TOLERANCE LIMITS FOR THE CLASS OF DISTRIBUTIONS[1] WITH INCREASING HAZARD RATES

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1. Introduction and summary. Let $\mathfrak{F}$ be an arbitrary class of distribution functions and let $Y_1, Y_2, \cdots, Y_N$ be a sample of size N from some $F \varepsilon \mathfrak{F}$. For $0 < P, \gamma < 1$, a statistic $U_N = U(Y_1, \cdots, Y_N)$ is said to be a (P, γ) upper tolerance limit for F relative to $\mathfrak{F}$, if

$$P_F[1 - F(U_N) \leqq P] \geqq \gamma \tag{0}$$

for all $F \varepsilon \mathfrak{F}$. (The interval $(-\infty, U_N]$ would be called a $1 - P$ content tolerance limit at confidence level γ in the terminology of [3].)

For certain parametric classes of distributions such as the normal family or the exponential family, tolerance limits are available for all sample sizes greater than or equal to $N = 2$ for all $0 < P, \gamma < 1$. (See, e.g. [5] for normal tolerance limits. In [4] exponential and double exponential tolerance limits are obtained based on the concept of exponential content.) The forms of the tolerance limits in these cases are heavily dependent upon the particular parametric class under consideration. Consequently, if tolerance limits are desired for larger classes of distributions it is necessary to abandon these results in favor of statistics U_N for which the distribution of $1 - F(U_N)$ can be appropriately bounded for all F's in the larger class.

Until now, the statistics used for this purpose were certain of the sample order statistics, $X_K = K$th smallest of $Y_1, Y_2, \cdots, Y_N$, $K = 1, 2, \cdots, N$. These are the traditional non-parametric tolerance limits. Fraser [2] and Robbins [7] have shown that they have desirable uniqueness and optimality properties when $\mathfrak{F}$ is the class of distributions absolutely continuous with respect to Lebesgue measure.

The non-parametric tolerance limits have one unfortunate disadvantage; namely, for given P and γ, there is a minimum sample size $N(P, \gamma, K)$ such that the condition

$$P_F[1 - F(X_{N-K}) \leqq P] \geqq \gamma$$

is met only if $N \geqq N(P, \gamma, K)$. Thus, in cases where sampling is very expensive and stringent requirements are made on the tolerance limit (small P and large γ) or where the statistician is presented with a sample of size $N < N(P, \gamma, 0)$ without the possibility of obtaining additional observations, the only recourse has

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been to use parametric tolerance limits with the distinct possibility that the limit obtained is meaningless.

In this paper we propose a compromise scheme whereby upper tolerance limits of the form $X_{N-K-j} + b(X_{N-K} - X_{N-K-j})$ are constructed which are valid for all $N \geqq 2$ and all $0 < P, \gamma < 1$. These limits satisfy (0) for the class, $\mathfrak{F}_U$, of absolutely continuous distribution functions for which $\varphi = -\log(1 - F)$ is convex and, thus, earn the title log-convex (L.C.) tolerance limits. The class $\mathfrak{F}_U$ has been studied by Barlow, Marshall and Proschan in [1] and was shown to contain most of the distributions commonly used as models in statistics including the normal and exponential. The distributions in this class are called increasing hazard rate distributions in reliability theory due to the fact that the instantaneous hazard rate $f/(1 - F)$, is non-decreasing on the support of the probability density f. We will elaborate on the properties of this class in Section 2. Lower (P, γ) tolerance limits of the form $L_N = X_{K+j+1} - b(X_{K+j+1} - X_{K+1})$, which satisfy $P_F[F(L_N) \leqq P] \geqq \gamma$ for the class $\mathfrak{F}_L$ of absolutely continuous distributions for which $\psi = -\log F$ is convex, will be obtained in Section 3. It is thus of interest to note (in Section 2) that the important class of distributions possessing densities which are Pólya frequency functions of order 2 (PF_2) is contained in $\mathfrak{F}_L \cap \mathfrak{F}_U$. Hence both upper and lower tolerance limits are available for members of this class.

Tables of the b-factors needed for both upper and lower tolerance limits are given in Section 4 for $j = 1$ and all combinations of $P = .500, .250, .100, .050$ and $\gamma = .90, .95, .99$. The sample sizes range from $N = 2$ to the smallest value of N for which the usual non-parametric tolerance limit can be used. At this point the L.C. tolerance limits are equal to the non-parametric ones. No questions of optimality are considered here; however, some comparisons of the L.C. tolerance limits with the usual normal and exponential tolerance limits are made by means of Monte Carlo sampling in Section 5. Some extensions of the theory will also be considered in Section 5.

2. Properties of the classes $\mathfrak{F}_U$ and $\mathfrak{F}_L$. A non-negative, measurable function f is said to be *positive of order two* (P_2) if for every two sets of increasing numbers $x_1 < x_2$, $t_1 < t_2$, $\det \|f(x_i - t_j)\|_{1,2} \geqq 0$. f will be called a *Pólya frequency function of order two* (TP_2) if, in addition, $\int_{-\infty}^{\infty} f(x)\,dx = 1$. Probability densities which are TP_2 abound. The normal, exponential, double exponential, gamma, beta, logistic and uniform distributions have densities which are members of the TP_2 family both in their original forms or truncated. However, the probability densities of the Cauchy, Student-t, F and Burr distributions, which decrease to zero in the tails at an algebraic rate, are not TP_2.

The scope of the family of distributions with TP_2 densities makes it admirably suited as an alternative to the usual non-parametric classes of distributions in situations where it is undesirable to use the parametric theory. Both upper and lower L.C. tolerance limits can be obtained for this family by virtue of the following theorem which is proved in [1], page 378.

THEOREM 1. *Let F be an absolutely continuous distribution with probability density f. Then if f is TP_2, $F \varepsilon \mathfrak{F}_U \cap \mathfrak{F}_L$.*

Examples of distributions in $\mathfrak{F}_U$ or $\mathfrak{F}_L$ exist for which the corresponding densities are not TP_2 (see, e.g. [1]). However, the distributions with TP_2 densities constitute, by far, the most important subclass of $\mathfrak{F}_U$ and $\mathfrak{F}_L$.

Lemma 2 of [8] implies that a TP_2 function $f(x)$ tends to zero exponentially as $x \to \pm\infty$. Consequently distributions with TP_2 densities possess moments of all orders. Similarly, [1], if $F \varepsilon \mathfrak{F}_U$, then $1 - F(x)$ tends to zero exponentially as $x \to \infty$ and if $F \varepsilon \mathfrak{F}_L$, $F(x)$ tends to zero exponentially as $x \to -\infty$. It is for this reason that distributions with densities which decrease to zero algebraically in one or both tails, such as the Cauchy distribution, are excluded from one or both of the classes $\mathfrak{F}_U$ and $\mathfrak{F}_L$.

3. Derivation of the tolerance limits. We will first establish a result which is more general than required to obtain the upper and lower L.C. tolerance limits. This theorem will provide the basis for the extensions of the theory which we will discuss in Section 5.

THEOREM 2. *Let H be a continuous and strictly monotone (either increasing or decreasing) function on the (closed) interval $[0, 1]$ into $[-\infty, \infty]$, and let $\mathcal{C}_H$ be the class of all continuous probability distribution functions F for which the composed functions HF are convex. Let $X_1, X_2, \cdots, X_N$ be the order statistics for a sample of size N from $F \varepsilon \mathcal{C}_H$, and $U_1, U_2, \cdots, U_N$ the order statistics for a sample of size N from the uniform distribution on $[0, 1]$. Then for every ρ in $H[(0, 1)]$ and $b \geqq 1$,*

$$(1)\quad \begin{aligned} &P_F[F(X_{N-K-j} + b(X_{N-K} - X_{N-K-j})) \geqq, \leqq H^{-1}(\rho)] \\ &\qquad \geqq P[bH(U_{N-K}) - (b-1)H(U_{N-K-j}) \geqq \rho], \end{aligned}$$

and

$$(2)\quad \begin{aligned} &P_F[F(X_{K+j+1} - b(X_{K+j+1} - X_{K+1})) \geqq, \leqq H^{-1}(\rho)] \\ &\qquad \geqq P[bH(U_{K+1}) - (b-1)H(U_{K+j+1}) \geqq \rho], \end{aligned}$$

for all $F \varepsilon \mathcal{C}_H$ and $0 \leqq K < K + j \leqq N - 1$. The inequality $\geqq$ applies in (1) and (2) when H is increasing and $\leqq$ when H is decreasing.

PROOF. Only the proof of (1) will be given, since the proof of (2) follows from it with only trivial modifications.

Let $\varphi(x) = HF(x)$ and let x and y be elements of $S_F = \{x \mid 0 < F(x) < 1\}$ with $x < y$. The chord to the curve $v = \varphi(u)$ which passes through the points $(x, \varphi(x)), (y, \varphi(y))$ is

$$l(u) = \varphi(x) + \{[\varphi(y) - \varphi(x)]/(y - x)\}(u - x),$$

and, because φ is convex and continuous on $[0, 1]$, $\varphi(z) \geqq l(z)$ for all $z \geqq y$. If we write $z = x + b(y - x)$ with $b \geqq 1$, it follows that

$$\varphi(x + b(y - x)) \geqq \varphi(x) + b(\varphi(y) - \varphi(x)) = b\varphi(y) - (b - 1)\varphi(x).$$

Thus, if $\rho \varepsilon H[(0, 1)]$, the inequality

$$bHF(y) - (b - 1)HF(x) \geqq \rho$$

implies $HF(x + b(y - x)) \geqq \rho$, or, equivalently, $F(x + b(y - x)) \geqq H^{-1}(\rho)$ when H is increasing and $F(x + b(y - x)) \leqq H^{-1}(\rho)$ when H is decreasing. Now, set $y = X_{N-K}$ and $x = X_{N-K-j}$. With F probability 1, $X_{N-K} \varepsilon S_F$, $X_{N-K-j} \varepsilon S_F$ and $X_{N-K-j} < X_{N-K}$. Thus,

$$P_F[F(X_{N-K-j} + b(X_{N-K} - X_{N-K-j}) \geqq H^{-1}(\rho)] \geqq P_F[bHF(X_{N-K}) - (b - 1)HF(X_{N-K-j}) \geqq \rho].$$

But, since F is continuous, the joint distribution of the random variables $F(X_1)$, $\cdots$, $F(X_N)$ is the same as that of $U_1, \cdots, U_N$, and (1) is proved.

Corollary 1. *Let* $0 < P < 1$ *and*

$$\Pi(b) = \Pi(b: N, K, j; P) = P[U_{K+1} \leqq P^{1/b} U_{K+j+1}^{(b-1)/b}].$$

Then for all $F \varepsilon \mathfrak{F}_U$, $b \geqq 1$ *and* $0 \leqq K < K + j \leqq N - 1$,

$$P_F[1 - F(X_{N-K-j} + b(X_{N-K} - X_{N-K-j})) \leqq P] \geqq \Pi(b).$$

Proof. Set $H(x) = -\log(1 - x)$ in the theorem. Then, with $\rho = -\log P$, it follows from (1) that

$$\begin{aligned} &P_F[1 - F(X_{N-K-j} + b(X_{N-K} - X_{N-K-j})) \leqq P] \\ (3) \qquad &\geqq P[-b \log(1 - U_{N-K}) + (b - 1)\log(1 - U_{N-K-j}) \geqq \rho] \\ &= P[1 - U_{N-K} \leqq P^{1/b}(1 - U_{N-K-j})^{(b-1)/b}]. \end{aligned}$$

But the joint distribution of $U_l' = 1 - U_{N-l+1}$, $l = 1, 2, \cdots, N$, is the same as the of $U_1, \cdots, U_N$. Thus, the last probability in (3) is $\Pi(b)$ and the proof of the corollary is completed with the observation that $\mathfrak{C}_H = \mathfrak{F}_U$ for this assignment of H.

In the same way, with $H(x) = -\log x$, we obtain by means of (2) the following result for distributions in the class $\mathfrak{F}_L$:

Corollary 2. *Let* $0 < P < 1$. *Then for all* $F \varepsilon \mathfrak{F}_L$, $b \geqq 1$ *and* $0 \leqq K < K + j \leqq N - 1$,

$$(4) \qquad P_F[F(X_{K+j+1} - b(X_{K+j+1} - X_{K+1})) \leqq P] \geqq \Pi(b).$$

Now, for fixed N, K and j, if $b = b(N, K, j, P, \gamma)$ can be chosen in the following way;

(a) when $\Pi(1) < \gamma$ select b to satisfy $\Pi(b) = \gamma$,

(b) when $\Pi(1) \geqq \gamma$ set $b = 1$,

then it would follow from (3) and (4) that $X_{N-K-j} + b(X_{N-K} - X_{N-K-j})$ and $X_{K+j+1} - b(X_{K+j+1} - X_{K+1})$ are the desired (P, γ) upper and lower tolerance limits over $\mathfrak{F}_U$ and $\mathfrak{F}_L$ respectively. When $\Pi(1) = P[U_{k+1} \leqq P] = P[U_{N-K} \geqq P] \geqq \gamma$, the selection $b = 1$ reduces the L.C. tolerance limits to the standard non-parametric tolerance limits which are distribution free over the class of all continuous distributions. Thus, to validate the construction of the L.C. tolerance limits it only remains to be shown that when $\Pi(1) < \gamma$ it is possible to find a number b such that $\Pi(b) = \gamma$. This will now be done in the process of obtaining an explicit expression for $\Pi(b)$.

From [9], the joint density of U_{K+1} and U_{K+j+1} is given as

$$f_{U_{K+1},U_{K+j+1}}(w,v) = \frac{N!}{(N-K-j-1)!(j-1)!K!} w^K(v-w)^{j-1}(1-v)^{N-K-j-1} \quad \text{for } 0 \leqq w \leqq v \leqq 1,$$
$$= 0, \qquad \text{otherwise.}$$

The region of integration determined by the inequality

$$U_{K+1} \leqq P^{1/b} U_{K+j+1}^{(b-1)/b}$$

is bounded by the lines $w = 0$ for $0 \leqq v \leqq 1$, $v = 1$ for $0 \leqq w \leqq 1$, the line $w = v$ for $0 \leqq v \leqq P$ and the curve

$$w = P^{(1/b)} v^{(b-1)/b}$$

for $P \leqq v \leqq 1$. Thus,

$$\Pi(b) = \frac{N!}{(N-K-j-1)!(j-1)!K!} \left\{ \int_0^P \int_0^v + \int_P^1 \int_0^{P^{(1/b)}v^{(b-1)/b}} \right\} \cdot w^K(v-w)^{j-1}(1-v)^{N-K-j-1} dw\, dv. \tag{5}$$

Now, for $v > P$, $P^{1/b}v^{(b-1)/b}$ is a continuous and strictly increasing function of b which tends to v as $b \to \infty$. Since the integrand of (5) is bounded by 1 for $0 \leqq K < K + j \leqq N - 1$ it follows that $\Pi(b)$ is continuous and strictly increasing with $\lim_{b\to\infty} \Pi(b) = 1$. Thus, if $\Pi(1) < \gamma < 1$, there always exists $b > 1$ for which $\Pi(b) = \gamma$.

The integrals in (5) can be reduced to one dimensional integrals in certain special cases the most important of which occurs for $j = 1$. This corresponds to the use of consecutive order statistics in the L.C. tolerance limits. Ie this case

$$\Pi(b) = \frac{N!}{(N-K-2)!(K+1)!} \left\{ \int_0^P v^{K+1}(1-v)^{N-K-2}\, dv + P^{(K+1)/b} \int_P^1 v^{(K+1)(b-1)/b}(1-v)^{N-K-2}\, dv \right\}$$

$$= I_P(K+2, N-K-1) + P^{(K+1)/b} \frac{\Gamma(N+1)\Gamma\left(\frac{(K+2)b-(K+1)}{b}\right)}{\Gamma(K+2)\Gamma\left(N-K+\frac{(K+1)(b-1)}{b}\right)} \cdot \left[1 - I_P\left(\frac{(K+2)b-(K+1)}{b}, N-K-1\right)\right]. \tag{6}$$

The fact that $\Pi(b)$ can be written in terms of the gamma function and incomplete beta distribution undoubtedly makes it possible to obtain simple and accurate approximations for this function. In fact, plots of available values of $b(N, K, 1, P, \gamma)$ versus N on semi log paper indicate that for even moderate values of N, $b \cong a \exp(-cN)$, where a and c are positive numbers depending on K, P and γ. This would indicate a relatively simple asymptotic expression for $\Pi(b)$ which would be most useful in obtaining approximate b values for untabulated parameter combinations. We have not succeeded in evaluating this approximation analytically. However graphical interpolation on semi log paper based on two or more scattered points computed from standard tables by means of the second expression in (6) will provide untabulated b values of sufficient accuracy for practical purposes.

The computations used to construct the tables in the next section were based on the integral expression in (6) and used quadrature formulae for the incomplete beta distribution due to D. E. Amos, Division 5421, Sandia Corporation, in a machine program for Sandia's CDC-1604. The computations were accurate to at least six significant figures over the given range of the arguments. However, for brevity, the entries have been rounded to four figures in Table 1.

4. Tables of b values for the case $j = 1$. For the specified values of P and γ, Table 1 provides the constants, b, such that

$$X_{N-K-1} + b(X_{N-K} - X_{N-K-1}) \tag{7}$$

is a (P, γ) upper tolerance limit and

$$X_{K+2} - b(X_{K+2} - X_{K+1}) \tag{8}$$

is a (P, γ) lower tolerance limit for $K = 0$ and, in certain cases, $K = 1$ and 2. A single asterisk following an entry indicates that the b value is to be used in (7) and (8) with $K = 1$ and two asterisks indicate that $K = 2$. All other entries are to be used with $K = 0$.

5. Comparisons with the parametric theory and extensions. A crude comparison of the L.C. upper tolerance limit $X_{N-1} + b(X_N - X_{N-1})$ with the usual parametric tolerance limits for the normal and exponential distributions was carried out as follows: 100 random samples of size $N = 10$ were selected from the normal distribution with zero mean and unit variance. The L.C. ($P = .10$, $\gamma = .90$) upper tolerance limit was calculated for each sample as was the usual normal tolerance limit $\bar{x} + ks$ (see [5]). Then, the sample means and standard deviations of the 100 L.C. and normal tolerance limits were computed.

The same procedure was followed for the exponential distribution with density $f(x) = \exp(-x)$, $x \geqq 0$. In this case, we used the upper tolerance limit $A\bar{X} + (1 - A)X_1$ which is based on the maximum likelihood estimates of μ and σ in the scale and translation parameter family of exponential distributions with densities

$$f_{\mu,\sigma}^{(x)} = (1/\sigma)f[(x - \mu)/\sigma].$$

As before, $X_1 = \min X_i$. The appropriate value of $A (A = 1.685)$ was supplied by D. B. Owen [6]. Upper tolerance limits of the same form but using a different coverage criterion were obtained by Guttman [4].

The results of these experiments are given in Table 2.

The influence of the rate of decrease of the upper tails of the distributions on the L.C. tolerance limits can be seen in part, from Table 2. The exponential distributions are the least favorable distributions in $\mathfrak{F}_U$ from the standpoint of "tail length" and correspondingly, produce the poorest L.C. to parameter tolerance limit comparison. The L.C. tolerance limit is, on the average, almost four times as large as the exponential tolerance limit. Apparently, this effect cannot be corrected by using order statistics with smaller indices. A sampling procedure identical to the one above but for $K = 1$ yielded a mean of 8.660 and a standard deviation of 7.542, and when $K = 2$ the mean increased to 12.067 and the standard deviation to 11.043.

In the normal case, the ratio of mean L.C. to mean normal tolerance limits is only 1.57. The variability of the L.C. tolerance limits, however, remains high relative to that of the normal limits.

The general nature of Theorem 2 suggests the following extensions of the theory. Since the function H in that theorem is monotone, it should be a straightforward matter for a given assignment of H to obtain the joint distribution of $H(U_1), \cdots, H(U_N)$ from that of $U_1, \cdots, U_N$ and to compute probabilities such as those on the right-hand side of inequalities (1) and (2). This will make it possible to obtain (P, γ) upper and lower tolerance limits via (1) and (2) relative to the class $\mathcal{C}_H$. A wide variety of subsets of the class of continuous distributions can be generated by varying one's choice of H.

The above mentioned difficulty with "long tailed" distributions motivates the following choice of functions, H_α, $0 < \alpha \leqq 1$, which progressively restrict $\mathfrak{F}_U$ to distributions with more rapidly decreasing tails:

$$H_\alpha(x) = [-\log (1 - x)]^\alpha.$$

The classes $\mathfrak{F}_U(\alpha)$ for which $H_\alpha F$ are convex exclude the exponential distributions for $\alpha < 1$ and, as $\alpha \to 0$, exclude progressively "shorter tailed" distributions. When it is reasonable to restrict the class of possible underlying distributions to some $\mathfrak{F}_U(\alpha)$ for $\alpha < 1$ it will be possible to obtain less conservative tolerance limits from (1) and (2) than were obtained here.

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TABLE 1

b Values for Log-Convex Tolerance Limits for Selected Values of P, γ, N and K
(For each entry, K is equal to the number of asterisks following the entry.)

N	γ = .90	γ = .95	γ = .99
	P = .50		
2	2.920	6.105	31.56
3	1.305	2.824	14.94
4	1.000	1.295	7.167
5	2.224*	1.000	3.441
6	1.126*	2.580*	1.636
7	1.000*	1.322*	1.000
	P = .25		
2	8.618	17.80	91.21
3	5.898	12.27	63.17
4	4.116	8.638	44.78
5	2.898	6.154	32.17
6	2.044	4.411	23.31
7	1.437	3.169	16.98
8	1.001	2.275	12.42
9	1.000	1.627	9.100
10	3.286*	1.156	6.677
11	2.496*	1.000	4.900
12	1.877*	4.227*	3.593
13	1.392*	3.232*	2.629
14	1.014*	2.454*	1.918
15	1.000*	1.846*	1.392
16	2.642**	1.372*	1.003
17	2.052**	1.003*	1.000
	P = .10		
2	17.09	35.18	179.8
3	13.98	28.82	147.5
4	11.70	24.17	123.9
5	9.931	20.57	105.6
6	8.512	17.67	90.90
7	7.344	15.29	78.80
8	6.368	13.30	68.68
9	5.541	11.61	60.10
10	4.835	10.17	52.77
11	4.227	8.924	46.46
12	3.701	7.849	40.99
13	3.244	6.914	36.23
14	2.845	6.097	32.07
15	2.496	5.382	28.43
16	2.190	4.754	25.23
17	1.920	4.202	22.41
18	1.683	3.714	19.92
19	1.474	3.284	17.72
20	1.289	2.903	15.77
21	1.125	2.566	14.05
22	1.000	2.267	12.51
23	4.766*	2.002	11.15
24	4.317*	1.767	9.943
25	3.907*	1.558	8.866
26	3.533*	1.372	7.908
27	3.192*	1.206	7.053
28	2.880*	1.059	6.291
29	2.595*	1.000	5.612
30	2.336*	5.230*	5.005
31	2.099*	4.743*	4.464
32	1.882*	4.298*	3.981
33	1.685*	3.891*	3.549
34	1.505*	3.520*	3.163
35	1.341*	3.182*	2.818
36	1.191*	2.872*	2.509
37	1.054*	2.590*	2.233
38	1.000*	2.333*	1.986
39	3.528**	2.098*	1.766
40	3.227**	1.884*	1.568
41	2.947**	1.689*	1.391
42	2.688**	1.511*	1.233
43	2.448**	1.349*	1.091
44	2.225**	1.201*	1.000
	P = .05		
2	23.65	48.63	248.4
3	20.48	42.15	215.4
4	18.12	37.32	190.9
5	16.24	33.49	171.4

TABLE 1—*Continued*

N	γ .90	.95	.99
	P = .05		
6	14.70	30.33	155.4
7	13.39	27.66	141.8
8	12.26	25.35	130.0
9	11.27	23.33	119.8
10	10.39	21.54	110.7
11	9.607	19.94	102.5
12	8.903	18.50	95.21
13	8.265	17.19	88.58
14	7.684	16.01	82.56
15	7.154	14.93	77.05
16	6.668	13.93	72.01
17	6.222	13.02	67.37
18	5.810	12.18	63.09
19	5.429	11.40	59.13
20	5.077	10.68	55.47
21	4.750	10.01	52.07
22	4.447	9.391	48.91
23	4.164	8.814	45.97
24	3.901	8.275	43.23
25	3.655	7.773	40.67
26	3.426	7.303	38.28
27	3.212	6.865	36.04
28	3.012	6.454	33.95
29	2.824	6.070	31.99
30	2.648	5.710	30.16
31	2.484	5.372	28.44
32	2.329	5.055	26.82
33	2.184	4.758	25.30
34	2.048	4.478	23.87
35	1.920	4.216	22.53
36	1.800	3.969	21.27
37	1.687	3.736	20.08
38	1.581	3.518	18.96
39	1.481	3.312	17.91
40	1.387	3.118	16.92
41	1.298	2.936	15.99
42	1.215	2.764	15.10
43	1.136	2.602	14.27
44	1.062	2.449	13.49

N	γ .90	.95	.99
	P = .05		
45	1.000	2.305	12.75
46	4.995*	2.169	12.05
47	4.760*	2.041	11.39
48	4.534*	1.920	10.77
49	4.319*	1.806	10.18
50	4.113*	1.698	9.629
51	3.916*	1.597	9.105
52	3.728*	1.500	8.610
53	3.548*	1.410	8.142
54	3.376*	1.324	7.700
55	3.211*	1.243	7.282
56	3.054*	1.167	6.887
57	2.903*	1.095	6.513
58	2.759*	1.026	6.160
59	2.622*	1.000	5.826
60	2.490*	5.567*	5.509
61	2.364*	5.308*	5.210
62	2.244*	5.060*	4.927
63	2.129*	4.823*	4.660
64	2.019*	4.597*	4.406
65	1.913*	4.380*	4.167
66	1.813*	4.173*	3.940
67	1.717*	3.974*	3.725
68	1.625*	3.785*	3.522
69	1.537*	3.604*	3.329
70	1.453*	3.430*	3.147
71	1.372*	3.264*	2.975
72	1.296*	3.106*	2.811
73	1.222*	2.954*	2.657
74	1.152*	2.809*	2.510
75	1.085*	2.670*	2.372
76	1.021*	2.538*	2.240
77	1.000*	2.411*	2.116
78	3.731**	2.290*	1.998
79	3.573**	2.174*	1.887
80	3.421**	2.063*	1.781
81	3.274**	1.957*	1.681
82	3.132**	1.856*	1.586
83	2.996**	1.759*	1.496

TABLE 1—*Continued*

N	γ .90	.95	.99	N	γ .90	.95	.99
	$P = .05$				$P = .05$		
84	2.864**	1.667*	1.411	87	2.498**	1.414*	1.182
85	2.738**	1.579*	1.330	88	2.384**	1.336*	1.113
86	2.615**	1.494*	1.254	89	2.275**	1.263*	1.048
				90	2.170**	1.192*	1.000

TABLE 2

Means and Standard Deviations of Parametric and Log-Convex (.10, .90) Upper Tolerance Limits for 100 Random Samples of Size $N = 10$

Tolerance Limit	Normal Mean	Normal St. Dev.	Exponential Mean	Exponential St. Dev.
Parametric	2.049	0.540	1.618	0.486
Log-Convex	3.209	1.656	5.876	3.512

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