ORTHANT PROBABILITIES FOR THE QUADRIVARIATE NORMAL DISTRIBUTION

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1. Summary. Let x_1, x_2, x_3, x_4 be jointly distributed with a quadrivariate normal distribution with mean 0, and correlation matrix $\{\rho_{ij}\}$. The orthant probability, i.e. the probability that all the x_i's will be simultaneously positive, is not, in general, given by a closed expression; but is easily computed in a special class of cases, here called orthoscheme probabilities. It is explicitly shown how the general orthant probability can be expressed as a linear combination of six orthoscheme probabilities. Orthoscheme probabilities have been tabulated by the author and instructions for the use of this table [1] are given. In addition, an abridged table is appended.

2. Introduction. Suppose $x_1, x_2, \cdots, x_n$ are jointly normally distributed with mean 0 and correlation matrix $\{\rho_{ij}\}$, and let P_n be the probability that the x_i's are simultaneously positive. For $n \geqq 4$, there is no general closed expression for P_n; but it is of considerable interest in several fields to know P_n or, at least, have a reliable approximation to it. Approximations to P_n, and even to more general probabilities, exist, but are ordinarily tedious to compute.

For example, some slowly convergent expansions are available [8], [11], [12]. David [5] has produced a reduction formula for odd n, which requires a knowledge of P_k for all $k < n$, and several more general (but less simple) reduction formulae are available (e.g. [3], [4], [7], [9], [14], [16]). Some special cases which prove more tractable (such as the equicorrelated case) have been investigated [2], [10], [13], [22] and Ruben [18] has investigated the probability content of various types of regions in n-space, under a normal distribution. In [19], I. R. Savage discusses sequences of bounds to normal orthant probabilities which are not centered at the origin.

To each matrix $\{\rho_{ij}\}$ corresponds an n-dimensional polyhedral half-cone, C_n, whose vertex is at the origin and whose n sides are $(n - 1)$-dimensional hyperplanes with the property that the cosine of the angle between the normals to the ith and jth hyperplanes is ρ_{ij}. Let S_n be the (hollow) unit n-sphere, centered at the origin, and let $T_n = S_n \cap C_n$. Then if t_n and s_n are the $(n - 1)$-dimensional contents of the regions T_n and S_n respectively, we have $P_n = t_n/s_n$ [18], [21]. Schläfli has shown that C_n can be dissected into $n!$ n-sided polyhedral half-cones for each of which the matrix of cosines between the normals to the bounding

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hyperplanes is of the form:

$$(2.1)\qquad \begin{bmatrix} 1 & a & 0 & 0 & \cdots & \cdots & 0 \\ a & 1 & b & 0 & \cdots & \cdots & 0 \\ 0 & b & 1 & c & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 1 & k \\ 0 & 0 & \cdots & \cdots & 0 & k & 1 \end{bmatrix}$$

(i.e. the only non-zero off diagonal terms are in the sub- and super-diagonals). Such a division of C_n implies a division of the spherical simplex T_n into $n!$ regions (called orthoschemes by Schläfli) each of which is a spherical simplex with $\frac{1}{2}(n-1)(n-2)$ right angles.

Let the content of an orthoscheme with cosine matrix given by (2.1) be denoted by $P_n(a, b, \cdots, k)$. $P_n(a, b, \cdots, k)$ is, of course, the probability that n jointly normal variables, with mean zero and correlation matrix given by (2.1), will be simultaneously positive. Rogers [15] has derived an expansion for $P_n(a, b, \cdots, k)$. $P_n(a, b, \cdots, k)$ has also been studied by van der Vaart who has derived a general formula for $n \geqq 4$ [23], [24].

For $n = 2, 3$ we have

$$P_2 = \tfrac{1}{4} + (1/2\pi) \arcsin \rho_{12}$$

$$P_3 = \tfrac{1}{8} + (1/4\pi)\{\arcsin \rho_{12} + \arcsin \rho_{13} + \arcsin \rho_{23}\}$$

For $n = 4$, the content of the orthoscheme (with cosines (a, b, c)) is given by van der Vaart as

$$(2.2)\qquad P_4(a, b, c) = \frac{1}{16}\Big\{1 + \frac{2}{\pi}(\arcsin a + \arcsin b + \arcsin c) + \left(\frac{2}{\pi}\right)^2 \int_0^c \int_0^a [(1-\gamma^2)(1-\alpha^2) - b^2]^{-\frac{1}{2}}\, d\alpha\, d\gamma\Big\}.$$

The integral involved in (2.2) is easily evaluated numerically, and by a suitable choice of a starting point, T_4 can be divided into $(n-1)! = 6$ (instead of $n! = 24$) orthoschemes. It is the purpose of this paper to show how P_4 can be expressed as a sum of the form

$$\sum_{i=1}^{6} \pm P_4(a_i, b_i, c_i)$$

by exhibiting the simplicial division explicitly and to demonstrate the use of the table of "orthoscheme probabilities" [1], to compute P_4 by this means.

In Section 3 the geometrical ideas behind the simplicial division of T_4 into six orthoschemes are examined and the division is explicitly carried out. In Sections 4 and 5 the use of the table is discussed and the reader with no theoretical interest may proceed directly to Section 4 without any loss.

3. Discussion and derivation of the simplicial division. Let $\mathbf{x}$ be a random vector in R^4. It is sufficient for our discussion here to assume that $\mathbf{x}$ is the vector sum of the random variables x_1, x_2, x_3, x_4 in the direction of the vectors $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4$ respectively, where $\boldsymbol{\varepsilon}_i$ is the ith unit vector (1 in the ith place and zeros elsewhere), so that the x_i's act orthogonally (in space) to one another. The concept of orthogonality is the one conferred by the use of the usual inner (or dot or scalar) product of two vectors $\mathbf{y}$ and $\mathbf{z}$, denoted by $(\mathbf{y}, \mathbf{z})$, so that

$$(\mathbf{y}, \mathbf{z}) = \sum_{i=1}^{4} y_i z_i \text{ where } \mathbf{y} = \sum_{i=1}^{4} y_i \boldsymbol{\varepsilon}_i \text{ and } \mathbf{z} = \sum_{i=1}^{4} z_i \boldsymbol{\varepsilon}_i .$$

Let $\mathbf{x}$ have mean 0 and nonsingular covariance matrix C. C has the property that for any two vectors $\mathbf{y}$ and $\mathbf{z}$, $\text{Cov}[(\mathbf{y}, \mathbf{x}), (\mathbf{z}, \mathbf{x})] = (\mathbf{y}, C\mathbf{z})$. C can always be written in the form $C = BB'$ where B is nonsingular and the prime denotes transposition. Let $\mathbf{z} = B^{-1}\mathbf{x}$. The covariance matrix of $\mathbf{z}$ is $(B^{-1})C(B^{-1})' = I$, the identity matrix.

Since $x_i = (\mathbf{x}, \boldsymbol{\varepsilon}_i) = (B\mathbf{z}, \boldsymbol{\varepsilon}_i) = (\mathbf{z}, B'\boldsymbol{\varepsilon}_i)$, the probability that the x_i's are simultaneously positive is the same as the probability that the $(\mathbf{z}, B'\boldsymbol{\varepsilon}_i)$'s will be simultaneously positive, i.e., that

$$(\mathbf{z}, B'\boldsymbol{\varepsilon}_i) \geqq 0 \quad \text{for} \quad i = 1, 2, 3, 4.$$

Let

$$\mathbf{a}_i = (B'\boldsymbol{\varepsilon}_i)/\|B'\boldsymbol{\varepsilon}_i\|,$$

$$H_i = \{\mathbf{z}: (\mathbf{z}, \mathbf{a}_i) = 0\}$$

and

$$Q = \{\mathbf{z}: (\mathbf{z}, \mathbf{a}_i) \geqq 0 \quad \text{for all} \quad i\}.$$

H_i is a hyperplane with normal $\mathbf{a}_i$ of unit length, such that $\mathbf{a}_i$ is directed into the half space $\{\mathbf{z}: (\mathbf{z}, \mathbf{a}_i) \geqq 0\}$ which we will call the positive side of H_i. Obviously $Q = \{\mathbf{z}: (\mathbf{z}, B'\boldsymbol{\varepsilon}_i) \geqq 0 \text{ for all } i\}$ so that the probability that the x_i's are simultaneously positive is the probability of lying in the region Q, which is a polyhedral half-cone, with vertex at the origin, bounded by the planes H_i and positive with respect to each. We notice that the cosine of the angle between the normals to H_i and H_j is

$$(\mathbf{a}_i, \mathbf{a}_j) = \frac{(\boldsymbol{\varepsilon}_i, C\boldsymbol{\varepsilon}_j)}{[(\boldsymbol{\varepsilon}_i, C\boldsymbol{\varepsilon}_i)(\boldsymbol{\varepsilon}_j, C\boldsymbol{\varepsilon}_j)]^{\frac{1}{2}}} = \frac{\text{Cov}(x_i, x_j)}{[\text{Var}(x_i), \text{Var}(x_j)]^{\frac{1}{2}}} = \rho_{ij}.$$

If the distribution of $\mathbf{x}$ is normal, the distribution of $\mathbf{z}$ will be spherical and hence $P_4 = P\{\mathbf{z}: \mathbf{z} \varepsilon Q\} = t_4/s_4$ where t_4 is the three-dimensional content of $R = T_4 = S_4 \cap Q$ and s_4 is the three-dimensional content of S_4.

The region R very closely resembles a tetrahedron whose bounding faces are the surfaces of spheres with unit radius, and for the purposes of illustration we shall consider it as a tetrahedron, bounded by planes, to which it is topologically equivalent, bearing in mind that certain restrictions on the angles of the Euclidean simplex of the illustrations do not actually apply.

Our task then reduces to evaluating the volume (t_4) of a "hyperspherical tetrahedron" (R) lying on the unit 4-sphere ,bounded by and on the positive side of each of the hyperplanes H_1, H_2, H_3 and H_4, where the angles between the normals to the hyperplanes are known.

R has four vertices, and we denote by V_i that vertex opposite H_i. Let $\mathbf{v}_i$ be the position vector of V_i. Since V_i is on the unit hypersphere, $\mathbf{v}_i$ is of unit length.

Van der Vaart's formula allows us to evaluate t_4 provided R is an orthoscheme; and Schläfli shows that, in general, R can be split into 24 orthoschemes by a four dimensional analogue of the following method for spherical triangles. Suppose the vertices of the spherical triangle T_3 are V_1, V_2, V_3. Then we can choose any point P on the sphere and by constructing great circles through P to V_1, V_2, V_3 and perpendicular to the planes bounding T_3, we can split T_3, in general, into six spherical right-angled triangles (see Fig. 1).

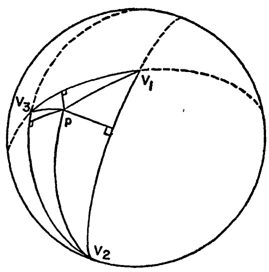

FIG. 1. Simplicial division of a spherical triangle $V_1V_2V_3$ into six orthoschemes.

However, if we choose one of the V_i as P, then T_3 will be split into two spherical right-angled triangles. (This will also happen if we choose P to be the antipodal point to one of the vertices.)

Our analogous procedure in four dimensions, therefore, is to "drop" a "perpendicular" from V_4 onto $H_4 \cap S_4$; i.e., we construct a great circle on S_4 which is orthogonal to $H_4 \cap S_4$ and passes through V_4. Because of the homogeneity present in the situation, we can do this by locating the projection of $\mathbf{v}_4$ on H_4, and multiplying by a scale factor to find the appropriate point, V_p, on S_4. But although the projection is unique, the intersection of $H_4 \cap S_4$ and the great circle perpendicular to $H_4 \cap S_4$ is not, since we have the choice of a positive or negative scale factor. If we settle on a V_p (with position vector $\mathbf{v}_p$) we find ourselves confronted with two possibilities:

(i) $(\mathbf{a}, \mathbf{v}_p) \geqq 0$ for at least one $i = 1, 2, 3$, i.e., the foot of the perpendicular falls on the positive side of or on at least one of the planes H_1, H_2, H_3.

(ii) $(\mathbf{a}_i, \mathbf{v}_p) < 0$ for all i, i.e., the foot of the perpendicular falls on the negative side of all the planes.

The reason for this becomes clear if we consider the three-dimensional case, where the corresponding simplicial division would be into two right-angled spherical triangles. In the first case, P_3 is a simple sum or difference of surface areas. In the second, multiplication by a positive constant results in the evaluation of $\frac{1}{2} - P_3$. Clearly it is most desirable to be in situation (i), and we can, as we shall show, always ensure that we are (see Fig. 2).

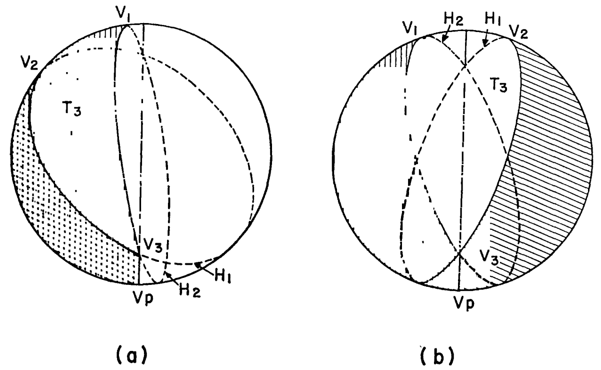

FIG. 2. Three dimensional case. H_3 (corresponding to H_4 in the four dimensional case) is in the plane of the paper. In each case, V_p is the point which results when the projection of V_3 onto H_3 is extended to the surface of the sphere in the "natural" way: i.e., by multiplication of its position vector by a positive constant. Evaluation of the area T_3 by a simple sum or difference of the areas of the orthoschemes $V_1V_3V_p$ (vertically hatched) and $V_2V_3V_p$ (diagonally hatched) is possible in (a) since V_p lies on the positive side of H_2; but is not possible in (b) since V_p lies on the negative sides of H_1 and H_2. Notice that the construction fails when V_3 is above the centre of the circle representing $H_3 \cap S_3$.

Having located V_p so that (i) holds, we can now construct great spheres through V_p and V_4 of two kinds: those which pass through some other V_i and those which are perpendicular to $H_i \cap H_4 \cap S_4$ for some i. In the first case this is done by finding the hyperplane K_i which passes through 0, V_p, V_4 and V_i, and in the second case by finding hyperplane M_i which passes through 0, V_p, V_4 and has a normal $(\mathbf{m}_i)$ orthogonal to $\mathbf{a}_i$. This accomplishes the simplicial division of R into six orthoschemes (see Fig. 3).

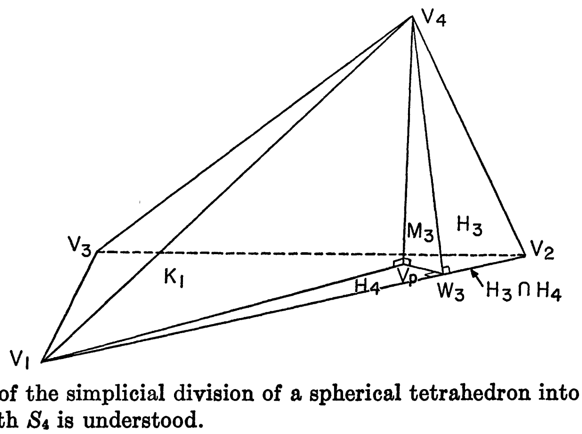

FIG. 3. Part of the simplicial division of a spherical tetrahedron into six orthoschemes. Intersection with S_4 is understood.

We now establish the method of finding the matrices of cosines of the angles between the normals to the planes bounding the orthoschemes, for each orthoscheme. The difficulty lies mainly in keeping track of the direction of the normal vectors.

Let h_i be the distance of V_i from H_i, so that h_i is given by

$$(\mathbf{v}_i, \mathbf{a}_j) = \delta_{ij} h_i \qquad i = 1, \cdots, 4.$$

Notice that this necessarily implies that $h_i > 0$.

Let

$$\begin{aligned} \tau_{ij\cdot k} &= \rho_{ij} - \rho_{ik}\rho_{jk} \\ u_{ij} &= +1 \text{ if } \rho_{ij} \geqq 0 \\ &= -1 \text{ if } \rho_{ij} < 0 \\ u_{ij\cdot k} &= +1 \text{ if } \tau_{ij\cdot k} \geqq 0 \\ &= -1 \text{ if } \tau_{ij\cdot k} < 0. \end{aligned}$$

We now drop a perpendicular from V_4 to H_4, by locating the projection of $\mathbf{v}_4$ on H_4, which is $\mathbf{v}_4 - h_4\mathbf{a}_4$, and extend it to S_4 as follows: let $V_p = \{\mathbf{v}_p\}$, where

$$\mathbf{v}_p = -u_{14}u_{24}u_{34}c(\mathbf{v}_4 - h_4\mathbf{a}_4),$$

where

$$c = (1 - h_4^2)^{-\frac{1}{2}}.$$

Notice that $\|\mathbf{v}_p\| = 1$ and $\mathbf{v}_p$ is uniquely defined provided we do not have $\rho_{14} = \rho_{24} = \rho_{34} = 0$ (for then $\mathbf{v}_4 = \mathbf{a}_4$ and $h_4 = 1$), and furthermore that

$$(3.1) \qquad (\mathbf{v}_p, \mathbf{a}_i) - u_{j4}u_{k4}ch_4|\rho_{i4}| \qquad i \neq j \neq k = 1, 2, 3.$$

Since these three quantities can never be simultaneously strictly negative, V_p cannot lie on the negative side of H_1, H_2 and H_3 simultaneously; hence we avoid situation (ii). V_p is of course, the foot of the perpendicular from V_4 to H_4 in R.

If $\rho_{14} = \rho_{24} = \rho_{34} = 0$, the construction fails, but we can deal with this by appealing to the 3-dimensional case since x_4 will be independent of x_1, x_2 and x_3. We assume this is not the case.

To locate (say) $W_3 = \{\mathbf{w}_3\}$, the foot of the perpendicular from V_4 to $H_4 \cap H_3$ in R, we project $\mathbf{v}_p$ onto $H_3 \cap H_4$ and conclude that $\mathbf{w}_3$ must be proportional to $(1 - \rho_{34}^2)\mathbf{v}_4 - h_4\mathbf{a}_4 - \rho_{34}h_4\mathbf{a}_3$. We do not want it to be possible that W_3 lies on the negative slide of both H_1 and H_2 simultaneously, so we define $\mathbf{w}_3$ uniquely as

$$\mathbf{w}_3 = -\mu_{12}d[(1 - \rho_{34}^2)\mathbf{v}_4 - h_4(\mathbf{a}_4 - \rho_{34}\mathbf{a}_3)]$$

where

$$\begin{aligned} \mu_{12} &= \tfrac{1}{2}[1 + u_{14\cdot 3} + u_{24\cdot 3} - u_{14\cdot 3}u_{24\cdot 3}] \\ &= -1 \text{ if } \tau_{14\cdot 3} \text{ and } \tau_{24\cdot 3} < 0 \\ &= +1 \text{ otherwise} \end{aligned}$$

and $d > 0$ is such that $\|w_3\| = 1$. Hence

$$(\mathbf{w}_3, \mathbf{a}_1) = \mu_{12} d \tau_{14\cdot 3}$$

and

$$(\mathbf{w}_3, \mathbf{a}_2) = \mu_{12} d \tau_{24\cdot 3} \tag{3.2}$$

cannot be simultaneously strictly negative.

Let us now consider the orthoscheme, R_{31}, whose vertices are V_4, V_p, W_3 and V_1. H_4 and H_3 are the bounding planes opposite V_4 and V_p respectively. Denote by K_1 the bounding plane through V_4, V_p, V_1 and opposite W_3, and let M_3 be the bounding plane through V_4, V_p, W_3 and opposite V_1 (see Fig. 3).

Since K_1 contains the origin, we express its equation as

$$(\mathbf{k}_1, \mathbf{z}) = 0 \text{ where } \|\mathbf{k}_1\| = 1,$$

and since K_1 passes through the intersection of H_2 and H_3, we must have for some α, β

$$\mathbf{k}_1 = \alpha \mathbf{a}_2 + \beta \mathbf{a}_3 .$$

But V_4 and V_p both lie in K_1, hence $\mathbf{v}_4$ and $\mathbf{v}_p$ both satisfy $(\mathbf{k}_1, \mathbf{z}) = 0$, hence $(\mathbf{k}_1, \mathbf{a}_4) = 0$ so $\alpha\rho_{24} + \beta\rho_{34} = 0$ and $\mathbf{k}_1$ is proportional to $\rho_{34}\mathbf{a}_2 - \rho_{24}\mathbf{a}_3$, unless $\rho_{34} = \rho_{24} = 0$, in which case $V_p = V_1$ and R_{31} is degenerate.

Similarly, M_3 contains the origin, and we may express its equation as

$$(\mathbf{m}_3, \mathbf{z}) = 0 \text{ where } \|\mathbf{m}_3\| = 1.$$

Now M_3 passes through the intersection of K_1 and K_2, and the normals to M_3 and H_3 are orthogonal, hence $\mathbf{m}_3$ is proportional to

$$\tau_{24\cdot 3}\mathbf{a}_1 - \tau_{14\cdot 3}\mathbf{a}_2 + (\rho_{14}\rho_{23} - \rho_{13}\rho_{24})\mathbf{a}_3 .$$

We now choose unit normals to these four bounding planes which are directed into the orthoscheme.

$H_4 : \mathbf{a}_4$

$H_3 : u_{14}u_{24}\mathbf{a}_3$

$K_1 : \mathbf{k}_1 = \mu_{12}u_{34}u_{24\cdot 3}g_1(\rho_{34}\mathbf{a}_2 - \rho_{24}\mathbf{a}_3)$

$M_3 : \mathbf{m}_3 = u_{24\cdot 3}f_3[\tau_{24\cdot 3}\mathbf{a}_1 - \tau_{14\cdot 3}\mathbf{a}_2 + (\rho_{14}\rho_{23} - \rho_{13}\rho_{24})\mathbf{a}_3]$,

where

$$g_1 = \|\rho_{34}\mathbf{a}_2 - \rho_{24}\mathbf{a}_3\|^{-1}$$

and

$$f_3 = \|\tau_{24\cdot 3}\mathbf{a}_1 - \tau_{14\cdot 3}\mathbf{a}_2 + (\rho_{14}\rho_{23} - \rho_{13}\rho_{24})\mathbf{a}_3\|^{-1} \tag{3.3}$$

provided that these two quantities are both finite, which will be the case when neither $\mathbf{k}_1$ nor $\mathbf{m}_3$ are the zero vector. Assuming that this is not the case, it may be checked that the normals do, in fact, point into R_{31}. This is done by taking the inner product of the normal to each face with the vector which defines the opposite vertex and checking that the result is positive or zero.

If $\mathbf{k}_1 = 0$, then we must have $\rho_{34} = \rho_{24} = 0$, or $\rho_{34} = \rho_{24}$ and $\mathbf{a}_2 = \mathbf{a}_3$. The first possibility implies that $\mathbf{a}_4$ is orthogonal to both $\mathbf{a}_3$ and $\mathbf{a}_2$, in which case it is obvious that V_p coincides with V_1 and R_{31} has no three-dimensional content. In the second case, R is degenerate.

If $\mathbf{m}_3 = 0$, then we must have $\tau_{24\cdot3} = \tau_{14\cdot3} = (\rho_{14}\rho_{23} - \rho_{13}\rho_{24}) = 0$, or that $\mathbf{a}_1$ is a linear combination of $\mathbf{a}_2$ and $\mathbf{a}_3$. The first case implies $(\mathbf{w}_3, \mathbf{a}_2) = (\mathbf{w}_3, \mathbf{a}_1) = 0$ and we certainly have $(\mathbf{w}_3, \mathbf{a}_3) = (\mathbf{w}_3, \mathbf{a}_4) = 0$ since $\mathbf{w}_3 \varepsilon H_3 \cap H_4$; so that $\mathbf{w}_3 = 0$, which is impossible, or, as is implied also by the second case, there is a linear relation between $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$, so that R has no three-dimensional content.

The matrix of cosines of the angles between the normals to the bounding planes of the orthoscheme R_{31}, is given by

$$\begin{array}{c} \\ H_4 \\ H_3 \\ K_1 \\ M_3 \end{array} \begin{array}{c} \begin{array}{cccc} H_4 & H_3 & K_1 & M_3 \end{array} \\ \begin{bmatrix} 1 & \theta_{31} & 0 & 0 \\ \theta_{31} & 1 & \varphi_{31} & 0 \\ 0 & \varphi_{31} & 1 & \psi_{31} \\ 0 & 0 & \psi_{31} & 1 \end{bmatrix} \end{array}$$

where

$$\begin{aligned} \theta_{31} &= (\mathbf{a}_4, u_{14}u_{24}\mathbf{a}_3) = u_{14}u_{24}u_{34}|\rho_{34}| \\ \varphi_{31} &= (u_{14}u_{24}\mathbf{a}_3, \mathbf{k}_1) = -\mu_{12}u_{14}u_{24}u_{34}g_1|\tau_{24\cdot3}| \\ \psi_{31} &= (\mathbf{k}_1, \mathbf{m}_3) = \mu_{12}g_1f_3|\rho_{34}|[\tau_{24\cdot3}\tau_{12\cdot3} - \tau_{14\cdot3}(1 - \rho_{23}^2)]. \end{aligned} \tag{3.4}$$

The volume (V_{31}) of R_{31} can now be found using (2.2) or the table, as discussed in the next section. The volumes of the five remaining orthoschemes can be found by substituting the permutations of (1, 2, 3) in (3.3) and (3.4).

Now the orthoschemes R_{31} and R_{32} have M_3 as a common face, and a common edge connecting V_1 and V_2. From (3.2) we see that W_3 lies in R, and hence between V_1 and V_2, if and only if $\tau_{14\cdot3}$ and $\tau_{24\cdot3}$ have the same sign. If W_3 does lie in R, R_{31} and R_{32} will be adjacent; but if W_3 does not lie in R, either $R_{31} \subset R_{32}$ or $R_{31} \supset R_{32}$ according as $\tau_{14\cdot3} < 0$ or $\tau_{24\cdot3} < 0$ respectively. Hence we may define X_3 as the volume of the region R_3 which has vertices V_1, V_2, V_p, V_4, and we see that

$$X_i = |u_{j4\cdot i}V_{ij} + u_{k4\cdot i}V_{ik}| \qquad i \neq j \neq k = 1, 2, 3.$$

Since we have defined V_p so that it cannot fall on the negative sides of all the hyperplanes H_1, H_2 and H_3 simultaneously, it follows that the contribution of X_i to the total volume is positive if $(\mathbf{a}_i, \mathbf{v}_p)$ is positive, and negative otherwise. From (3.1) we see that the condition that $(\mathbf{a}_i, \mathbf{v}_p)$ be positive is that u_{j4} and u_{k4} $(i \neq j \neq k)$ have the same sign. Hence the total volume, X, of the region

R is given by

$$X = |u_{24}u_{34}X_1 + u_{14}u_{34}X_2 + u_{14}u_{24}X_3|.$$

4. How to use the tables. In this and the following section we drop the subscript from $P_4(a, b, c)$.

Using van der Vaart's formula (2.2), $P(a, b, c)$ has been tabulated in [1] for all non-negative values of a and b (at intervals of .05) and c (at intervals of .01) which are compatible with the correlation matrix being positive definite, i.e. with

$$\begin{aligned} &a^2 + b^2 + c^2 - a^2c^2 < 1 \\ &a, b, c < 1. \end{aligned} \tag{4.1}$$

In addition, a shortened table is appended to this paper where the increments on a, b, c are .1, .1, .05 respectively.

For negative a, b or c, we notice, from (2.2), the relations, setting $Q(t) = (1/4\pi) \arcsin t$,

$$\begin{aligned} P(-a, b, c) &= \tfrac{1}{8} + Q(b) + Q(c) - P(a, b, c) \\ P(a, -b, c) &= P(a, b, c) - Q(b) \\ P(-a, -b, c) &= \tfrac{1}{8} + Q(c) - P(a, b, c) \\ P(-a, -b, -c) &= P(a, b, c) - \{Q(a) + Q(b) + Q(c)\} \\ P(-a, b, -c) &= P(a, b, c,) - \{Q(a) + Q(c)\} \\ P(a, b, c) &= P(c, b, a) \end{aligned}$$

The last identity has further uses in interpolation, since the increments on the third parameter are finer.

An explicit description of the use of the tables now follows. In practice, obvious symmetries may much reduce the labor, which, in the general case, is unfortunately tedious.

(1) Find the correlation matrix $\{\rho_{ij}\}$ of the four variates under examination. Check that it is nonsingular. If $\rho_{14} = \rho_{24} = \rho_{34} = 0$, the fourth variate is independent of the rest and

$$P_4 = \tfrac{1}{2}\{\tfrac{1}{8} + (1/4\pi)(\arcsin \rho_{12} + \arcsin \rho_{23} + \arcsin \rho_{13})\};$$

but if this is not the case, proceed as follows.

(2) For each permutation (i, j, k) of three of $(1, 2, 3, 4)$, omitting the cases where $k = 4$ and noting the symmetry in i and j, calculate

$$r_{ij\cdot k} = \rho_{ij} - \rho_{ik}\rho_{jk} .$$

(3) For each permutation (i, j, k) or $(1, 2, 3)$ calculate

(a)
$$\begin{aligned} u_{i4} &= +1 \quad \text{if} \quad \rho_{i4} \geqq 0 \\ &= -1 \quad \text{if} \quad \rho_{i4} < 0. \end{aligned}$$

(b)
$$u_{i4\cdot j} = +1 \quad \text{if} \quad \tau_{i4\cdot j} \geqq 0$$
$$= -1 \quad \text{if} \quad \tau_{i4\cdot j} < 0.$$

(c)
$$\mu_{ij} = \tfrac{1}{2}\{1 + u_{i4\cdot k} + u_{j4\cdot k} - u_{i4\cdot k}u_{j4\cdot k}\}.$$
(Note $\mu_{ij} = \mu_{ji}$.)

(d)
$$f_k = [\tau_{j4\cdot k}^2(1 - \rho_{ik}^2) + \tau_{i4\cdot k}^2(1 - \rho_{jk}^2) - 2\tau_{ij\cdot k}\tau_{i4\cdot k}\tau_{j4\cdot k}]^{-\frac{1}{2}}$$
(Note the symmetry in i and j.)

(e) If ρ_{j4} and ρ_{k4} are not both zero, find
$$g_i = [\rho_{j4}\tau_{j4\cdot k} + \rho_{k4}\tau_{k4\cdot j}]^{-\frac{1}{2}}$$
(Note the symmetry in j and k.)

If $\rho_{j4} = \rho_{k4} = 0$, but $\rho_{i4} \neq 0$, then $X_j = X_k = 0$ in (5).
(f) If $\rho_{k4} \neq 0$, find
$$\theta_{ki} = u_{i4}u_{j4}u_{k4}|\rho_{k4}|$$
$$\varphi_{ki} = -\mu_{ij}(u_{14}u_{24}u_{34})g_i|\tau_{j4\cdot k}|$$
$$\psi_{ki} = \mu_{ij}g_if_k|\rho_{k4}|[\tau_{j4\cdot k}\tau_{ij\cdot k} - \tau_{i4\cdot k}(1 - \rho_{jk}^2)].$$
If $\rho_{k4} = 0$, $X_k = 0$ in (5).
(4) For each $k = 1, 2, 3$ for which $\rho_{k4} \neq 0$, find X_k by the following method: For each permutation (i, j, k) of $(1, 2, 3)$, put
$$a = \theta_{ki}, \qquad b = \varphi_{ki}, \qquad c = \psi_{ki}.$$
Then find $V_{ki} = P(a, b, c)$, using the table. It will almost certainly be the case that (a, b, c) will not correspond to an entry in the table and some method of interpolation will be required. Some comments are made about this in the next section.

Now compute
$$X_k = |u_{i4\cdot k}V_{ki} + u_{j4\cdot k}V_{kj}|, \qquad i \neq j \neq k.$$
(5) $P_4 = X = |u_{24}u_{34}X_1 + u_{14}u_{34}X_2 + u_{14}u_{24}X_3|$.

5. Some comments about the table and examples of its use. The practical problem of tabulation lies mainly in the integration of the double integral of (2.2), which reduces to a single integral, for

$$\int_0^c \int_0^a \{(1 - \gamma^2)(1 - \alpha^2) - b^2\}^{-\frac{1}{2}}\, d\alpha\, d\gamma = \int_0^c (1 - \gamma^2)^{-\frac{1}{2}} \arcsin\left[a\left\{\frac{1 - \gamma^2}{1 - \gamma^2 - b^2}\right\}^{\frac{1}{2}}\right] d\gamma. \tag{5.1}$$

For the purposes of computation, the arcsin was converted to an arctan because the computer used could then deal with it automatically.

The comments which follow apply equally to the detailed table of [1] and the shortened version. Both tables give $P(a, b, c)$ for all permissible values of b and c for fixed a. Since the increments on c are finer than on a and b, it may be desirable to exploit the symmetry in a and c when interpolation is necessary.

No startling claims of accuracy are made. On the basis of a small number of checks, it seems reasonable to suppose that accuracy is to five decimal places except possibly for large values of the parameters and possibly to six for small values of the parameters. Obviously errors accumulate with interpolation and the addition of the six component probabilities.

It is obvious that if one of a, b, or c is zero, $P(a, b, c)$ will reduce to the form $\frac{1}{2}P_3(a_0, b_0)$ which can be evaluated exactly by the formula given in Section 2. Some checking shows that for large values of a_0 and b_0, the computed table may be off by nearly 10^{-5}, so that in such cases, at least, the easy exact computation is generally worthwhile.

Interpolation is a problem. Linear interpolation in three dimensions is not very appealing and is not unambiguous. However, let us suppose that we desire $P(a, b, c)$ where

$$\alpha_i \leqq a \leqq \alpha_j; \qquad \beta_k \leqq b \leqq \beta_l; \qquad \gamma_m \leqq c \leqq \gamma_n,$$

the α's, β's and γ's being values in the table and, letting each pair (i, j), (k, l), (m, n) be some permutation of $(1, 2)$ so that (a, b, c) is "nearest" to $(\alpha_1, \beta_1, \gamma_1)$, we have approximately

$$\begin{aligned} P(a, b, c) = {} & P(\alpha_1, \beta_1, \gamma_1) + \frac{a - \alpha_1}{\alpha_2 - \alpha_1}\,\delta_\alpha[P(\alpha_2, \beta_1, \gamma_1)] \\ & + \frac{b - \beta_1}{\beta_2 - \beta_1}\,\delta_\beta[P(\alpha_1, \beta_2, \gamma_1)] + \frac{c - \gamma_1}{\gamma_2 - \gamma_1}\,\delta_\gamma[P(\alpha_1, \beta_1, \gamma_2)], \end{aligned} \tag{5.2}$$

where $\delta_\alpha[P(\alpha_2, \beta_1, \gamma_1)] = P(\alpha_2, \beta_1, \gamma_1) - P(\alpha_1, \beta_1, \gamma_1)$, etc. Some terms of higher order might be warranted, but the computations involved could be lengthy unless one of the parameters is a tabulated value (in which case interpolation is two dimensional).

In an attempt to improve on linear interpolation, we might employ Taylor's theorem to obtain an approximation (easily derived from (5.1)), where (α, β, γ) is the "nearest" entry to (a, b, c):

$$\begin{aligned} P(a, b, c) = {} & P(\alpha, \beta, \gamma) + (1/8\pi)\{f(a, \alpha) + f(b, \beta) + f(c, \gamma)\} \\ & + (1/4\pi^2)[f(a, \alpha) \sin^{-1} \gamma\{(1 - \alpha^2)/(1 - \alpha^2 - \beta^2)\}^{\frac{1}{2}} \\ & + f(b, \beta) \sin^{-1} [\alpha\beta\gamma\{(1 - \alpha^2 - \beta^2)(1 - \beta^2 - \gamma^2)\}^{-\frac{1}{2}}] \\ & + f(c, \gamma) \sin^{-1} \alpha\{(1 - \gamma^2)/(1 - \beta^2 - \gamma^2)\}^{\frac{1}{2}}] \end{aligned} \tag{5.3}$$

where $f(x, \xi) = (x - \xi)/(1 - \xi^2)^{\frac{1}{2}}$.

It is not at all clear under what circumstances this is better than linear interpolation. The behavior of this approximation does not compare consistently one way or the other with linear interpolation, even in the equicorrelated case. When all factors are accounted for, it seems that an accuracy of at least $3\frac{1}{2}$ decimal places can be expected on the large table, and $2\frac{1}{2}$ decimal places on the shortened version.

As an example, consider the probability that four random variables jointly distributed with a quadrivariate normal distribution with mean 0 and correlation matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

are simultaneously positive.

Since $\rho_{14} = \rho_{24} = 0$, we have immediately that $X_1 = X_2 = 0$, so we have only to consider the permutations $(i, j, k) = (1, 2, 3)$ and $(2, 1, 3)$. We then find

$$\begin{aligned}
\tau_{12\cdot3} &= \tau_{13\cdot2} = \tau_{23\cdot1} = \tfrac{1}{4} \\
\tau_{34\cdot1} &= \tau_{34\cdot2} = \tfrac{1}{2} \\
\tau_{14\cdot3} &= \tau_{24\cdot3} = -\tfrac{1}{4} \\
\tau_{14\cdot2} &= \tau_{24\cdot1} = 0 \\
u_{14} &= u_{24} = u_{34} = +1 \\
u_{14\cdot3} &= u_{24\cdot3} = +1 \\
\mu_{12} &= -1 \\
f_3 &= \{\tfrac{1}{16}\cdot\tfrac{3}{4} + \tfrac{1}{16}\cdot\tfrac{3}{4} - \tfrac{1}{2}\cdot\tfrac{1}{16}\}^{-\frac{1}{2}} = 4 \\
g_1 &= g_2 = [\tfrac{1}{2}\cdot\tfrac{1}{2}]^{-\frac{1}{2}} = 2. \\
\theta_{31} &= \theta_{32} = \tfrac{1}{2} \\
\varphi_{31} &= \varphi_{32} = \tfrac{1}{2} \\
\psi_{31} &= \psi_{32} = -\tfrac{1}{2}.
\end{aligned}$$

From the table we find $V_{31} = V_{32} = .075000$, so that $P_4 = .15000$ (obtained by both McFadden and Plackett).

A special case is that where the variables are equicorrelated, i.e., $\rho_{ij} = \rho$ when $i \neq j$, and $\rho_{ii} = 1$. Then we always have

$$a = \rho; \qquad b = -[\tfrac{1}{2}(1 - \rho)]^{\frac{1}{2}}; \qquad c = -\tfrac{1}{2}$$

and $P_4 = 6P(a, b, c)$. Actually, the equicorrelated case can be much more precisely dealt with in other ways (see, for example, [22]) so that the brute-force method given here is of no great value; but, as an example, let us suppose $\rho = \frac{1}{3}$.

TABLE 1

A=0.

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.062500	.066486	.070512	.074623	.078874	.083333	.088104	.093352	.099396	.107054
.05	.064490	.068476	.072502	.076614	.080864	.085324	.090094	.095342	.101386	.109044
.10	.066486	.070471	.074497	.078609	.082859	.087319	.092090	.097338	.103381	.111040
.15	.068491	.072476	.076503	.080614	.084865	.089324	.094095	.099343	.105387	.113045
.20	.070512	.074497	.078524	.082635	.086886	.091345	.096116	.101364	.107408	.115066
.25	.072554	.076539	.080566	.084677	.088928	.093387	.098158	.103406	.109450	.117108
.30	.074623	.078609	.082635	.086747	.090997	.095457	.100227	.105475	.111519	.119178
.35	.076727	.080713	.084739	.088851	.093101	.097561	.102331	.107579	.113623	.121282
.40	.078874	.082859	.086886	.090997	.095247	.099707	.104478	.109726	.115770	.123428
.45	.081072	.085058	.089084	.093195	.097446	.101905	.106676	.111924	.117968	
.50	.083333	.087319	.091345	.095457	.099707	.104167	.108937	.114185	.120229	
.55	.085672	.089657	.093683	.097795	.102045	.106505	.111276	.116542	.122567	
.60	.088104	.092090	.096116	.100227	.104478	.108937	.113708	.118956		
.65	.090654	.094639	.098666	.102777	.107028	.111487	.116258	.121506		
.70	.093352	.097338	.101364	.105475	.109726	.114185	.118956	.124204		
.75	.096243	.100229	.104255	.108367	.112617	.117077	.121847			
.80	.099396	.103381	.107408	.111519	.115770	.120229				
.85	.102925	.106910	.110937	.115048	.119298	.123758				
.90	.107054	.111040	.115066	.119178	.123428					
.95	.112365	.116350	.120376	.124488						

A= .10

C	B= 0.	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.066486	.070471	.074497	.078609	.082859	.087319	.092090	.097338	.103381	.111040
.05	.068603	.072589	.076617	.080732	.084988	.089456	.094239	.099506	.105584	.113324
.10	.070725	.074712	.078742	.082861	.087122	.091598	.096393	.101680	.107793	.115617
.15	.072858	.076846	.080878	.085000	.089267	.093752	.098560	.103866	.110015	.117929
.20	.075008	.078996	.083031	.087157	.091429	.095922	.100743	.106071	.112258	.120271
.25	.077180	.081169	.085206	.089336	.093614	.098116	.102951	.108300	.114529	.122657
.30	.079382	.083372	.087410	.091544	.095829	.100340	.105189	.110563	.116838	.125107
.35	.081620	.085610	.089652	.093790	.098081	.102602	.107467	.112867	.119195	.127658
.40	.083903	.087895	.091938	.096081	.100379	.104912	.109793	.115223	.121614	.130409
.45	.086242	.090234	.094280	.098428	.102734	.107278	.112179	.117643	.124114	
.50	.088647	.092640	.096690	.100842	.105157	.109715	.114638	.120143	.126726	
.55	.091135	.095129	.099181	.103340	.107664	.112238	.117187	.122745	.129514	
.60	.093722	.097718	.101774	.105939	.110275	.114866	.119848	.125479		
.65	.096435	.100431	.104492	.108665	.113014	.117628	.122653	.128403		
.70	.099305	.103303	.107369	.111552	.115918	.120561	.125650	.131696		
.75	.102381	.106381	.110453	.114648	.119036	.123722	.128924			
.80	.105734	.109737	.113817	.118030	.122449	.127207				
.85	.109488	.113495	.117587	.121825	.126299	.131252				
.90	.113881	.117893	.122007	.126293	.130925					
.95	.119531	.123556	.127722	.132260						

TABLE 2

A= .20

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.070512	.074497	.078524	.082635	.086886	.091345	.096116	.101364	.107408	.115066
.05	.072757	.076744	.080774	.084893	.089155	.093631	.098426	.103714	.109829	.117661
.10	.075008	.078996	.083031	.087157	.091429	.095922	.100743	.106071	.112258	.120271
.15	.077271	.081260	.085299	.089432	.093716	.098226	.103073	.108442	.114704	.122909
.20	.079551	.083541	.087584	.091725	.096021	.100548	.105422	.110833	.117176	.125594
.25	.081854	.085847	.089894	.094042	.098350	.102896	.107797	.113255	.119683	.128351
.30	.084189	.088183	.092235	.096391	.100712	.105277	.110208	.115714	.122239	.131220
.35	.086563	.090558	.094615	.098780	.103114	.107700	.112662	.118223	.124858	.134283
.40	.088984	.092981	.097043	.101217	.105566	.110174	.115172	.120793	.127562	
.45	.091465	.095463	.099530	.103715	.108079	.112711	.117749	.123441	.130380	
.50	.094016	.098016	.102089	.106285	.110666	.115326	.120409	.126187	.133366	
.55	.096654	.100656	.104736	.108943	.113345	.118036	.123173	.129061	.136655	
.60	.099398	.103403	.107490	.111712	.116136	.120864	.126068	.132111		
.65	.102275	.106283	.110378	.114616	.119067	.123842	.129135	.135430		
.70	.105319	.109330	.113436	.117693	.122179	.127014	.132439			
.75	.108581	.112596	.116715	.120997	.125527	.130450	.136115			
.80	.112137	.116158	.120294	.124610	.129206	.134278				
.85	.116119	.120147	.124308	.128676	.133385	.138873				
.90	.120778	.124818	.129021	.133489	.138526					
.95	.126770	.130837	.135148	.140156						

A= .30

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.074623	.078609	.082635	.086747	.090997	.095457	.100227	.105475	.111519	.119178
.05	.077000	.080987	.085020	.089142	.093410	.097895	.102705	.108015	.114173	.122132
.10	.079382	.083372	.087410	.091544	.095829	.100340	.105189	.110563	.116838	.125107
.15	.081776	.085768	.089813	.093958	.098261	.102799	.107688	.113126	.119522	.128126
.20	.084189	.088183	.092235	.096391	.100712	.105277	.110208	.115714	.122239	.131220
.25	.086627	.090623	.094682	.098850	.103190	.107783	.112758	.118336	.125001	.134435
.30	.089098	.093096	.097162	.101343	.105702	.110326	.115347	.121003	.127824	.137859
.35	.091610	.095611	.099684	.103878	.108258	.112914	.117984	.123726	.130729	
.40	.094173	.098176	.102257	.106466	.110868	.115558	.120684	.126523	.133745	
.45	.096798	.100804	.104893	.109117	.113544	.118271	.123459	.129411	.136921	
.50	.099498	.103507	.107605	.111846	.116300	.121070	.126329	.132419	.140349	
.55	.102290	.106302	.110410	.114670	.119154	.123974	.129318	.135586		
.60	.105194	.109210	.113330	.117611	.122130	.127009	.132459	.138980		
.65	.108238	.112258	.116392	.120698	.125260	.130211	.135803	.142757		
.70	.111460	.115485	.119635	.123971	.128586	.133632	.139437			
.75	.114912	.118943	.123113	.127488	.132172	.137357	.143561			
.80	.118676	.122716	.126912	.131339	.136126	.141549				
.85	.122890	.126941	.131175	.135683	.140648	.146810				
.90	.127821	.131890	.136189	.140854	.146358					
.95	.134162	.138274	.142740							

TABLE 3

A= .40

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.078874	.082859	.086886	.090997	.095247	.099707	.104478	.109726	.115770	.123428
.05	.081385	.085374	.089409	.093536	.097810	.102306	.107132	.112470	.118685	.126896
.10	.083903	.087895	.091938	.096081	.100379	.104912	.109793	.115223	.121614	.130407
.15	.086434	.090428	.094481	.098639	.102962	.107532	.112471	.117994	.124567	.134018
.20	.088984	.092981	.097043	.101217	.105566	.110174	.115172	.120793	.127562	
.25	.091561	.095561	.099632	.103824	.108198	.112846	.117906	.123632	.130614	
.30	.094173	.098176	.102257	.106466	.110868	.115558	.120684	.126523	.133745	
.35	.096828	.100835	.104926	.109153	.113584	.118319	.123516	.129480	.136987	
.40	.099537	.103547	.107649	.111896	.116359	.121142	.126418	.132524	.140384	
.45	.102311	.106325	.110438	.114707	.119204	.124041	.129405	.135678	.144026	
.50	.105165	.109183	.113309	.117601	.122136	.127033	.132500	.138979		
.55	.108116	.112138	.116279	.120596	.125174	.130141	.135731	.142481		
.60	.111186	.115213	.119370	.123717	.128345	.133395	.139139	.146290		
.65	.114403	.118437	.122612	.126995	.131681	.136835	.142788			
.70	.117809	.121849	.126047	.130472	.135232	.140523	.146796			
.75	.121457	.125506	.129732	.134211	.139070	.144561	.151499			
.80	.125436	.129496	.133759	.138311	.143316	.149161				
.85	.129889	.133966	.138281	.142948	.148209					
.90	.135101	.139203	.143610	.148497						
.95	.141805	.145965	.150606							

A= .50

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.083333	.087319	.091345	.095457	.099707	.104167	.108937	.114185	.120229	
.05	.085987	.089976	.094014	.098146	.102429	.106937	.111783	.117159	.123469	
.10	.088647	.092640	.096690	.100842	.105157	.109715	.114638	.120143	.126726	
.15	.091321	.095318	.099379	.103553	.107901	.112508	.117510	.123148	.130018	
.20	.094016	.098016	.102089	.106285	.110666	.115326	.120409	.126187	.133366	
.25	.096738	.100743	.104828	.109046	.113463	.118177	.123345	.129272	.136796	
.30	.099498	.103507	.107605	.111846	.116300	.121070	.126329	.132419	.140348	
.35	.102303	.106316	.110428	.114694	.119186	.124018	.129375	.135646	.144085	
.40	.105165	.109183	.113309	.117601	.122136	.127033	.132500	.138979		
.45	.108096	.112119	.116261	.120581	.125161	.130132	.135722	.142450		
.50	.111111	.115139	.119298	.123649	.128281	.133333	.139067	.146111		
.55	.114229	.118263	.122441	.126827	.131516	.136664	.142571	.150056		
.60	.117472	.121513	.125713	.130138	.134894	.140156	.146286			
.65	.120872	.124921	.129145	.133617	.138453	.143858	.150297			
.70	.124470	.128528	.132781	.137311	.142247	.147843	.154790			
.75	.128325	.132394	.136685	.141286	.146357	.152239				
.80	.132528	.136613	.140952	.145653	.150923	.157347				
.85	.137234	.141339	.145750	.150604	.156238					
.90	.142740	.146880	.151412	.156570						
.95	.149823	.154040	.158889							

TABLE 4

A= .60

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.088104	.092090	.096116	.100227	.104478	.108937	.113708	.118956		
.05	.090910	.094900	.098941	.103080	.107373	.111898	.116773	.122211		
.10	.093722	.097718	.101774	.105939	.110275	.114866	.119848	.125479		
.15	.096549	.100549	.104621	.108814	.113193	.117852	.122943	.128774		
.20	.099398	.103403	.107490	.111712	.116136	.120864	.126068	.132110		
.25	.102277	.106287	.110390	.114641	.119111	.123913	.129236	.135505		
.30	.105194	.109210	.113330	.117611	.122130	.127009	.132459	.138980		
.35	.108160	.112181	.116319	.120632	.125204	.130164	.135754	.142561		
.40	.111186	.115213	.119370	.123717	.128345	.133395	.139139	.146290		
.45	.114284	.118318	.122495	.126880	.131568	.136718	.142638	.150231		
.50	.117472	.121513	.125712	.130138	.134894	.140155	.146285			
.55	.120768	.124817	.129041	.133512	.138345	.143738	.150127			
.60	.124197	.128255	.132507	.137031	.141952	.147503	.154238			
.65	.127792	.131860	.136144	.140729	.145758	.151511	.158782			
.70	.131595	.135675	.139998	.144659	.149824	.155852				
.75	.135671	.139766	.144137	.148894	.154243	.160709				
.80	.140115	.144230	.148665	.153554	.159182					
.85	.145090	.149232	.153760	.158856	.165028					
.90	.150912	.155098	.159787	.165304						
.95	.158401	.162687	.167805							

A= .70

C	B= 0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.093352	.097338	.101364	.105475	.109726	.114185	.118956	.124204		
.05	.096325	.100317	.104362	.108510	.112817	.117369	.122297	.127935		
.10	.099305	.103303	.107369	.111552	.115918	.120561	.125649	.131691		
.15	.102300	.106305	.110391	.114610	.119035	.123773	.129026	.135503		
.20	.105319	.109330	.113436	.117693	.122178	.127014	.132439			
.25	.108369	.112387	.116514	.120810	.125358	.130296	.135903			
.30	.111460	.115485	.119635	.123971	.128585	.133631	.139436			
.35	.114603	.118635	.122808	.127187	.131872	.137034	.143058			
.40	.117809	.121849	.126047	.130472	.135232	.140522	.146795			
.45	.121092	.125141	.129365	.133840	.138683	.144116	.150686			
.50	.124470	.128528	.132781	.137310	.142246	.147842	.154788			
.55	.127962	.132030	.136316	.140907	.145948	.151737				
.60	.131595	.135675	.139998	.144658	.149824	.155852				
.65	.135404	.139497	.143863	.148606	.153921	.160268				
.70	.139434	.143543	.147960	.152804	.158313	.165145				
.75	.143753	.147881	.152362	.157336	.163113					
.80	.148462	.152615	.157181	.162337	.168548					
.85	.153733	.157923	.162612	.168059						
.90	.159902	.164150	.169055	.175150						
.95	.167837	.172218	.177730							

TABLE 5

A= .80

C \ B=	0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.099396	.103381	.107408	.111519	.115770	.120229				
.05	.102561	.106555	.110608	.114770	.119104	.123712				
.10	.105734	.109737	.113817	.118029	.122449	.127207				
.15	.108923	.112935	.117043	.121306	.125812	.130724				
.20	.112137	.116158	.120294	.124610	.129205	.134277				
.25	.115385	.119415	.123580	.127951	.132639	.137879				
.30	.118676	.122715	.126912	.131339	.136125	.141547				
.35	.122022	.126072	.130300	.134788	.139679	.145299				
.40	.125436	.129496	.133758	.138311	.143315	.149159				
.45	.128932	.133003	.137303	.141926	.147055	.153158				
.50	.132528	.136612	.140952	.145653	.150923	.157343				
.55	.136247	.140345	.144729	.149517	.154950	.161807				
.60	.140115	.144229	.148665	.153553	.159181					
.65	.144170	.148302	.152797	.157805	.163679					
.70	.148461	.152615	.157181	.162337	.168546					
.75	.153060	.157240	.161895	.167245						
.80	.158073	.162289	.167063	.172692						
.85	.163686	.167950	.172901	.179017						
.90	.170254	.174599	.179864							
.95	.178703	.183232								

A= .90

C \ B=	0	.10	.20	.30	.40	.50	.60	.70	.80	.90
	.107054	.111040	.115066	.119178	.123428					
.05	.110463	.114462	.118532	.122730	.127168					
.10	.113881	.117893	.122006	.126293	.130921					
.15	.117316	.121342	.125499	.129875	.134699					
.20	.120777	.124817	.129020	.133488	.138518					
.25	.124275	.128329	.132579	.137143	.142393					
.30	.127820	.131889	.136188	.140852	.146343					
.35	.131424	.135509	.139860	.144631	.150396					
.40	.135100	.139202	.143608	.148495						
.45	.138866	.142986	.147452	.152465						
.50	.142739	.146879	.151410	.156566						
.55	.146744	.150906	.155510	.160830						
.60	.150911	.155097	.159785	.165299						
.65	.155278	.159493	.164279	.170036						
.70	.159900	.164148	.169053	.175141						
.75	.164853	.169143	.174199							
.80	.170253	.174598	.179861							
.85	.176298	.180721	.186308							
.90	.183372	.187924								
.95	.192473	.197328								

We then have to find $P(\frac{1}{3}, -3^{\frac{1}{2}}/3, -\frac{1}{2})$. As a first approximation we might enter the unabridged table to find $P(.33, .6, .5) = P(.5, .6, .33) = .128149$. Then $P(\frac{1}{3}, 3^{\frac{1}{2}}/3, \frac{1}{2}) = .127069$ (from (5.3)) and $P(\frac{1}{3}, -3^{\frac{1}{2}}/3, -\frac{1}{2}) = .024975$, so that $P_4 = .149848$. Linear interpolation, in this case only two-dimensional, gives $P_4 = .149585$. The exact value of P_4 is .149738 (McFadden obtains .14976 and Ruben obtains .14974). Using the abridged table, (5.3) gives .15043 and linear interpolation gives .15115.

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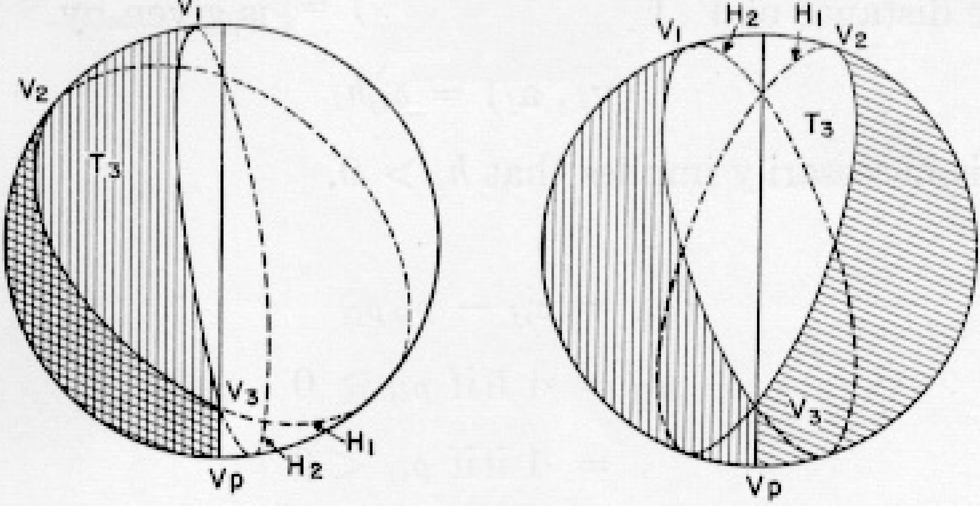

(a) (b)