MINIMAX CHARACTER OF THE R^2-TEST IN THE SIMPLEST CASE

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Summary. In the first nontrivial case, dimension $p = 3$ and sample size $N = 3$ or 4 (depending on whether or not the mean is known), it is proved that the classical level α normal test of independence of the first component from the others, based on the squared sample multiple correlation coefficient R^2, maximizes, among all level α tests, the minimum power on each of the usual contours where the R^2-test has constant power. A corollary is that the R^2-test is most stringent of level α in this case.

1. Introduction. Let $X_1, \cdots, X_N$ be independent normal p-vectors with common mean vector ξ and common nonsingular covariance matrix Σ. Write $N\bar{X} = \sum_1^N X_i$ and $S = \sum_1^N (X_i - \bar{X})(X_i - \bar{X})'$. Partition Σ and S as

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

respectively, where Σ_{22} and S_{22} are $(p-1) \times (p-1)$. Write $\rho^2 = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}/\Sigma_{11}$. Let $\delta (0 < \delta < 1)$ be specified. For testing the hypothesis $H_0 : \rho^2 = 0$ against $H_1 : \rho^2 = \delta$ at significance level α, a commonly employed procedure is the test based on the squared sample multiple correlation coefficient R^2, which rejects H_0 when $R^2 = S_{12}S_{22}^{-1}S_{21}/S_{11} > C$, where C is chosen so as to yield a test of level α. Throughout this paper $0 < \alpha < 1$, so that $0 < C < 1$.

In this paper we are interested in a minimax question regarding the R^2-test, namely, whether or not that test maximizes, among all level α tests, the minimum power under H_1. We succeed in proving that, for each possible choice of δ and α, the answer is affirmative in the first nontrivial case, $p = 3$, $N = 4$ (or $N = 3$ for the corresponding problem where ξ is known).

Our method of proof parallels that of Giri, Kiefer, and Stein (1963) (hereafter referred to as GKS) for the corresponding T^2-test result; the steps are the same, the detailed calculations in the present case being slightly more complicated. The remarks in GKS on the indications that the result holds for general p and N (in particular, from the local results of Giri and Kiefer (1962)), but of the inadequacy of the present method, apply also here. The reader is referred to GKS for a discussion of the Hunt-Stein theorem, its validity under the group of real lower-triangular matrices and its failure under the full linear group, and for other comments. Anderson (1958) is referred to for multivariate theory and Lehmann (1959) for testing (including invariance and minimax) theory.

Received January 14, 1964.

[1] Research supported by ONR contract No. Nonr-401(03).

[2] Fellow of the John Simon Guggenheim Memorial Foundation; research supported in part by ONR contract No. Nonr-266(04) (NRO 47-005).

It is well known that among tests based on the sufficient statistic $(\bar{X}, S)$, the R^2-test is best invariant under the group G of transformations of the form $(\xi, \Sigma, \bar{X}, S) \to (A\xi + b, A\Sigma A', A\bar{X} + b, ASA')$ where A is nonsingular and $A_{12} = A_{21}' = 0$. (For $p > 2$, this does not imply our minimax result, because of the failure of the Hunt-Stein theorem.) Simaika (1941) showed that the R^2-test is uniformly most powerful among all level α tests whose power is a function only of ρ^2, a result which is also implied by stronger results of Wolfowitz (1945) as well as by the best invariant character just mentioned. When $p = 2$ we have the well known properties of the standard two-tailed test based on the sample correlation coefficient; and when $N \leqq p$ if ξ is unknown or $N \leqq p - 1$ if ξ is known, it is easy to see that the infimum over H_1 of the power of every test equals the size of the test (for example, in this case $z_p = 1$ in (2.2), so that the distribution of z does not depend on the correlation between first and last components). Hence, the case $p = 3$, $N = 4$ (or $N = 3$ if ξ is assumed known) is the simplest one to be considered.

We now outline briefly our method of proof. We may restrict attention to the space of the minimal sufficient statistic $(\bar{X}, S)$. In the next section we shall first reduce the problem to the case where ξ is known and N is reduced by one, and shall then apply the Hunt-Stein theorem for an appropriate group G_T of $p \times p$ matrices (essentially the direct sum of the $(p - 1) \times (p - 1)$ lower triangular matrices and the nonzero reals), which is solvable. (See Kiefer (1957), Lehmann (1959), p. 345.) Thus, there is a test of level α which is almost invariant (hence, in the present problem, there is such a test which is invariant; see Lehmann (1957), p. 225) under G_T and which maximizes, among *all* level α tests, the minimum power over H_1. Whereas R^2 was a maximal invariant under G, with a single distribution under each of H_0 and H_1, the maximal invariant under G_T is a $(p - 1)$ dimensional statistic $R = (R_2, \cdots, R_p)'$ with a single distribution under H_0 but with a distribution which depends continuously on a $(p - 2)$-dimensional parameter $\Delta = (\delta_2, \cdots, \delta_p)$, $\delta_i \geqq 0$, $\sum_2^p \delta_i = \delta$ (fixed), under H_1. Thus, when $N > p > 2$ (or $N \geqq p > 2$ if ξ is known), there is no UMP invariant test under G_T as there was under G. We compute the Lebesgue densities f_Δ^* and f_0^* of R, under H_0 and H_1. Because of the compactness of the reduced parameter spaces $\{0\}$ and $\Gamma = \{(\delta_2, \cdots, \delta_p): \delta_i \geqq 0, \sum_2^p \delta_i = \delta\}$ and the continuity of f_Δ^* in Δ, it follows (see Wald (1950)) that every minimax test for the reduced problem in terms of R, is Bayes. In particular, the R^2-test, $\sum_2^p R_i > C$, (where $\sum_2^p R_i$ is what is usually called R^2), which is G_T-invariant, maximizes the minimum power over H_1 if and only if there is a probability measure λ on Γ such that, for some constant K,

$$\int_\Gamma \frac{f_\Delta^*(r_2, \cdots, r_p)}{f_0^*(r_2, \cdots, r_p)} \lambda\,(d\Delta) >, =, < K \tag{1.1}$$

according to whether $\sum_2^p r_i >, =, < C$, except possibly for a set of measure zero. (Here C depends on the specified α, and λ and K may depend only on C and the specified value $\delta > 0$.) An examination of the integrand in (1.1) will allow us to replace (1.1) by the equivalent

$$\int_\Gamma \frac{f_\Delta^*(r_2, \cdots, r_p)}{f_0^*(r_2, \cdots, r_p)} \lambda\,(d\Delta) = K \quad \text{if} \quad \textstyle\sum_2^p r_i = C. \tag{1.2}$$

We are able to evaluate the unique value which K must take on in order that (1.2) can be satisfied, and are then faced with the question of whether or not there exists a probability measure λ satisfying the left half of (1.2). The development thus far, which holds for general p and $N > p$, is carried out in Sections 2 and 3. In Section 4 we then obtain a λ and carry out the proof that it satisfies the left half of (1.2) in the special case $p = 3$, $N = 4$ (or $N = 3$ if ξ is known).

2. Reduction of the Problem to (1.2). Throughout this paper, we shall find it convenient to index the components of $(p - 1)$-vectors by subscripts $2, 3, \cdots, p$, with a corresponding convention for $(p - 1) \times (p - 1)$ matrices.

For testing H_0 against H_1 we need only consider test functions which depend on the statistic $(\bar{X}, S)$, sufficient for (ξ, Σ). It can easily be verified that the group H of transformations $(\xi, \Sigma, \bar{X}, S) \to (\xi + b, \Sigma, \bar{X} + b, S)$ leaves the testing problem in question invariant, that H is normal in the group G^* generated by H and the group of transformations G_T considered below, and that G_T and H (and hence G^*) satisfy the Hunt-Stein conditions; the action of the transformations in H is to reduce the problem to that where $\xi = 0$ (known) and $S = \sum_{i=1}^N X_i X_i'$ is sufficient for Σ, where N has been reduced by 1 from what it was originally. Using the standard method of reduction in steps, we can therefore treat this latter formulation, considering $X_1, X_2, \cdots, X_N$ to have zero mean. We assume also $N \geqq p \geqq 2$, it having been shown in Section 1 that, in the degenerate case $N < p$, the maximin value of the power equals the size. Furthermore, with this formulation, we need only consider test functions which depend on the sufficient statistic $S = \sum_{i=1}^N X_i X_i'$, the Lebesgue density of which is

$$f_\Sigma(s_{11}, s_{12}, s_{22}) = c(\det \Sigma)^{-N/2} \exp\,[-(\tfrac{1}{2}) \operatorname{tr} \Sigma^{-1} s] \times (\det s)^{(N-p-1)/2} \tag{2.1}$$

where

$$c^{-1} = 2^{Np/2} \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma((N + 1 - i)/2).$$

We now consider the group G_T of nonsingular lower triangular matrices (zero above the main diagonal) whose first column contains only zeros except for the first element. A typical element g of G_T can be represented as $g = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$ where g_{22} is $(p - 1) \times (p - 1)$ lower triangular. As we have stated earlier, it is easily seen that this group operating as $(S; \Sigma) \to (gSg'; g\Sigma g')$, leaves the problem invariant. We now compute a maximal invariant of S under the action of the group G_T in the usual fashion: If a function ϕ of S is invariant under G_T, then $\phi(S) = \phi(gSg')$ for all S and all $g \in G_T$, i.e., $\phi(S_{11}, S_{12}, S_{22}) = \phi(g_{11}S_{11}g_{11}, g_{11}S_{12}g_{22}', g_{22}S_{22}g_{22}')$. We may consider the domain of S to be symmetric positive definite matrices, which have probability one for all Σ; then there is an F in G_T

with positive diagonal elements such that $FF' = \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$. Putting $g = LF^{-1}$ where L is any diagonal matrix with values ± 1 in any order on the main diagonal, we see that ϕ is a function only of $L_{22}F_{22}^{-1}S_{21}L_{11}/F_{11}$ and hence, because of the freedom of choice of L, of $|F_{22}^{-1}S_{21}/F_{11}|$; or, equivalently, of the $(p-1)$-vector whose ith component $Z_i (2 \leqq i \leqq p)$ is the sum of squares of the first i components[3] of $|F_{22}^{-1}S_{21}/F_{11}|$ (whose components are indexed 2, 3, $\cdots$, p). Write $b_{[i]}$ for the $(i-1)$-vector consisting of the first $i-1$ components of the $(p-1)$-vector b, and $c_{[i]}$ for the upper left hand $(i-1) \times (i-1)$ submatrix of a $(p-1) \times (p-1)$ matrix c. Then Z_i can be written as $(F_{22}^{-1}S_{21}/F_{11})'_{[i]} \cdot (F_{22}^{-1}S_{21}/F_{11})_{[i]}$. Since $(F_{22[i]})^{-1} = (F_{22}^{-1})_{[i]}$, we have, for $2 \leqq i \leqq p$,

$$Z_i = \frac{S_{12[i]}(F_{22[i]}^{-1})'(F_{22[i]})^{-1}S'_{12[i]}}{S_{11}} = \frac{S_{12[i]}S_{22[i]}^{-1}S_{21[i]}}{S_{11}} \tag{2.2}$$

The vector $Z = (Z_2, \cdots, Z_p)'$ is thus a maximal invariant under G_T if it is invariant under G_T, and it is easily seen to be the latter. Z_i is essentially the squared sample multiple correlation computed from the first i coordinates of the X_i. Let us define a $(p-1)$-vector $R = (R_2, R_3, \cdots, R_p)'$ by

$$\sum_2^i R_j = Z_i \qquad 2 \leqq i \leqq p; \tag{2.3}$$

i.e., $R_i = Z_i - Z_{i-1}$ where we define $Z_1 = 0$. It follows trivially from above that R is also a maximal invariant under G_T. It is easily verified that $R_j \geqq 0$ for each j, $\sum_2^p R_j \leqq 1$, and of course $\sum_2^p R_j = S_{12}S_{22}^{-1}S_{21}/S_{11}$ is the squared sample multiple correlation coefficient between the first and other components (usually denoted by R^2). We shall find it more convenient to work with the equivalent statistic R instead of with Z.

A corresponding maximal invariant $\Delta = (\delta_2, \cdots, \delta_p)'$ in the parametric space of Σ under G_T when H_1 is true is given by

$$\sum_2^i \delta_j = \Sigma_{12[i]}(\Sigma_{22[i]})^{-1}\Sigma_{21[i]}/\Sigma_{11}, \qquad 2 \leqq i \leqq p. \tag{2.4}$$

It is clear that $\delta_j \geqq 0$ and $\sum_2^p \delta_j = \rho^2$, the squared population multiple correlation coefficient. The corresponding maximal invariant under H_0 takes on the single value $0 = (0, \cdots, 0)'$. It is well-known that the Lebesgue density function f_Δ^* of the maximal invariant depends only on Δ under H_1 and is a fixed f_0^* under H_0. We must now compute f_Δ^* and f_0^*. Actually we need only obtain the ratio f_Δ^*/f_0^* for use in (1.2), so we could proceed without keeping track of factors

[3] On page 1527 of GKS, Z_i should be defined similarly, instead of as the square of the ith component.

not depending on Δ in this computation. However, it is not much extra work to keep track of these factors, so we shall do so. There are several ways of computing f_Δ^*. For example, one method different from that which we shall use, but parallel to the method used by Anderson for computing the distribution of R^2, is to use the Bartlett decomposition to write

$$R_i \Big/ \left(1 - \sum_2^p R_j\right) = \left[\left(1 - \sum_2^i \delta_j\right)^{\frac{1}{2}} N_i + \delta_i^{\frac{1}{2}} \chi_{N-i+2}\right]^2 \Big/ (1 - \delta)\chi_{N-p+1}^2$$

where N_i are normal, χ_j is a chi-variable with j degrees of freedom, and all N_i and χ_j are independent, and then to integrate out on the χ_{N-i+2} and, finally, on χ_{N-p+1}^2.

We can assume $\Sigma_{11} = 1$, $\Sigma_{22} = I$ (the $(p-1) \times (p-1)$ identity matrix), and $\Sigma_{21} = (\delta_2^{\frac{1}{2}}, \cdots, \delta_p^{\frac{1}{2}})' = \delta^*$ (say) in (2.1), since f_Δ^* depends on Σ only through Δ. For this choice Σ^* (say) of Σ, (2.1) can be rewritten as

$$\begin{aligned} &f_{\Sigma^*}(s_{11}, s_{12}, s_{22}) \\ (2.5) \qquad &= c(1-\rho^2)^{-N/2} \exp\left[-\tfrac{1}{2} \operatorname{tr}(A_{11}s_{11} + A_{12}s_{12}' + A_{12}'s_{12} + A_{22}s_{22})\right] \\ &\quad \times (\det(s))^{(N-p-1)/2} \end{aligned}$$

where

$$\begin{aligned} A_{11} &= (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1} = (1-\rho^2)^{-1}, \\ A_{12} &= (\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \Sigma_{11})^{-1}\Sigma_{12}\Sigma_{22}^{-1} = -(1-\rho^2)^{-1}\delta^{*\prime}, \\ A_{22} &= (\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})^{-1} = (I - \delta^*\delta^{*\prime})^{-1}. \end{aligned}$$

Let B be the unique lower triangular $p \times p$ matrix belonging to G_T with positive diagonal elements $b_{ii} (1 \leqq i \leqq p)$ and such that $S_{22} = B_{22}B_{22}'$, $S_{11} = b_{11}^2$, and let $V = B_{22}^{-1}S_{21}$. One can easily compute the Jacobians

$$\partial S_{22}/\partial B_{22} = 2^{p-1}\prod_2^p (b_{ii})^{p+1-i}, \qquad \partial S_{21}/\partial V = \prod_2^p b_{ii},$$

and $\partial S_{11}/\partial b_{11} = 2b_{11}$, so that the joint density of b_{11}, V, and B_{22} is

$$(2.6) \qquad h_{\Sigma^*}(b_{11}, v, b_{22}) = 2^p f_{\Sigma^*}(b_{11}^2, v'b_{22}', b_{22}b_{22}')b_{11}\prod_{i=2}^p b_{ii}^{p+2-i}.$$

Putting $W = (W_2, \cdots, W_p)'$ with $W_i = |V_i| \ (2 \leqq i \leqq p)$, and noting that the $(p-1)$-vector W can arise from any of the 2^{p-1} vectors $V = M_{22}W$ where M_{22} is a $(p-1) \times (p-1)$ diagonal matrix with diagonal entries ± 1, we can write $g = bM$ where $M = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}$ with $M_{11} = \pm 1$ and g ranging over *all* matrices in G_T; we obtain for the density of W, writing $\bar{g}_{ij}$ $(i \geqq j \geqq 2)$ for the components of g_{22},

$$h^*_{\Sigma^*}(w) = 2^p \int f_{\Sigma^*}(g_{11}^2, w'g_{22}', g_{22}g_{22}') \prod_{i=2}^p |\bar{g}_{ii}|^{p+2-i}|g_{11}| \times \prod_{i \geqq j \geqq 2} d\bar{g}_{ij}\, dg_{11}$$

$$(2.7) \quad = (1-\rho^2)^{-N/2} 2^p c \int \exp\{-[2(1-\rho^2)]^{-1} \operatorname{tr} (g_{11}^2 - \delta^* w' g_{22}' - \delta^{*\prime} g_{22} w + (1-\rho^2)(I - \delta^*\delta^{*\prime})^{-1} g_{22} g_{22}')\} \times \prod_{i=2}^p |\bar{g}_{ii}|^{N+1-i} |g_{11}|^{N-p} (1 - w'w/g_{11}^2)^{(N-p-1)/2} \prod_{i \geqq j \geqq 2} d\bar{g}_{ij}\, dg_{11}$$

Writing $W = g_{11}U$, we obtain from (2.7) that the density of U is

$$(2.8) \quad h^{**}_{\Sigma^*}(u) = (1-\rho^2)^{-N/2} 2^p c \int \exp\{-[2(1-\rho^2)]^{-1} \operatorname{tr} (g_{11}^2 - g_{11}\delta^* u' g_{22}' - g_{11}\delta^{*\prime} g_{22} u + (1-\rho^2)(I - \delta^*\delta^{*\prime})^{-1} g_{22} g_{22}')\} \times |g_{11}|^{N-1} \prod_{i=2}^p |\bar{g}_{ii}|^{N+1-i} (1 - u'u)^{(N-p-1)/2} \prod_{i \geqq j \geqq 2} d\bar{g}_{ij}\, dg_{11},$$

the range of integration being from $-\infty$ to ∞ in each variable. It is easily checked that $U_j^2 = R_j (2 \leqq j \leqq p)$. Hence, the density of $R = (R_2, \cdots, R_p)'$ is given by

$$(2.9) \quad f^*_\Delta(r) = \frac{(1-\rho^2)^{-N/2} 2c}{\prod_{i=2}^p r_i^{\frac{1}{2}}} \int \exp\{-[2(1-\rho^2)]^{-1} \operatorname{tr} (g_{11}^2 - 2g_{11}\delta^{*\prime} g_{22} r^* + (1-\rho^2)(I - \delta^*\delta^{*\prime})^{-1} g_{22} g_{22}')\} \times \left(1 - \sum_{j=2}^p r_j\right)^{(N-p-1)/2} |g_{11}|^{N-1} \prod_{i=2}^p |\bar{g}_{ii}|^{N+1-i} \prod_{i \geqq j \geqq 2} d\bar{g}_{ij}\, dg_{11}$$

where $r^* = (r_2^{\frac{1}{2}}, \cdots, r_p^{\frac{1}{2}})'$. Let $\bar{C} = (1-\rho^2)^{-1}(I - \delta^*\delta^{*\prime})$. Since $\bar{C}$ is positive definite, there exists a lower triangular $(p-1) \times (p-1)$ matrix T with positive diagonal elements $T_{ii} (2 \leqq i \leqq p)$ such that $T\bar{C}T' = I$. Writing $h = Tg_{22}$, we obtain $\partial h / \partial g_{22} = \prod_2^p T_{ii}^{i-1}$. Let us define $\gamma_i (2 \leqq i \leqq p)$ by

$$(2.10) \quad \gamma_i = 1 - \sum_{j=2}^i \delta_j, \qquad \gamma_1 = 1$$

(so that $\gamma_p = 1 - \rho^2$) and $\alpha_i (2 \leqq i \leqq p)$ by

$$(2.11) \quad \alpha_i = [\delta_i \gamma_p / \gamma_{i-1}\gamma_i]^{\frac{1}{2}}.$$

Writing $\alpha = (\alpha_2, \cdots, \alpha_p)'$, a simple calculation (similar to that used to obtain (2.3) of GKS) shows that $(T_{[i]}\delta^*_{[i]})'(T_{[i]}\delta^*_{[i]}) = \gamma_p(1 - \gamma_i)/\gamma_i$, so that $\alpha = T\delta^*$. Since $\bar{C}\delta^* = \delta^*$ by direct computation, we obtain $\alpha = T\bar{C}\delta^* = T^{-1\prime}\delta^*$. From this and the easy computation $\det \bar{C} = (1-\rho^2)^{2-p}$, we obtain

$$f^*_\Delta(r) = 2c(1-\rho^2)^{-N(p-1)/2} \prod_2^p r_i^{-\frac{1}{2}}$$

$$\times \int \exp\{-[2(1-\rho^2)]^{-1} \operatorname{tr}(g_{11}^2 - 2g_{11}\alpha' h r^* + hh')\}$$

$$\times \left(1 - \sum_{j=2}^p r_j\right)^{(N-p-1)/2} |g_{11}|^{N-1} \prod_{i=2}^p |h_{ii}|^{N+1-i} \prod_{i \geqq j \geqq 2} dh_{ij}\, dg_{11}$$

$$(2.12) \qquad = 2c(1-\rho^2)^{-N(p-1)/2} \prod_2^p r_i^{-\frac{1}{2}} \left(1 - \sum_{j=2}^p r_j\right)^{(N-p-1)/2}$$

$$\times \int \exp\{-g_{11}^2/2(1-\rho^2)\}|g_{11}|^{N-1}$$

$$\times \left\{\int \exp\{-[2(1-\rho^2)]^{-1} \sum_{i \geqq j \geqq 2} [h_{ij}^2 - 2\alpha_i r_j^{\frac{1}{2}} h_{ij} g_{11}]\}\right.$$

$$\left.\times \prod_{i=2}^p |h_{ii}|^{N+1-i} \prod_{i \geqq j \geqq 2} dh_{ij}\right\} dg_{11},$$

the integration again being from $-\infty$ to ∞ in each variable. For $i > j$ the integration with respect to h_{ij} yields a factor $(2\pi)^{\frac{1}{2}}(1-\rho^2)^{\frac{1}{2}} \exp[\alpha_i^2 r_j g_{11}^2/2(1-\rho^2)]$. For $i = j$, we obtain a factor

$$(2\pi)^{\frac{1}{2}}(1-\rho^2)^{(N+2-i)/2} \exp[\alpha_i^2 r_i g_{11}^2/2(1-\rho^2)]$$

$$(2.13) \qquad \times E(\chi_1^2(\alpha_i^2 r_i g_{11}^2/(1-\rho^2))^{(N+1-i)/2}) = [2(1-\rho^2)]^{(N-i+2)/2}$$

$$\times \Gamma((N-i+2)/2)\phi((N-i+2)/2, \tfrac{1}{2}, r_i\alpha_i^2 g_{11}^2/2(1-\rho^2)),$$

where $\chi_1^2(\beta)$ is a noncentral chi-square variable with one degree of freedom and noncentrality parameter $\beta = E\chi_1^2(\beta) - 1$ and where ϕ is the confluent hypergeometric function (sometimes denoted by ${}_1F_1$),

$$\phi(a, b; x) = \sum_{j=0}^\infty \frac{\Gamma(a+j)\Gamma(b)x^j}{\Gamma(a)\Gamma(b+j)j!}.$$

Thus for $r \varepsilon H = \{r : r_i \geqq 0, 2 \leqq i \leqq p; \sum_2^p r_i < 1\}$ we have (noting that the exponent of the factor $(1-\rho^2)$ vanishes)

$$f^*_\Delta(r) = (2\pi)^{(p-1)(p-2)/4} \left(1 - \sum_{j=2}^p r_j\right)^{(N-p-1)/2}$$

$$(2.14) \qquad \times 2c \prod_2^p r_i^{-\frac{1}{2}} \int_{-\infty}^\infty \exp\left\{[-[2(1-\rho^2)]^{-1} g_{11}^2 \left(1 - \sum_{j=2}^p r_j \sum_{i>j} \alpha_i^2\right)\right\}$$

$$\times \prod_{i=2}^p [2^{(N+2-i)/2}\Gamma((N-i+2)/2)\phi((N-i+2)/2, \tfrac{1}{2}; r_i \alpha_i^2 g_{11}^2/2(1-\rho^2))]$$

$$\times |g_{11}|^{N-1}\, dg_{11}.$$

Integrating with respect to g_{11}, the density of r can be written as

$$f_\Delta^*(r) = \frac{(1-\rho^2)^{N/2}\left(1-\sum_{i=1}^p r_i\right)^{(N-p-1)/2}}{\left(1+\sum_{i=2}^p r_i((1-\rho^2)/\gamma_i - 1)\right)^{N/2}\Gamma((N-p+1)/2)\pi^{(p-1)/2}}$$

$$(2.15)\qquad \times \frac{1}{\prod_{i=2}^p \{r_i^{\frac{1}{2}}\Gamma((N-i+2)/2)\}} \times \sum_{\beta_2=0}^\infty \cdots \sum_{\beta_p=0}^\infty \Gamma\left(\sum_{j=2}^p \beta_j + N/2\right)$$

$$\times \prod_{i=2}^p \left\{\frac{\Gamma((N-i+2)/2+\beta_i)}{(2\beta_i)!} \times \left[\frac{4r_i\alpha_i^2}{1+\sum_{j=2}^p r_j((1-\rho^2)/\gamma_j - 1)}\right]^{\beta_i}\right\}.$$

Hence

$$\frac{f_\Delta^*(r)}{f_0^*(r)} = \frac{(1-\delta)^{N/2}}{\left(1+\sum_{i=2}^p r_i((1-\delta)/\gamma_i - 1)\right)^{N/2}}$$

$$(2.16)\qquad \times \sum_{\beta_2=0}^\infty \cdots \sum_{\beta_p=0}^\infty \frac{\Gamma\left(\sum_{j=2}^p \beta_j + N/2\right)}{\Gamma(N/2)}$$

$$\times \prod_{i=2}^p \left\{\frac{\Gamma((N-i+2)/2+\beta_i)}{\Gamma((N-i+2)/2)(2\beta_i)!} \times \left[\frac{4r_i\alpha_i^2}{1+\sum_{j=2}^p r_j((1-\delta)/\gamma_j - 1)}\right]^{\beta_i}\right\}.$$

The continuity of f_Δ^* in Δ over its compact domain $\Gamma = \{(\delta_2, \cdots, \delta_p): \delta_i \geqq 0, \sum_{i=2}^p \delta_i = \delta\}$ is evident, so we conclude that the minimax character of the critical region $\sum_{i=2}^p R_i \geqq C$ is equivalent to the existence of a probability measure λ satisfying (1.1). Clearly (1.1) implies (1.2). On the other hand, if there is a λ and a K for which (1.2) is satisfied and if $\bar{r} = (\bar{r}_2, \cdots, \bar{r}_p)$ is such that $\sum_2^p r_i = C' > C$, writing $f = f_\Delta^*/f_0^*$ and $\bar{\bar{r}} = C\bar{r}/C'$, we see at once that

$$f(\bar{r}) = f(C'\bar{\bar{r}}/C) > f(\bar{\bar{r}}) = K,$$

because of the form of f_Δ^*/f_0^* and the fact that $C'/C > 1$ and $\sum_2^p \bar{\bar{r}}_i = C$. It is to be noted that $\gamma_i^{-1}(1-\delta) - 1 = -\sum_{j>i}\delta_j/\gamma_i$ and that $\gamma_i > 0$. This and a similar argument for the case $C' < C$ show that (1.1) implies (1.2). The remaining computations of the paper are somewhat simplified by the fact that for fixed C and δ we can at this point easily compute the unique value of K for which (1.2) can possibly be satisfied.

3. Evaluation of K. Let $\hat{R} = (R_2, \cdots, R_{p-1})'$ and write $f_\Delta^*(\hat{r} \mid u)$ for the version of the conditional Lebesgue density of $\hat{R}$ given that $\sum_2^p R_i = u$, which is continuous in $\hat{r}$ and u for $r_i > 0$, $\sum_2^{p-1} r_i < u < 1$, and is 0 elsewhere; also write $f_\delta^{**}(u)$ for the Lebesgue density of $R^2 = \sum_2^p R_i$ which is continuous for $0 < u < 1$ and vanishes elsewhere (and which depends on Δ only through δ). Then (1.2) can be written as

$$\int f_\Delta^*(r \mid C)\, d\lambda(\Delta) = \left[\frac{Kf_0^{**}(C)}{f_\delta^{**}(C)}\right] f_0^*(\hat{r} \mid C) \tag{3.1}$$

for $r_i > 0$ and $\sum_2^{p-1} r_i < C$. The integral of (3.1), being a probability mixture of probability densities, is itself a probability density in $\hat{r}$, as is $f_0^*(\hat{r} \mid C)$. Hence the expression in square brackets equals one. It is well known that for $0 < C < 1$

$$\begin{aligned} f_\delta^{**}(C) = \frac{(1-\delta)^{N/2}\Gamma(N/2)}{\Gamma((N-p+1)/2)\Gamma((p-1)/2)} C^{(p-3)/2}(1-C)^{(N-p-1)/2} \\ \times F(N/2, N/2; (p-1)/2; C\delta), \end{aligned} \tag{3.2}$$

where $F(a, b; c; x)$ is the ordinary (${}_2F_1$) hypergeometric series, given by

$$\begin{aligned} F(a, b; c; x) &= \sum_{r=0}^{\infty} \frac{x^r\Gamma(a+r)\Gamma(b+r)\Gamma(c)}{r!\,\Gamma(a)\Gamma(b)\Gamma(c+r)} \\ &= \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r r!} x^r, \end{aligned} \tag{3.3}$$

where we write $(a)_r = \Gamma(a+r)/\Gamma(a)$. (See Anderson (1958) or use (2.15).) Hence from (3.1) the value of K which satisfies (1.2) is given by

$$K = (1-\delta)^{N/2}F(N/2, N/2; (p-1)/2; C\delta). \tag{3.4}$$

Hence by (3.4) and (2.16) the condition (1.2) becomes

$$\begin{aligned} \int_\Gamma \left[1 + \sum_2^p r_i((1-\delta)/\gamma_i - 1)\right]^{-N/2} \sum_{\beta_2=0}^{\infty} \cdots \sum_{\beta_p=0}^{\infty} \frac{\Gamma\left(\sum_2^p \beta_j + N/2\right)}{\Gamma(N/2)} \\ \times \prod_{i=2}^{p} \left\{\frac{\Gamma((N-i+2)/2+\beta_i)}{\Gamma((N-i+2)/2)(2\beta_i)!} \times \left[\frac{4r_i\alpha_i^2}{1 + \sum_2^p r_j((1-\delta)/\gamma_j - 1)}\right]^{\beta_i}\right\} d\lambda(\Delta) \\ = F(N/2, N/2; (p-1)/2; C\delta) \end{aligned} \tag{3.5}$$

for all r with $r_i > 0$ and $\sum_2^p r_i = C$. Unlike the corresponding equation (2.8) of GKS, (3.5) does not yield an obvious conclusion regarding the dependence of λ on C and δ only through $C\delta$, although we shall obtain this conclusion in the case treated in the next paragraph.

4. The case $p = 3$, $N = 3$ (or $N = 4$ if ξ is unknown). In this case (3.5) can be written (as can be seen, for example, by writing $\phi(\frac{3}{2}, \frac{1}{2}; x) = (1+2x)e^x$ for $i = 2$ in (2.14), and then carrying out the integration) as

$$\int_\Gamma \sum_{n=0}^{\infty} \left\{ \frac{(1+2n)}{(1-r_2\delta)^{\frac{3}{2}}} \left(\frac{r_3\delta_3}{(1-\delta_2)(1-r_2\delta)} \right)^n + \frac{r_2\delta_2(1-\delta)(2n+1)(2n+3)}{(1-\delta_2)(1-r_2\delta)^{\frac{5}{2}}} \left(\frac{r_3\delta_3}{(1-\delta_2)(1-r_2\delta)} \right)^n \right\} d\lambda(\Delta) = F(\tfrac{3}{2}, \tfrac{3}{2}; 1; C\delta). \tag{4.1}$$

One could presumably try to solve for λ by using the theory of Meijer transforms with kernel $F(\frac{3}{2}, \frac{3}{2}; 1; x)$. We proceed instead, as in GKS for the T^2 problem, by expanding (4.1) in an appropriate power series. Write Γ_1 for the unit one-dimensional simplex $\{(\beta_1, \beta_2): \beta_i \geqq 0, \sum_1^2 \beta_i = 1\}$ and make the change of variables $t_1 = r_2/(1-r_2)$, $t_1 + t_2 = (r_2 + r_3)/[1-(r_2+r_3)]$, $\eta_1 = \delta_2/(1-\delta_2)$, $\eta_1 + \eta_2 = \delta/(1-\delta) = \delta'$ (say), $C^* = C/(1-C)$, $y = t_2\delta'/(1+t_1+\delta')(1+C^*)$, and $\beta_i = \eta_i/\delta' (i = 1, 2)$. Write λ^* for a measure for β_2 on Γ_1 associated with λ in the obvious way, and denote by $\mu_i = \int_0^1 \beta_2^i \, d\lambda^*(\beta_2)$ the ith moment of λ^*. Finally, write $z = C\delta = C^*\delta'/(1+C^*)(1+\delta')$. We then obtain from (4.1)

$$(1-z)\sum_{n=0}^{\infty} y^n(2n+1)\mu_n + (z-y)\sum_{n=0}^{\infty} y^n(2n+1)(2n+3) \times (\mu_n - \mu_{n+1}) = (1-z)^{\frac{5}{2}} F(\tfrac{3}{2}, \tfrac{3}{2}; 1; z)(1-y)^{-\frac{3}{2}}. \tag{4.2}$$

Writing $B_z = (1-z)^{\frac{5}{2}} F(\frac{3}{2}, \frac{3}{2}; 1; z)$, we obtain upon equating coefficients of like powers of y on the two sides of (4.2), the following set of equations as equivalent to (4.1):

$$\text{(a) } 1 + 2z - 3z\mu_1 = B_z$$

$$\text{(b) } -(2n-1)\mu_{n-1} + (2n + z(2n+2))\mu_n - z(2n+3)\mu_{n+1} = B_z \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})n!}, \qquad n \geqq 1. \tag{4.3}$$

(Of course $\mu_0 = 1$ for λ^* to be a probability measure.) It is clear from (4.3) that λ^*, if it exists, depends on C and δ only through their product. One could now try to show that the sequence $\{\mu_i\}$ defined by $\mu_0 = 1$ and (4.3) satisfies the classical necessary and sufficient conditions for it to be a moment sequence of a probability measure on [0, 1] or, equivalently, that the Laplace transform $\sum_0^\infty \mu_j(-t)^j/j!$ is completely monotone on $[0, \infty)$, but we have been unable to proceed successfully in this way. Instead, we shall obtain, in the next paragraph, a function $m_z(x)$, which we then prove in the succeeding paragraphs below to be the Lebesque density $d\lambda^*(x)/dx$ of an absolutely continuous probability measure λ^* satisfying (4.3) (and hence (4.1)). That proof does not rely on the somewhat heuristic development of the next paragraph, but we nevertheless sketch that development to give an idea of where the $m_z(x)$ of (4.8) comes from.

The generating function $\phi(t) = \sum_{j=0}^\infty \mu_j t^j$ of the sequence $\{\mu_j\}$ satisfies a differential equation which is obtained in the usual way by multiplying (4.3) (b) by t^{n-1} and summing with respect to n from 1 to ∞:

$$(4.4)\qquad 2(1-t)(t-z)\phi'(t) - t^{-1}(t^2 - 2zt + z)\phi(t) = B_z(1-t)^{-\frac{1}{2}} - 1 - zt^{-1}.$$

This is solved by treatment of the corresponding homogeneous equation and by variation of parameter, to yield

$$(4.5)\qquad \phi(t) = \left[\frac{t-z}{(1-t)t}\right]^{\frac{1}{2}} \int_0^t \left[\frac{B_z \tau^{\frac{1}{2}}}{2(1-\tau)(\tau-z)^{\frac{3}{2}}} - \frac{\tau^{\frac{1}{2}}}{(\tau-z)^{\frac{3}{2}}(1-\tau)^{\frac{1}{2}}} + \frac{1}{2[\tau(1-\tau)(\tau-z)]^{\frac{1}{2}}}\right] d\tau.$$

The constant of integration has been chosen to make ϕ continuous at 0 with $\phi(0) = 1$, and (4.5) defines a single-valued analytic function on the complex plane cut from 0 to z and from 1 to ∞. Now, if there did exist an absolutely continuous λ^* whose suitably regular derivative m_z satisfied

$$(4.6)\qquad \int_0^1 m_z(x)\, dx/(1-tx) = \phi(t),$$

we could obtain m_z by using the simple inversion formula

$$(4.7)\qquad m_z(x) = \frac{1}{2\pi i x} \lim_{\epsilon \downarrow 0} [\phi(x^{-1} + i\epsilon) - \phi(x^{-1} - i\epsilon)].$$

Since there is nothing in the theory of Stieltjes transforms which tells us that an m_z satisfying (4.7) does satisfy (4.6) (and hence (4.1)), we will use (4.7) only as a formal device to obtain m_z which we shall then prove, in the remaining paragraphs, satisfies (4.1). From (4.5) and (4.7) we obtain, for $0 < x < 1$,

$$(4.8)\qquad \begin{aligned} m_z(x) = \frac{(1-zx)^{\frac{1}{2}}}{2\pi x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \Bigg\{ & B_z \int_0^x \frac{du}{(1-u)(1-zu)^{\frac{3}{2}}} \\ & + \int_0^\infty \left[\frac{B_z u^{\frac{1}{2}}}{(1+u)(z+u)^{\frac{3}{2}}} + \frac{1}{[u(1+u)(z+u)]^{\frac{1}{2}}} - 2\frac{u^{\frac{1}{2}}}{(1+u)^{\frac{1}{2}}(z+u)^{\frac{3}{2}}}\right] du \Bigg\} = \frac{(1-zx)^{\frac{1}{2}}}{2\pi(x(1-x))^{\frac{1}{2}}} \{B_z Q_z(x) + c_z\} \quad \text{(say)}. \end{aligned}$$

c_z can be evaluated by making the change of variables $v = (1+u)^{-1}$ and using (4.11) below. We obtain

$$c_z = \tfrac{2}{3} B_z F(\tfrac{3}{2}, 1; \tfrac{5}{2}; 1-z) + \pi F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-z) - \pi F(\tfrac{3}{2}, \tfrac{1}{2}; 2; 1-z)$$

and

$$Q_z(x) = 2(1-z)^{-1}[1 - (1-zx)^{-\frac{1}{2}}] + (1-z)^{-\frac{3}{2}} \log\left[\frac{(1-zx)^{\frac{1}{2}} + (1-z)^{\frac{1}{2}}}{(1-zx)^{\frac{1}{2}} - (1-z)^{\frac{1}{2}}} \cdot \frac{1-(1-z)^{\frac{1}{2}}}{1+(1-z)^{\frac{1}{2}}}\right].$$

Now to show that $d\lambda^*(x) = m_z(x)\, dx$ (with m_z defined by (4.8)) satisfies

(4.1) with λ^* a probability measure, we must show that

$$
\text{(4.9)}\quad
\begin{aligned}
&\text{(a) } m_z(x) \geqq 0 \text{ for almost all } x, 0 < x < 1;\\
&\text{(b) } \int_0^1 m_z(x)\,dx = 1;\\
&\text{(c) } \mu_1 = \int_0^1 x m_z(x)\,dx \text{ satisfies (4.3)(a);}\\
&\text{(d) } \mu_n = \int_0^1 x^n m_z(x)\,dx \text{ satisfies (4.3)(b) for } n \geqq 1.
\end{aligned}
$$

Condition (4.9) (a) will follow from (4.8) and the positivity of B_z and c_z for $0 < z < 1$. The former is obvious. To prove the positivity of c_z, we first note that $F(\frac{3}{2}, \frac{3}{2}; 1; z) \geqq (1 - z)^{-2}$; this is seen by comparing the two power series, the coefficients of z^j being $[(\frac{3}{2})_j/j!]^2$ and $(j + 1)$, and the ratio of the former to the latter being $\prod_{i=1}^{j} (i + \frac{1}{2})^2/i(i + 1) \geqq 1$. We thus have $B_z \geqq 1 - z$. Substituting this lower bound into the expression for c_z and writing $u = 1 - z$, the resulting lower bound for c_z has a power series in u (convergent for $|u| < 1$) whose constant term is 0 and whose coefficient of u^j for $j \geqq 1$ is $(j + \frac{1}{2})^{-1} - \Gamma^2(j + \frac{1}{2})/\Gamma(j)\Gamma(j + 1)(j + 1)$; by the well known logarithmic convexity of the Γ-function, $\Gamma^2(j + \frac{1}{2}) < \Gamma(j)\Gamma(j + 1)$, so the coefficient of u^j for $j \geqq 1$ is $> (j + \frac{1}{2})^{-1} - (j + 1)^{-1} > 0$. Hence, $c_z > 0$ for $0 < z < 1$.

To prove (4.9) (d) we note that $m_z(x)$ defined by (4.3) satisfies the differential equation

$$
\text{(4.10)}\quad m_z'(x) + \tfrac{1}{2}m_z(x)\left[\frac{1 - 2x + zx^2}{x(1 - x)(1 - zx)}\right] = B_z/2\pi x^{\frac{1}{2}}(1 - x)^{\frac{3}{2}}(1 - zx),
$$

so that an integration by parts yields, for $n \geqq 1$,

$$
\begin{aligned}
-z(n + 2)\mu_{n+1} &+ (1 + z)(n + 1)\mu_n - n\mu_{n-1}\\
&= \int_0^1 \{-z(n + 2)x^{n+1} + (1 + z)(n + 1)x^n - nx^{n-1}\}m_z(x)\,dx\\
&= \int_0^1 x^n(1 - (1 + z)x + zx^2)m_z'(x)\,dx\\
&= \tfrac{1}{2}\left\{-\mu_{n-1} + 2\mu_n - z\mu_{n+1} + B_z\frac{\Gamma(n + \frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}\right\},
\end{aligned}
$$

which is (4.3) (b).

The proofs of (4.9) (b) and (c) rely on certain identities involving hypergeometric functions. In the next paragraph we list some of the properties of hypergeometric functions which will be used in these proofs.

The material presented in this paragraph can be found in Erdélyi (1953), Chapter 2. The hypergeometric function $F(a, b; c; x)$ has the following integral representation when $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$:

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\, dt. \tag{4.11}$$

We will also use the representation

$$\log\left(\frac{1+x}{1-x}\right) = 2xF(\tfrac{1}{2}, 1; \tfrac{3}{2}; x^2) \tag{4.12}$$

and the identities

$$F(a, b; c; x) = F(b, a; c; x); \tag{4.13}$$

$$\begin{aligned}(c-a-1)F(a, b; c; x) + aF(a+1, b; c; x)& \\ - (c-1)F(a, b; c-1; x) &= 0,\end{aligned} \tag{4.14}$$

$$\begin{aligned}&\lim_{c\to -n}[\Gamma(c)]^{-1}F(a, b; c; x) \\ &= \frac{(a)_{n+1}(b)_{n+1}}{(n+1)!} x^{n+1} F(a+n+1, b+n+1; n+2; x) \text{ for } n = 0, 1, 2, \cdots.\end{aligned} \tag{4.15}$$

$$\begin{aligned}c(1-x)F(a, b; c; x) - cF(a-1, b; c; x)& \\ + (c-b)xF(a, b; c+1; x) &= 0;\end{aligned} \tag{4.16}$$

$$\begin{aligned}&F(\tfrac{1}{2}+\lambda, -\tfrac{1}{2}-\nu; 1+\lambda+\mu; x)F(\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu; 1+\nu+\mu; 1-x) \\ &+ F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu; 1+\lambda+\mu; x)F(-\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu; 1+\nu+\mu; 1-x) \\ &- F(\tfrac{1}{2}+\lambda, \tfrac{1}{2}-\nu; 1+\lambda+\mu; x)F(\tfrac{1}{2}-\lambda, \tfrac{1}{2}+\nu; 1+\nu+\mu; 1-x) \\ &= \frac{\Gamma(1+\lambda+\mu)\Gamma(1+\nu+\mu)}{\Gamma(\lambda+\mu+\nu+\tfrac{3}{2})\Gamma(\tfrac{1}{2}+\mu)};\end{aligned} \tag{4.17}$$

$$F(a, b; c; x) = (1-x)^{c-a-b}F(c-a, c-b; c; x). \tag{4.18}$$

We are now in a position to prove (4.9) (b) and (c). From (4.8), using (4.11) and (4.12), we obtain

$$\begin{aligned}\int_0^1 m_z(x)\, dx &= (1-z)^{\frac{3}{2}}F(\tfrac{3}{2}, \tfrac{3}{2}; 1; z)F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; z) \\ &\times [1 - F(\tfrac{1}{2}, 1; \tfrac{3}{2}; 1-z) + \tfrac{1}{3}(1-z)F(\tfrac{3}{2}, 1; \tfrac{5}{2}; 1-z)] \\ &- (1-z)^{\frac{3}{2}}F(\tfrac{3}{2}, \tfrac{3}{2}; 1; z) + (\pi/2)[F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-z) - F(\tfrac{3}{2}, \tfrac{1}{2}; 2; 1-z)] \\ &\times F(-\tfrac{1}{2}, \tfrac{1}{2}; 1; z) \\ &+ B_z \int_0^1 \frac{(1-zx)^{\frac{1}{2}}}{2\pi(1-z)^{\frac{3}{2}}x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \log\left(\frac{1+(1-z)^{\frac{1}{2}}(1-zx)^{-\frac{1}{2}}}{1-(1-z)^{\frac{1}{2}}(1-zx)^{-\frac{1}{2}}}\right) dx.\end{aligned} \tag{4.19}$$

The first expression in square brackets in (4.19) vanishes, as is easily seen from the power series (3.3). Using the power series for log $(1+u^{\frac{1}{2}})/(1-u^{\frac{1}{2}})$, the integral of (4.19) can be written as

$$\frac{1}{\pi(1-z)}\sum_{n=0}^{\infty}\frac{(1-z)^n}{2n+1}\int_0^1\frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}(1-zx)^n}$$

$$= (1-z)^{-1} + (1-z)^{-1}\sum_{n=1}^{\infty}\frac{(1-z)^n}{2n+1}F(n,\tfrac{1}{2};1;z)$$

$$= (1-z)^{-1} + \sum_{n=0}^{\infty}\frac{(1-z)^n}{2n+3}F(n+1,\tfrac{1}{2};1;z)$$

$$(4.20)\qquad = (1-z)^{-1} + \tfrac{1}{2}\sum_{m=0}^{\infty}\frac{z^m(\frac{1}{2})_m}{(m!)^2}\sum_{n=0}^{\infty}\frac{(n+m)!\,(1-z)^n}{n!\,(n+\frac{3}{2})}$$

$$= (1-z)^{-1} + \tfrac{1}{2}\sum_{m=0}^{\infty}\frac{z^m(\frac{1}{2})_m}{(m!)^2}(1-z)^{-\frac{3}{2}}\int_0^{1-z}\frac{m!\,t^{\frac{1}{2}}}{(1-t)^{m+1}}\,dt$$

$$= (1-z)^{-1} + \tfrac{1}{2}(1-z)^{-\frac{3}{2}}\int_0^{1-z}\frac{t^{\frac{1}{2}}\,dt}{(1-t)^{\frac{1}{2}}(1-z-t)^{\frac{1}{2}}}$$

$$= (1-z)^{-1} + (\pi/4)(1-z)^{-\frac{1}{2}}F(\tfrac{1}{2},\tfrac{3}{2};2;1-z).$$

Hence, from (4.19) and (4.20) after cancellation one gets

$$(4.21)\qquad \int_0^1 m_z(x)\,dx = (\pi/2)F(-\tfrac{1}{2},\tfrac{1}{2};1;z)[F(\tfrac{1}{2},\tfrac{1}{2};1;1-z) - F(\tfrac{3}{2},\tfrac{1}{2};2;1-z)] + (\pi/4)(1-z)^2F(\tfrac{3}{2},\tfrac{3}{2};1;z)F(\tfrac{3}{2},\tfrac{1}{2};2;1-z).$$

Using (4.14) with $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 2$, and (4.18) with $a = b = \frac{3}{2}$, $c = 1$, we obtain from (4.21)

$$(4.22)\qquad \int_0^1 m_z(x)\,dx = (\pi/4)\{F(-\tfrac{1}{2},\tfrac{1}{2};1;z)F(\tfrac{1}{2},\tfrac{1}{2};2;1-z) + F(\tfrac{3}{2},\tfrac{1}{2};2;1-z)[F(-\tfrac{1}{2},-\tfrac{1}{2};1;z) - F(-\tfrac{1}{2},\tfrac{1}{2};1;z)]\}.$$

By the use of (4.14) with $a = b = -\frac{1}{2}$, $c = 1+\epsilon$ and (4.15) with $n = 0$, $c = \epsilon \to 0$, (4.22) reduces to

$$(4.23)\qquad (\pi/4)\{F(-\tfrac{1}{2},\tfrac{1}{2};1;z)F(\tfrac{1}{2},\tfrac{1}{2};2;1-z) + (z/2)F(\tfrac{3}{2},\tfrac{1}{2};2;1-z)F(\tfrac{1}{2},\tfrac{1}{2};2;z)\}.$$

Now, by (4.17) with $\mu = 1$, $\lambda = -1$, $\nu = 0$, the expression (4.23) equals one if we have

$$F(\tfrac{3}{2},\tfrac{1}{2};2;1-z)[F(-\tfrac{1}{2},\tfrac{1}{2};1;z) - F(-\tfrac{1}{2},-\tfrac{1}{2};1;z) + (z/2)F(\tfrac{1}{2},\tfrac{1}{2};2;z)] = 0.$$

The expression inside the square brackets is easily seen to be zero by using (3.3) and computing the coefficient of z^n. Thus (4.9) (b) is proved.

We now verify (4.9) (c). We proceed from (4.8) in a manner parallel to that used to obtain (4.21). The integrand of (4.19) is altered by multiplication by x, and in place of (4.20) we obtain $(1 - z)^{-1}/2 - z^{-1}/3 + [\pi/4z(1 - z)^{\frac{1}{2}}] \cdot F(-\frac{1}{2}, \frac{3}{2}; 2; 1 - z)$. The analogue of (4.21) is

$$\begin{aligned} \mu_1 = \int_0^1 x m_z(x)\, dx &= (\pi/4)F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; z) \\ &\times [F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - z) - F(\tfrac{3}{2}, \tfrac{1}{2}; 2; 1 - z)] + (\pi/4)(1 - z)^2/z \\ &\times F(\tfrac{3}{2}, \tfrac{3}{2}; 1; z)F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; 1 - z) - [(1 - z)^{\frac{5}{2}}/3z]F(\tfrac{3}{2}, \tfrac{3}{2}; 1; z). \end{aligned} \tag{4.24}$$

To verify (4.9) (c) we then have to prove the following identity (using (4.3) (a)):

$$\begin{aligned} (1 + 2z)/3z &= (\pi/4)F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; z)[F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - z) \\ &\quad - F(\tfrac{3}{2}, \tfrac{1}{2}; 2; 1 - z)] + (\pi/4)[(1 - z)^2/z] \\ &\quad \times F(\tfrac{3}{2}, \tfrac{3}{2}; 1; z)F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; 1 - z). \end{aligned} \tag{4.25}$$

Using (4.18) with $c = 1$, $a = b = \frac{3}{2}$, then (4.16) with $a = \frac{1}{2}$, $b = -\frac{1}{2}$, $c = 1$, and then (4.17) with $\lambda = \mu = 0$, $\nu = 1$, (4.25) can be reduced to

$$\begin{aligned} (4/3\pi)(1 + 2z) &= zF(-\tfrac{1}{2}, \tfrac{3}{2}; 2; z)[F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - z) \\ &\quad - F(\tfrac{3}{2}, \tfrac{1}{2}; 2; 1 - z)] + (3z/2)F(\tfrac{1}{2}, -\tfrac{1}{2}; 2; z)F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; 1 - z) \\ &\quad + (4/3\pi)(1 - z) + (1 - z)F(\tfrac{1}{2}, \tfrac{3}{2}; 2; 1 - z)[F(\tfrac{1}{2}, -\tfrac{1}{2}; 1; z) \\ &\quad - F(\tfrac{1}{2}, -\tfrac{3}{2}; 1; z)]. \end{aligned} \tag{4.26}$$

Using (4.14) with $a = -\frac{3}{2}$, $b = \frac{1}{2}$, $c = 1 + \epsilon$, and then (4.15) with $n = 0$, $c = \epsilon \to 0$, the expression inside the square brackets in the last term of (4.26) can be further reduced to $\frac{1}{2}zF(-\frac{1}{2}, \frac{3}{2}; 2; z)$. Hence we are faced with the problem of establishing the identity

$$\begin{aligned} 4/\pi &= F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; z)F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - z) + \tfrac{3}{2}F(\tfrac{1}{2}, -\tfrac{1}{2}; 2; z) \\ &\times F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; 1 - z) - [(z + 1)/2]F(\tfrac{3}{2}, -\tfrac{1}{2}, 2; z) \\ &\times F(\tfrac{1}{2}, \tfrac{3}{2}; 2; 1 - z), \end{aligned} \tag{4.27}$$

which finally by (4.11) with $\lambda = \nu = -1$, $\mu = 2$ reduces to

$$\begin{aligned} 0 &= F(-\tfrac{1}{2}, \tfrac{3}{2}; 2; z)[F(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - z) + \tfrac{3}{2}F(\tfrac{3}{2}, -\tfrac{1}{2}; 2; 1 - z) \\ &\quad - \tfrac{3}{2}F(\tfrac{1}{2}, -\tfrac{1}{2}; 2; 1 - z) + ((1 - z)/2 - 1)F(\tfrac{3}{2}, \tfrac{1}{2}; 2; 1 - z)]. \end{aligned} \tag{4.28}$$

The expression inside the square brackets in (4.28) has a power series in $1 - z$, the value of which is easily seen to be zero by computing the coefficients of various power of $1 - z$. Hence (4.9) (c) is proved.

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