THE TAIL σ-FIELD OF A MARKOV CHAIN AND A THEOREM OF OREY[1]

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1. Introduction. Orey (1962) proved that, if p is a transition probability matrix with one ergodic class of recurrent and aperiodic states, then

$$\lim_{n\to\infty} \sum_j |p^n(i_1, j) - p^n(i_2, j)| = 0.$$

We present here a somewhat different proof which may give additional insight. Of course, Orey's result implies the corollary to our Theorem 2, as well as our Theorem 1 and its corollaries.

Let $\{x_n : 0 \leqq n < \infty\}$ be a sequence of random variables on the probability triple $(\Omega, \mathfrak{F}, P)$. Let $\mathfrak{F}^{(n)}$ be the smallest σ-field over which the $x_\nu : \nu \geqq n$ are measurable. The *tail σ-field* $\mathfrak{F}^{(\infty)}$ of $\{x_n : 0 \leqq n < \infty\}$ is $\bigcap_{n=0}^{\infty} \mathfrak{F}^{(n)}$. The main result of this paper is: If $\{x_n : 0 \leqq n < \infty\}$ is a Markov chain with stationary transition probabilities, countable state space I, and all states recurrent, then $\mathfrak{F}^{(\infty)}$ is atomic under P. More precisely, let $\{I_c : c \,\varepsilon\, C\}$ be the partition of I into its cyclically moving subclasses. It is equivalent to the usual definition (Chung (1960) Section I.3) that i and j are in the same I_c if and only if there is an $n \geqq 0$ and a k in I with the n-step transition probabilities from i to k and from j to k both positive. Then each $\mathfrak{F}^{(\infty)}$-set differs from some union of sets $[x_0 \,\varepsilon\, I_c]$ by a set of P-measure 0. In particular, if $\{x_n : 0 \leqq n < \infty\}$ is aperiodic and has only one recurrent class, its tail σ-field is trivial and Orey's result follows. These results are proved in Section 2 which concludes by describing the tail σ-field of a random walk on a countable Abelian group, and the σ-field of exchangeable sets defined on the recurrent Markov chain $\{x_n : n \geqq 0\}$ with countable state space and stationary transitions. A set is *exchangeable* if it depends measurably on the $x_n : n \geqq 0$ and is invariant under finite permutations of them.

Section 3 contains three examples. Example 1 is an aperiodic chain with only one recurrent class, in which two independent particles, starting from different states, may never meet. Example 2 is a transient chain $\{x_n : 0 \leqq n < \infty\}$ with nonatomic tail σ-field but trivial *invariant σ-field*. An event A is *invariant* if there is a Borel set B of I-sequences, with $(i_0, i_1, \cdots)$ in B if and only if $(i_1, i_2, \cdots)$ is in B, and $A = (x_0, x_1, \cdots)^{-1}B$. Example 3 is a stationary, three state Markov chain with one aperiodic, ergodic class, but a non-trivial σ-field of exchangeable events.

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2. The results.

THEOREM 1. *If* $\{x_n : 0 \leqq n < \infty\}$ *is a recurrent Markov chain with countable state space, stationary transition probabilities, and* $[x_0 = i]$ *has probability* 1 *for some state* i, *then each set in the tail* σ*-field of* $\{x_n : 0 \leqq n < \infty\}$ *has probability* 0 *or* 1.

PROOF. Let $\{y_\nu : 0 \leqq \nu < \infty\}$ be the *inter-i blocks* of $\{x_n : 0 \leqq n < \infty\}$: if τ_ν is the time of the νth return to i, and $\tau_0 = 0$, then $y_\nu = (x_{\tau_\nu}, \cdots, x_{\tau_{\nu+1}-1})$. The y_ν are independent and identically distributed (Chung (1960) Section I.3). Each set in the tail σ-field of $\{x_n : 0 \leqq n < \infty\}$ depends measurably on the $y_\nu : \nu \geqq 0$, and is invariant under finite permutations of them. The 0-1 law in (Hewitt and Savage (1955) Theorem 11.3) completes the proof.

COROLLARY 1. *If* $\{x_n : 0 \leqq n < \infty\}$ *is a recurrent Markov chain with countable state space, stationary transition probabilities, and* $[x_0 \,\varepsilon\, I_c]$ *has probability* 1 *for some cyclically moving subclass* I_c, *then each set in the tail* σ*-field of* $\{x_n : 0 \leqq n < \infty\}$ *has probability* 0 *or* 1.

PROOF (by V. Strassen). Let J be the set of all values assumed by x_0 with positive probability. Let $p^{(n)}(i, j)$ be the n-step transition probability from i to j; if i_1, i_2 are in J, there is an $n \geqq 0$ and a k in I with $p^{(n)}(i_1, k) > 0$ and $p^{(n)}(i_2, k) > 0$. Suppose $\{x_n : 0 \leqq n < \infty\}$ is defined on the probability triple $(\Omega, \mathfrak{F}, P)$. If $i \,\varepsilon\, J$ and $A \,\varepsilon\, \mathfrak{F}^{(\infty)}$, then

$$P(A \mid x_0 = i) = p^{(n)}(i, k)P(A \mid x_n = k) + [1 - p^{(n)}(i, k)]P(A \mid x_n \neq k).$$

If $P(A \mid x_0 = i_1) > 0$, then $P(A \mid x_0 = i_1) = 1$ by the theorem; hence $P(A \mid x_n = k) > 0$, so $P(A \mid x_0 = i_2)$ is positive and therefore 1 by the theorem. Thus, either $P(A \mid x_0 = i) = 0$ for all $i \,\varepsilon\, J$ and $P(A) = 0$, or $P(A \mid x_0 = i) = 1$ for all $i \,\varepsilon\, J$ and $P(A) = 1$.

COROLLARY 2. *Let* $\{x_n : 0 \leqq n < \infty\}$ *be a recurrent Markov chain with countable state space* I *and stationary transition probabilities. Let* $\{I_c : c \,\varepsilon\, C\}$ *partition* I *into its cyclically moving subclasses. Then the tail* σ*-field of* $\{x_n : 0 \leqq n < \infty\}$ *is equivalent to the* σ*-field generated by the sets* $[x_0 \,\varepsilon\, I_c]$ *for* $c \,\varepsilon\, C$.

PROOF. If $c \,\varepsilon\, C$, the set $[x_0 \,\varepsilon\, I_c]$ differs from an x_n-measurable set by a set of probability 0, for each n; hence $[x_0 \,\varepsilon\, I_c]$ differs from a set in $\mathfrak{F}^{(\infty)}$ by a set of probability 0. Conversely, let $\{x_n : 0 \leqq n < \infty\}$ be defined on the probability triple $(\Omega, \mathfrak{F}, P)$. If $A \,\varepsilon\, \mathfrak{F}^{(\infty)}$ and $P[x_0 \,\varepsilon\, I_c] > 0$, then $P(A \mid [x_0 \,\varepsilon\, I_c])$ is 0 or 1 by the previous corollary. Thus, A differs from a union of sets $[x_0 \,\varepsilon\, I_c]$ by a set of P-measure 0.

COROLLARY 3. *Let* $\{x_n : 0 \leqq n < \infty\}$ *be a recurrent Markov chain with countable state space* I *and stationary transition probabilities. Let* $\{I_e : e \,\varepsilon\, E\}$ *partition* I *into ergodic classes. Then the invariant* σ*-field of* $\{x_n : 0 \leqq n < \infty\}$ *is equivalent to the* σ*-field generated by the sets* $[x_0 \,\varepsilon\, I_e]$ *for* $e \,\varepsilon\, E$.

PROOF. Immediate from Corollary 2.

THEOREM 2. *Let* $\{x_n : 0 \leqq n < \infty\}$ *be a sequence of random variables on the probability triple* $(\Omega, \mathfrak{F}, P)$. *The tail* σ*-field of* $\{x_n : 0 \leqq n < \infty\}$ *is trivial under* P *if and only if*

(1) $$\lim_{n\to\infty} \sup_{A\varepsilon\mathfrak{F}^{(n)}} |P(A \cap B) - P(A)P(B)| = 0$$

for each B in $\mathfrak{F}$.

PROOF. If (1) holds, apply it with $A = B \varepsilon \mathfrak{F}^{(\infty)}$ to obtain $P(A) = P(A)^2$, so $\mathfrak{F}^{(\infty)}$ is trivial under P. Conversely, suppose $\mathfrak{F}^{(\infty)}$ is trivial and fix B in $\mathfrak{F}$. Let $1_B(\omega) = 1$ or 0 according as $\omega \varepsilon B$ or $\omega \varepsilon \Omega - B$. Then for $A \varepsilon \mathfrak{F}^{(n)}$,

$$|P(A \cap B) - P(A)P(B)| = |\int_A [1_B - P(B)]\, dP|$$
$$= |\int_A [P(B \mid \mathfrak{F}^{(n)}) - P(B)]\, dP| \leqq \int |P(B \mid \mathfrak{F}^{(n)}) - P(B)|\, dP.$$

The backward martingale convergence theorem (Doob (1953) Theorem 4.2, p. 328) completes the proof of (1).

COROLLARY. *Let* $[p^{(n)}(i, j) : i, j \varepsilon I]$ *be the matrix of n-step stationary transition probabilities for a recurrent Markov chain with countable state space I. For any two states* i_1 *and* i_2 *in the same cyclically moving subclass,*

$$\lim_{n\to\infty} \sum_{j\varepsilon I} |p^{(n)}(i_1, j) - p^{(n)}(i_2, j)| = 0.$$

PROOF. Let $\{x_n : 0 \leqq n < \infty\}$ be a Markov chain with matrix of 1-step stationary transition probabilities $[p^{(1)}(i, j); i, j \varepsilon I]$, and $x_0 = i_1$ or i_2 with probability $\frac{1}{2}$ each. The tail σ-field of $\{x_n : 0 \leqq n < \infty\}$ is trivial by Corollary 1 of Theorem 1. After a slight manipulation, Theorem 2 yields:

$$\lim_{n\to\infty} \sup_{S\subset I} |P(x_n \varepsilon S \mid x_0 = i_1) - P(x_n \varepsilon S \mid x_0 = i_2)| = 0.$$

But $\sum_{j\varepsilon I} |Q_1(j) - Q_2(j)| = 2 \sup_{S\subset I} |Q_1(S) - Q_2(S)|$ for any two probabilities Q_1, Q_2 on the subsets of I, completing the proof.

These results apply also to random walks. Let G be a countable, Abelian group and π a probability on G. Let V be the set of i in G with $\pi(i) > 0$. Let $\{G_c : c \varepsilon C\}$ partition G into the cosets of the group spanned by V-V. Let $\{G_e : e \varepsilon E\}$ partition G into the cosets of the group spanned by V. Let $\{z_n : 0 \leqq n < \infty\}$ be independent random variables with values in G; the z_n with $n \geqq 1$ having common distribution π. Let $x_n = \sum_{\nu=0}^{n} z_\nu$. The tail σ-field of $\{x_n : 0 \leqq n < \infty\}$ is equivalent to the σ-field generated by the sets $[x_0 \varepsilon G_c]$ for $c \varepsilon C$. The invariant σ-field of $\{x_n : 0 \leqq n < \infty\}$ is equivalent to the σ-field generated by the sets $[x_0 \varepsilon G_e]$ for $e \varepsilon E$. Proofs are omitted, being virtually identical to those of Theorem 1 and its corollaries.

Call a set *exchangeable* if it depends measurably on $x_n : n \geqq 0$ and is invariant under finite permutations of them. The argument for Theorem 1 proves: if $\{x_n : n \geqq 0\}$ is a recurrent Markov chain with countable state space, stationary transitions, and x_0 degenerate, then any exchangeable set has probability 0 or 1. Call $i \sim j$ if and only if there exists a state k, finite sequences δ and σ of states with $i\delta$ a permutation of $j\sigma$, and all transitions in $i\delta k$ and in $j\sigma k$ possible. Then $i \sim j$ if and only if conditional probabilities of exchangeable sets given $x_0 = i$ coincide with those given $x_0 = j$. Here is a sketch of the argument.

Suppose $i \sim j$, and A is an exchangeable subset of the Borel space of state sequences with $P\{(x_n : n \geqq 0) \varepsilon A \mid x_0 = i\} = 1$ and $P\{(x_n : n \geqq 0) \varepsilon A \mid x_0 = j\} = 0$. If φ is a finite sequence of states, A_φ is the set of infinite sequences ψ with $\varphi\psi \varepsilon A$. If φ^* is a permutation of φ, exchangeability implies $A_\varphi = A_{\varphi^*}$. Consider the δ, σ, k promised by the definition of $i \sim j$, and call N the length of δ. Since $P\{(x_n : n \geqq 0) \varepsilon A \mid x_0 = i\} = 1$, therefore $P\{(x_n : n \geqq 0) \varepsilon A \mid x_0 \cdots x_{N+1} = i\delta k\} = 1$. But $P\{(x_n : n \geqq 0) \varepsilon A \mid x_0 \cdots x_{N+1} = i\delta k\} = P\{(x_n : n \geqq 1) \varepsilon A_{i\delta k} \mid x_0 = k\} = P\{(x_n : n \geqq 1) \varepsilon A_{j\sigma k} \mid x_0 = k\}$. So $P\{(x_n : n \geqq 0) \varepsilon A \mid x_0 = j\} \geqq P\{(x_n : n \geqq 1) \varepsilon A_{j\sigma k} \mid x_0 = k\} P\{x_0 \cdots x_{N+1} = j\sigma k \mid x_0 = j\} > 0$, a contradiction.

Conversely, let $i \nsim j$. Fix any state k, and let A be the set of all infinite state sequences such that k occurs infinitely often, and for all remote k's, the preceding part of the sequence is obtained by permuting a sequence $i\sigma$, where all transitions of $i\sigma k$ are possible. Then A is exchangeable, $P[(x_n : n \geqq 0) \varepsilon A \mid x_0 = i] = 1$, and $P[(x_n : n \geqq 0) \varepsilon A \mid x_0 = j] = 0$, completing the proof.

Thus $\sim$ is an equivalence relation. If $\{x_n : 0 \geqq n < \infty\}$ is a recurrent Markov chain with countable state space and stationary transitions, then the σ-field of exchangeable events is atomic, and the atoms are of the form: x_0 is in a particular $\sim$ equivalence-class. As an application if $\{x_n : n \geqq 0\}$ is a recurrent k-step Markov chain with countable state space, stationary transitions, and no forbidden transitions, its σ-field of exchangeable events is trivial.

3. Three examples. Example 1 is a transition matrix p in which all states are recurrent, aperiodic, and in the same ergodic class; but two independent particles, starting in different states and moving according to p, may never meet. The states are the nonnegative integers, and $p(i, i - 1) = 1$ for $i \geqq 1$, while $p(0, n) = a_n > 0$ to be defined below. Consider two independent Markov processes $\{x_n : 0 \leqq n < \infty\}$ and $\{y_n : 0 \leqq n < \infty\}$ with p for matrix of stationary transition probabilities and initial states respectively 0 and 1. If the two processes ever meet, they stay together until reaching 0. Let S_ν be the time of the νth return of $\{x_n\}$ to 0; while T_ν is the time of the νth visit of $\{y_n\}$ to 0. Then $S_k = \sum_{\nu=1}^{k} U_\nu$ and $T_k = 1 + \sum_{\nu=1}^{k-1} V_\nu$, where $\{U_\nu : 1 \leqq \nu < \infty; V_\nu : 1 \leqq \nu < \infty\}$ are independent, with common distribution $\{a_n\}$. The $\{a_n\}$ will be chosen to make the event $[S_k = T_m$ for some $k \geqq 1$ and $m \geqq 1]$ have probability less than 1. Consider the random walk on the planar lattice where a point moves to each of its four neighbors with probability 1/5 and stays fixed with probability 1/5. Let a_n be the probability of a first return to the origin at time n. If two independent such walks start simultaneously from the origin, the probability that for some n the first returns to the origin at time n, while the second returns at time $n - 1$, is precisely the probability of $[S_k = T_m$ for some $k \geqq 1$ and $n \geqq 1]$. This probability cannot be 1; for if it were 1, the two walks would return to the origin at the same time with probability 1, which is known to be false (Chung and Fuchs (1951) Theorem 6).

Example 2 is a transient Markov chain $\{x_n : 0 \leqq n < \infty\}$ with countable state space, matrix p of stationary transition probabilities, trivial invariant σ-field, and nonatomic tail σ-field. The states are pairs (m, n) of natural numbers with $n \leqq 2 \cdot 3^m$ and x_0 is $(1, 1)$ with probability 1. The matrix p is defined by: $p[(m, 1), (m, 3^m)] = p[(m, 1), (m, 2 \cdot 3^m)] = \frac{1}{2}$; while $p[(m, n), (m, n-1)] = 1$ for $n \geqq 3$; and $p[(m, 2), (m + 1, 1)] = 1$. For $m \geqq 1$, let U_m be 1 or 2 according as $(m, 1)$ is followed by $(m, 3^m)$ or $(m, 2 \cdot 3^m)$ in $\{x_n : 0 \leqq n < \infty\}$. The time T_m at which $(m + 1, 1)$ is reached equals $\sum_{\nu=1}^{m} U_\nu 3^\nu$, and T_m determines $U_1 \cdots U_m$. For any n, the variables $x_n, x_{n+1}, \cdots$ determine all T_m with large enough m, and hence $x_1, \cdots, x_{n-1}$. The tail σ-field of $\{x_n : 0 \leqq n < \infty\}$ is therefore the σ-field determined by $\{U_n : 1 \leqq n < \infty\}$, which is nonatomic.

Let A be an invariant event on $\{x_n : 0 \leqq n < \infty\}$. There is a Borel subset B of the space of state sequences $\{i_0, i_1, \cdots\}$ with $1_B(i_0, i_1, \cdots) = 1_B(i_1, i_2, \cdots)$, and $1_A = 1_B(x_0, x_1, \cdots)$. Hence $1_A = 1_B(x_{T_m}, x_{T_m+1}, \cdots)$ is measurable on $U_{m+1}, U_{m+2}, \cdots$, and A has probability 0 or 1 by the Kolmogorov 0-1 law.

Example 3 is a stationary Markov chain with states 1, 2, 3, aperiodic and ergodic, having a non-trivial σ-field of exchangeable events. Its matrix of transition probabilities is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Let δ and σ be finite sequences of states with forbidden transitions in neither $1\delta3$ nor $3\sigma3$. Plainly, in 1δ there is one more 2 than 3. In 3σ, however, there are as many 2's as 3's. Thus 1δ cannot be permuted into 3σ, and in view of the last part of Section 2, the σ-field of exchangeable events is not trivial.

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