$$

Then

(13) $V_n(x, h, a) =$

$$h^d(2\pi)^{-d} \int_{\|\theta\| \leqq a^{-1}} e^{-ix\cdot\theta} k_a(\theta) \prod_{k=1}^d (1 - e^{-ih\theta_k})(ih\theta_k)^{-1} e^{-iA_n\cdot\theta} f_1{}^n(B_n{}^{-1}\theta)\,d\theta.$$

It follows from Levy ([5], pp. 221–223) that the logarithm ψ of f is such that $R\psi(\theta) = -|\theta|^\alpha C(\theta/|\theta|)$, where $\alpha \in (0, 2]$ is the index of the stable distribution and C is a continuous, strictly positive function on the unit sphere in R^d. We have that $|f(\theta)| \leqq 1$ for $\theta \in R^d$, $|f(\theta)| < 1$ unless $\theta = 0$, $f(\theta) \to 0$ as $|\theta| \to \infty$, and $f(\theta)$ is integrable. It follows from the integrability of f, by the Fourier inversion formula and the Riemann-Lebesgue lemma, that the density p of F is continuous and that $p(x) \to 0$ as $|x| \to \infty$. The density p is therefore uniformly continuous on R^d, as stated in the introduction.

Since the Theorem and Corollary 1 are clearly equivalent, it suffices to prove Corollary 1. We first investigate the behavior of $V_n(x, h, a)$ as $n \to \infty$, $h \to 0$, and $a \to 0$.

LEMMA 1. *If F_1 is nonlattice, then for any $N > 0$*

(14) $$V_n(x, h, a) = h^d(p(x) + o_{n,h,a}(1)), \qquad a \geqq (NB_n)^{-1}.$$

If F_1 is strongly nonlattice, then for any fixed $c < c_1$

(15) $$V_n(x, h, a) = h^d(p(x) + o_{n,h,a}(1)), \qquad a \geqq e^{-cn}.$$

In proving this lemma we need to estimate the following integrals:

$$I_1 = \int_{\epsilon B_n \leqq \|\theta\| \leqq a^{-1}} e^{-ix\cdot\theta} k_a(\theta) \prod_{k=1}^d (1 - e^{-ih\theta_k})(ih\theta_k)^{-1} e^{-iA_n\cdot\theta} f_1^n(B_n^{-1}\theta)\, d\theta;$$

$$I_2 = \int_{A < \|\theta\| < \epsilon B_n} e^{-ix\cdot\theta} k_a(\theta) \prod_{k=1}^d (1 - e^{-ih\theta_k})(ih\theta_k)^{-1} e^{-iA_n\cdot\theta} f_1^n(B_n^{-1}\theta)\, d\theta;$$

$$I_3 = \int_{\|\theta\| \leqq A} e^{-ix\cdot\theta} (k_a(\theta) \prod_{k=1}^d (1 - e^{-ih\theta_k})(ih\theta_k)^{-1} e^{-iA_n\cdot\theta} f_1^n(B_n^{-1}\theta) - f(\theta))\, d\theta;$$

$$I_4 = \int_{A < \|\theta\|} e^{ix\cdot\theta} f(\theta)\, d\theta.$$

We note that

$$|(1 - e^{-ih\theta_k})(ih\theta_k)^{-1}| \leqq 1$$

and

$$|1 - (1 - e^{-ih\theta_k})(ih\theta_k)^{-1}| \leqq \tfrac{1}{2} h\theta_k .$$

Thus for any fixed $\epsilon > 0$

$$I_1 \leqq \int_{\epsilon B_n < \|\theta\| \leqq a^{-1}} |f_1(B_n^{-1}\theta)|^n\, d\theta.$$

If F_1 is nonlattice (and in particular if F_1 is strongly nonlattice), then for any fixed $N > 0$ there is a $\delta > 0$ such that $|f_1(\theta)| \leqq e^{-\delta}$ for $\epsilon < \|\theta\| \leqq N$. Since (1) clearly necessitates that $B_{n+1}/B_n \to 1$ as $n \to \infty$, we obtain

$$\int_{\epsilon B_n < \|\theta\| \leqq NB_n} |f_1(B_n^{-1}\theta)|^n\, d\theta \leqq (NB_n)^d e^{-\delta n} = o_n(1).$$

Thus if F_1 is nonlattice, then $a^{-1} \leqq NB_n$ and $I_1 = o_n(1)$. If F_1 is strongly nonlattice and $c < c_1$, then N can be made large enough so that

$$|f_1(\theta)| \leqq \exp(-d(c + c_1)/2) \qquad \text{for} \quad \|\theta\| > N.$$

Consequently

$$\int_{NB_n < \|\theta\| \leqq e^{cn}} |f_1(B_n^{-1}\theta)|^n\, d\theta \leqq \exp(-dn(c_1 + c)/2 + dnc)$$
$$= \exp(-dn(c_1 - c)/2) = o_n(1).$$

Hence if F_1 is strongly nonlattice, then $a^{-1} \leqq e^{cn}$ and $I_1 = o_n(1)$. We see therefore that $I_1 = o_n(1)$ in either the nonlattice or strongly nonlattice case.

Since the strongly nonlattice condition is not involved in estimating I_2, I_3, and I_4, we assume throughout the rest of the proof of the lemma simply that F_1 is nonlattice.

Let $A > 0$ be fixed. Then $k_a(\theta) = 1 + o_a(1)$ as $a \to 0$, $e^{-iA_n\cdot\theta} f_1^n(B_n^{-1}\theta) = f(\theta) + o_n(1)$ as $n \to \infty$, and

$$\prod_{k=1}^d (1 - e^{-ih\theta_k})(ih\theta_k)^{-1} = 1 + o_h(1) \qquad \text{as } h \to 0,$$

where $o_a(1)$, $o_n(1)$, and $o_h(1)$ converge to 0 uniformly in $\|\theta\| \leqq A$. Thus for fixed $A > 0$, $I_3 = o_{n,h,a}(1)$.

Since F is integrable, A may be chosen so that I_4 is as small as desired.

We have left only to estimate I_2. In particular we have to show that for sufficiently small ϵ and sufficiently large A and n, $\int_{A < \|\theta\| \leqq \epsilon B_n} |f_1(B_n^{-1}\theta)|^n\, d\theta$ can be made as small as desired. A proof of exactly this result appears in Rvačeva [6], pp. 203–4, and will not be repeated here.

Since

$$p(x) = (2\pi)^{-d} \int_{R^d} e^{-ix\cdot\theta} f(\theta)\, d\theta, \qquad x \varepsilon R^d,$$

the proof of the lemma is complete.

Corollary 1 can be reformulated as

LEMMA 2. *If F_1 is nonlattice, then for any $\epsilon > 0$ and $N > 0$ there exist $n_0 > 0$ and $h_0 > 0$ such that if $n \geqq n_0$, $x \varepsilon R^d$, and $(NB_n)^{-1} \leqq h \leqq h_0$, then*

$$h^d(p(x) - \epsilon) \leqq P_n(B_n(x + A_n), B_n h) \leqq h^d(p(x) + \epsilon). \tag{16}$$

If F_1 is strongly nonlattice, then for any $\epsilon > 0$ and $c < c_1$, there exist $n_0 > 0$ and $h_0 > 0$ such that if $n \geqq n_0$, $x \varepsilon R^d$, and $e^{-cn} \leqq h \leqq h_0$, then (16) *holds.*

We shall prove only the first statement. The slight modifications necessary to prove the second statement will be obvious.

Let p_0 denote the finite maximum of $p(x)$, $x \varepsilon R^d$. Choose $\epsilon > 0$ and $N > 0$. Since $p(x)$, $x \epsilon R^d$, is uniformly continuous, there is an $h_1 \epsilon (0, 1)$ such that $|p(x) - p(y)| \leqq \frac{1}{4}\epsilon$ if $\|x - y\| \leqq h_1$. There is a $\delta > 0$ such that $(1 + 2\delta)^d \leqq \frac{4}{3}$, $(1 + 2\delta)^d - 1 = \epsilon_1$ and

$$\int_{\|x\|>1/\delta} K(x)\, dx = \epsilon_2,$$

where

$$(p_0 + \epsilon_1 p_0 + \tfrac{1}{2}\epsilon)(1 - \epsilon_2)^{-1} - p_0 \leqq \epsilon$$

and

$$\epsilon_1 p_0 + \epsilon_2(p_0 + \epsilon) \leqq \tfrac{1}{2}\epsilon.$$

Set $i = (1, \cdots, 1) \varepsilon R^d$. By Lemma 1 we can find $n_0 > 0$ and $h_0 \varepsilon (0, h_1)$ such that for $n \geqq n_0$, $x \varepsilon R^d$, and $(NB_n)^{-1} \leqq h \leqq h_0$

$$\begin{aligned} V_n(x - \delta h i, h(1 + 2\delta), \delta^2 h) &\leqq h^d(1 + 2\delta)^d p(x - \delta h i) + \tfrac{1}{6}\epsilon h^d \\ &\leqq h^d(1 + 2\delta)^d(p(x) + \tfrac{1}{4}\epsilon) + \tfrac{1}{6}\epsilon h^d \\ &\leqq h^d(p(x) + \epsilon_1 p_0 + \tfrac{1}{2}\epsilon) \end{aligned}$$

and

$$\begin{aligned} V_n(x + \delta h i, h(1 - 2\delta), \delta^2 h) &\geqq h^d(1 - 2\delta)^d p(x + \delta h i) - \tfrac{1}{4}\epsilon h^d \\ &\geqq h^d(1 - 2\delta)^d(p(x) - \tfrac{1}{4}\epsilon) - \tfrac{1}{4}\epsilon h^d \\ &\geqq h^d(p(x) - \epsilon_1 p_0 - \tfrac{1}{2}\epsilon). \end{aligned}$$

Now

$$P_n(B_n(x - \delta h i - y + A_n), B_n h(1 + 2\delta)) \geqq P_n(B_n(x + A_n), B_n h), \quad \|y\| \leqq \delta h,$$

and

$$P_n(B_n(x + \delta h i - y + A_n), B_n h(1 - 2\delta)) \leqq P_n(B_n(x + A_n), B_n h), \quad \|y\| \leqq \delta h.$$

Consequently

$$\begin{aligned} V_n(x - \delta hi, h(1 + 2\delta), \delta^2 h) &\geqq \int_{\|y\| \leqq \delta h} K_{\delta^2 h}(y) P_n(B_n(x - \delta hi - y + A_n), h(1 + 2\delta))\, dy \\ &\geqq \int_{\|y\| \leqq \delta h} K_{\delta^2 h}(y) P_n(B_n(x + A_n), B_n h)\, dy \\ &= (1 - \epsilon_2) P_n(B_n(x + A_n), B_n h). \end{aligned}$$

Therefore

$$\begin{aligned} P_n(B_n(x + A_n), B_n h) &\leqq h^d(p(x) + \epsilon_1 p_0 + \tfrac{1}{2}\epsilon)(1 - \epsilon_2)^{-1} \\ &\leqq h^d(p(x) + \epsilon). \end{aligned}$$

Similarly

$$\begin{aligned} V_n(x + \delta hi, h(1 - 2\delta), \delta^2 h) &\leqq \int_{\|y\| \leqq \delta h} K_{\delta^2 h}(y) P_n(B_n(x + \delta hi - y + A_n), B_n h(1 - 2\delta))\, dy \\ &\quad + \epsilon_2(p_0 + \epsilon) h^d \\ &\leqq P_n(B_n(x + A_n), B_n h) + \epsilon_2(p_0 + \epsilon) h^d. \end{aligned}$$

Thus

$$P_n(B_n(x + A_n), B_n h) \geqq h^d(p(x) - \epsilon_1 p_0 - \epsilon_2(p_0 + \epsilon) - \tfrac{1}{2}\epsilon) \geqq h^d(p(x) - \epsilon).$$

This completes the proof of Lemma 2, from which Corollary 1 and the Theorem follow immediately.

REFERENCES

[1] Esseen, C. G. (1945). Fourier analysis of distribution functions. A mathematical study of the Laplace-Gaussian law. *Acta Math.* **77** 1–125.

[2] Gnedenko, B. V. (1954). On a local limit theorem for identically distributed independent summands. *Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe* **3** 287–293. (Russian)

[3] Gnedenko, B. V. and Kolmogorov, A. N. (1954). *Limit Theorems for Sums of Independent Random Variables.* Addison-Wesley, Cambridge. (English translation).

[4] Kallianpur, G. and Robbins, H. (1954). The sequence of sums of independent random variables. *Duke Math J.* **21** 285–307.

[5] Lévy, P. (1954). *Théorie de l'Addition des Variables Aléatoires.* Gauthier-Villars, Paris.

[6] Rvačeva, E. L. (1954). On domains of attraction of multi-dimensional distributions. *L'vov. Gos. Univ. Uč. Zap. Ser. Meh.-Mat.* **6** 5–44. (*Select. Transl. Math. Statist. and Probability,* (1962) **2** 183–205.

[7] Shepp, L. A. (1964). A local limit theorem. *Ann. Math. Statist.* **35** 419–423.