PRESENT VALUE OF A RENEWAL PROCESS[1]

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0. Summary. This paper studies the present cost $C(\rho)$ of a renewal process, defined as the sum of the values of the costs of the replacements, considered at the starting time of the renewal process, with a compound interest.

The characteristic function of $C(\rho)$ is found when the inter-arrival times X_j are negatively exponentially distributed. The asymptotic properties of $C(\rho)$ as the force of interest ρ tends to zero are studied in the general case, obtaining that if X_1 has finite moments of all orders, $C(\rho)$ is asymptotically normal; moreover, if we assume $EX_1^2 < \infty$, then the existence of all the moments of X_1 is necessary in order that the moments of $C(\rho)$ converge to the moments of the normal distribution.

1. Introduction. Let us consider a renewal process with inter-arrival times $X_1, X_2 \cdots$ identically and independently distributed with distribution function $F(x)$; we assume $F(0-) = 0$, and $F(0) < 1$, in order to avoid the trivial case $P(X = 0) = 1$. Starting at time $T_0 = 0$ we will have a replacement at each of the instants $T_1 = X_1$, $T_2 = X_1 + X_2$, $\cdots$.

We suppose that each renewal has a constant cost, which we assume equal to 1. So one will have to pay one (dollar, say) at time T_1, one at time T_2, and so on. It is of interest to study the present value, at time 0, of these payments, assuming compound interest.

The present value A_0, at time 0, of a sum A at time T, is given by:

$$A_0 = e^{-\rho T} A \tag{1}$$

where $\rho > 0$ is the rate of interest. So the value C_i, at time 0, of a replacement which will take place at time T_i, is given by $C_i = \exp(-\rho T_i)$. The present value, at time 0, of the total cost of the renewal process, which will be denoted by $C(\rho)$, is

$$C(\rho) = \sum_{i=1}^{\infty} C_i = \sum_{i=1}^{\infty} \exp\left(-\rho \sum_{j=1}^{i} X_j\right). \tag{2}$$

If we put $N(t)$ equal to the number of $T_i \leqq t$, and set

$$C(t, \rho) = \sum_{i=1}^{N(t)} \exp\left(-\rho \sum_{j=1}^{i} X_j\right),$$

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then

$$\lim_{\rho \to 0} C(t, \rho) = N(t) \quad \text{and} \quad \lim_{t \to +\infty} C(t, \rho) = C(\rho).$$

This gives a link between the variables $C(\rho)$ and $N(t)$. Some aspects of the random variable $C(t, \rho)$ have been studied (see [1] and [2]), mainly with respect to its first moments and (in the latter paper) their asymptotic properties.

Let us consider the random variables Y_i defined by $Y_i = \exp(-\rho X_i)$. They are independent and identically distributed; we will denote by α_r their moments. If X is a random variable with distribution function $F(x)$, and ϕ_X is the characteristic function of X, we have

$$\alpha_r = EY_j^r = Ee^{-\rho r X} = \phi_X(ir\rho); \tag{3}$$

where it is always $0 < \alpha_r < 1$.

If we express $C(\rho)$ by means of the Y_j's, the moments of $C(\rho)$ can be obtained in terms of the α_r's by simple but cumbersome computations. On the other hand, from the definition of $C(\rho)$, we have

$$C(\rho) = e^{-\rho X_1}(1 + C'(\rho)) \tag{4}$$

where $C'(\rho)$ has the same distribution as $C(\rho)$, and is independent of $X_1 \cdot$ A recurrence relation based on (4) enables one to obtain easily the moments of $C(\rho)$. We will denote by γ_r these moments. We obtain, in particular

$$\gamma_1 = EC(\rho) = \frac{\alpha_1}{1 - \alpha_1}, \qquad \sigma^2[C(\rho)] = \frac{\alpha_2 - \alpha_1^2}{(1 - \alpha_1)^2(1 - \alpha_2)}. \tag{5}$$

The same recurrence method shows that all the moments of $C(\rho)$ are finite. This result could be obtained also by an argument similar to the one used to prove that $N(t)$ has finite moments of all orders (see [5]; p. 245).

2. The case of a negative exponential distribution. A particular case of great importance in applications is given by the negative exponential distribution, which will be studied in this section. In this case, we have $F(x) = 1 - e^{-\lambda x}$ for $x > 0$ and $\lambda > 0$. Also $\phi_X(t) = (1 - it/\lambda)^{-1}$ and $\alpha_r = (1 + r\rho/\lambda)^{-1}$. Hence putting $\mu = \lambda/\rho$, we have $\gamma_1 = \mu$ and $\sigma^2 = \sigma^2[C(\rho)] = \mu/2$.

Let us study now the distribution of $C(\rho)$. From (4) it follows that the characteristic function $\phi_C(t)$ of $C(\rho)$ satisfies the integral equation:

$$\phi_C(t) = \int_0^{+\infty} \exp(ite^{-\rho x})\phi_C(te^{-\rho x})\, dF(x), \tag{6}$$

which, in the case of negative exponential distribution, becomes, by a change of variables:

$$\phi_C(t) = \mu \int_0^t e^{iy}\phi_C(y)y^{\mu-1}t^{-\mu}\, dy.$$

By multiplying both sides by t^μ and differentiating, we have the differential equation:

$$\phi_C'(t) = \mu[(e^{it} - 1)/t]\phi_C(t),$$

with the condition $\phi_C(0) = 1$. Therefore

$$\phi_C(t) = \exp\left[\mu \int_0^t \frac{e^{ix} - 1}{x}\, dx\right]. \tag{7}$$

An interesting question concerning the random variable $C(\rho)$ is its asymptotic behaviour as ρ tends to zero. For the negative exponential distribution the study of this problem is made very easy by formula (7). Let us consider the normalized variable $Z_\rho = [C(\rho) - \gamma_1]/\sigma$. Using (7), the characteristic function $\phi_\rho(t)$ of Z_ρ is given by:

$$\phi_\rho(t) = \exp\left[-\frac{\mu}{\sigma} it + \mu \int_0^{t/\sigma} \frac{e^{ix} - 1}{x}\, dx\right].$$

Hence by expanding and taking limits, $\phi_\rho(t) \to \exp(-t^2/2)$ as $\rho \to 0$. Thus we have proved:

THEOREM 1. *If X has a negative exponential distribution, $C(\rho)$ is asymptotically normally distributed as ρ tends to zero.*

3. The general case. It can be easily seen that the negative exponential distribution is the only one which permits a simple solution of the integral equation (6) by a transformation into a differential equation. In the general case, since the integral equation (6) is a homogeneous Volterra equation of the second kind, the solution must be found among the singular solutions. Thus it appears rather difficult to find by this method the distribution of $C(\rho)$ for distributions of X other than the negative exponential one.

On the other hand, if we want to study the asymptotic properties of the distribution of $C(\rho)$, other means are available; for instance, the investigation of the behaviour of the moments of $C(\rho)$.

Let us first establish a lemma.

LEMMA 1. *If $\beta_k = EX^k < \infty$ $(k \geqq 1)$, then, for every $t \geqq 0$,*

$$\lim_{\rho\to 0} \rho^{-k} E e^{-\rho t X}(e^{-\rho X} - \alpha_1)^h = E(\beta_1 - X)^k \qquad \textit{if } h = k$$
$$= 0 \qquad \textit{if } h > k.$$

PROOF. We can write, by a standard expansion:

$$e^{-\rho x} = 1 - \rho x \theta_1(x, \rho), \qquad (e^{-\rho x} \leqq \theta_1(x, \rho) \leqq 1)$$

and

$$\alpha_1 = 1 - \rho\beta_1\theta(\rho), \qquad (0 < \theta(\rho) < 1;\ \lim_{\rho\to 0}\theta(\rho) = 1)$$

then:

$$E\rho^{-k} e^{-\rho t X}(e^{-\rho X} - \alpha_1)^h = E e^{-\rho t X}(e^{-\rho X} - \alpha_1)^{h-k}[\beta_1\theta(\rho) - X\theta_1(X, \rho)]^k.$$

Since $E|\beta_1\theta(\rho) - X\theta_1(X, \rho)|^k \leqq (\beta_1 + X)^k$, the expectation and limit can be interchanged, and the lemma is proved.

Let us consider now the central moments $\bar{\gamma}_r$ of $C(\rho)$. By means of (4) and (5), since $C'(\rho)$ and X_1 are independent, we obtain:

$$\begin{aligned}\bar{\gamma}_r &= E[C(\rho) - \gamma_1]^r = E[e^{-\rho X_1}(C'(\rho) + 1) - \gamma_1]^r \\ &= E[e^{-\rho X_1}(C'(\rho) - \gamma_1) + (e^{-\rho X_1} - \alpha_1)(1 - \alpha_1)^{-1}]^r \\ &= \sum_{j=0}^{r} \binom{r}{j} \bar{\gamma}_j (1 - \alpha_1)^{-r+j} E[e^{-j\rho X}(e^{-\rho X} - \alpha_1)^{r-j}]\end{aligned}$$

from which it follows that

$$\bar{\gamma}_r = (1 - \alpha_r)^{-1} \sum_{j=0}^{r-1} \binom{r}{j} \bar{\gamma}_j (1 - \alpha_1)^{-r+j} E[e^{-j\rho X}(e^{-\rho X} - \alpha_1)^{r-j}]. \tag{8}$$

We can now prove:

LEMMA 2. *If, for* $r > 0$, $EX^{\frac{1}{2}r+1} < \infty$, *then*

$$\lim_{\rho\to 0} \rho^{\frac{1}{2}r} \bar{\gamma}_r = K_r \tag{9}$$

where

$$\begin{aligned} K_r &= (\beta_2 - \beta_1^2)^{\frac{1}{2}r} r!/2^r(\tfrac{1}{2}r)!\beta_1^{3r/2} && \textit{if } r \textit{ is even} \\ &= 0 && \textit{if } r \textit{ is odd.}\end{aligned}$$

PROOF. We will prove this lemma by induction. Clearly (9) holds for $r = 0, 1$, since $\gamma_0 = 1$, $\gamma_1 = 0$. Now let us assume that it holds for $0, 1, \cdots r - 1$, with $r \geqq 2$.

By means of the expansion used in Lemma 1, we obtain: $\lim_{\rho\to 0} \rho(1 - \alpha_r)^{-1} = (r\beta_1)^{-1}$. Hence, using (8), we have by the induction assumption that

$$\lim_{\rho\to 0} \rho^{\frac{1}{2}r} \bar{\gamma}_r = \sum_{j=0}^{r-1} \binom{r}{j} \frac{1}{r} \beta_1^{-r+j-1} K_j \lim_{\rho\to 0} \rho^{-(\frac{1}{2}r - \frac{1}{2}j + 1)} E[e^{-j\rho X}(e^{-\rho X} - \alpha_1)^{r-j}].$$

Now $EX^{\frac{1}{2}r - \frac{1}{2}j + 1} < \infty$, since $j \geqq 0$; so for $\frac{1}{2}r - \frac{1}{2}j + 1 \leqq r - j$, i.e. $j \leqq r - 2$, Lemma 1 holds. Moreover, for $j = r - 1$,

$$E[e^{-j\rho X}(e^{-\rho X} - \alpha_1)^{r-j}] = \alpha_r - \alpha_{r-1}\alpha_1 = O(\rho^2),$$

as can be easily seen by an expansion as in Lemma 1. Then all the terms of the sum, except possibly for $j = r - 2$, vanish in the limit, and for $j = r - 2$ we obtain K_r. Thus the lemma is proved.

Lemma 2 enables us to establish a sufficient condition for the asymptotic normality of $C(\rho)$.

THEOREM 2. *If all the moments of* X *are finite, then the distribution of* $Z_\rho = (C(\rho) - \gamma_1)/\bar{\gamma}_2^{-\frac{1}{2}}$ *tends to the standard normal distribution as* ρ *decreases to zero.*

PROOF. Lemma 2 gives, for $r = 2$:

$$\lim_{\rho\to 0} \rho\bar{\gamma}_2 = (\beta_2 - \beta_1^2)/2\beta_1^3 .$$

Thus, applying Lemma 2, we obtain that the moments of Z_ρ converge to the moments of the normal distribution, and this is sufficient to ensure that the distribution of the variate converge to the normal distribution (see, for instance, [4] p. 110). The theorem is thus proved.

As it has been observed, the random variable $C(t, \rho)$ gives a link between $C(\rho)$ and $N(t)$. It is known ([3] and [6]) that, if $EX^2 < \infty$, $N(t)$ is asymptotically normal; i.e. $C(t, \rho)$ is asymptotically normal if we let ρ tend to zero first, and then t tend to ∞. Theorem 2 assures (although under more severe restrictions) that the result is true if the limits are interchanged.

It would be interesting to have some necessary condition in order that the asymptotic distribution be normal; unfortunately a condition of this kind appears rather difficult to establish.

However, the proof given above requires as an essential condition the existence of all moments of X; so that one should expect that, if this condition is dropped, the moments of Z_ρ do not converge to the moments of the normal variate. That this is actually true will be seen later.

LEMMA 3. *If* $EX < \infty$ *and, for* t *and* k *with* $0 < t \leqq k$,

$$\lim_{\rho \to 0} \rho^{-t} E(e^{-\rho X} - \alpha_1)^k = 0$$

then, for every $q < t$: $EX^q < \infty$.

PROOF. We will show first that, under the hypotheses above, we have:

$$\lim_{\rho \to 0} \rho^{-t} E|e^{-\rho X} - \alpha_1|^k = A \tag{10}$$

where $0 \leqq A < \infty$. In fact, writing $b_\rho = -\rho^{-1} \log \alpha_1$,

$$\int_0^{b_\rho} (e^{-\rho X} - \alpha_1)^k \rho^{-t} \, dF(x) + \int_{b_\rho}^{+\infty} (e^{-\rho X} - \alpha_1)^k \rho^{-t} \, dF(x) = E|e^{-\rho X} - \alpha_1|^k \rho^{-t} < \infty.$$

But, as ρ decreases to zero, $\rho^{-1}(1 - \alpha_1)$ tends to β_1; thus also b_ρ tends to β_1. Also $(e^{-\rho x} - \alpha_1)\rho^{-1}$ converges to $\beta_1 - x$. Hence, by dominated convergence theorem and the hypothesis of the Lemma, (10) is proved.

Now, since $\alpha_1 \to 1$, given K with $0 < K < 1$, by making ρ small enough, we can make $\alpha_1 > K$. Then we have:

$$E|e^{-\rho X} - \alpha_1|^k \geqq \int_{-\rho^{-1}\log K}^{+\infty} (\alpha_1 - e^{-\rho X})^k \, dF(x) > (\alpha_1 - K)^k [1 - F(-\rho^{-1} \log K)].$$

Hence, putting $x = -\rho^{-1} \log K$, it follows from (10) that:

$$\limsup_{x \to +\infty} x^t [1 - F(x)] \leqq (-\log K)^t (1 - K)^{-k} A$$

and this establishes the lemma.

THEOREM 4. *If* $EX^2 < \infty$, *then in order that all the moments of* Z_ρ *converge to the moments of the normal distribution* $N(0, 1)$, *it is necessary that all the moments of* X *be finite.*

PROOF. The proof will be by induction. Since $EX^2 < \infty$, the convergence of EZ_ρ^k to the kth central moment of the normal variate $N(0, 1)$ is equivalent to

$$\lim_{\rho \to 0} \rho^{\frac{1}{2}k} \bar{\gamma}_k = K_k . \tag{11}$$

We will assume that $EX^{\frac{1}{2}r} < \infty$ for an integer $r \geqq 4$, and that (11) holds for every integer k, and we will prove that $EX^{\frac{1}{2}(r+1)} < \infty$. The theorem will thus be proved.

Let us go back to the proof of Lemma 2. For $j \geqq 2$, we have:

$$\tfrac{1}{2}r + 1 - \tfrac{1}{2}j \leqq \tfrac{1}{2}r \quad \text{and} \quad EX^{\frac{1}{2}r+1-\frac{1}{2}j} < \infty,$$

and for $j = 1$, $\bar{\gamma}_1 = 0$; so we obtain:

$$\lim_{\rho\to 0} \rho^{\frac{1}{2}r}\bar{\gamma}_r = K_r + (1/r)\beta_1^{-r-1} \lim_{\rho\to 0} \rho^{-(\frac{1}{2}r+1)}E(e^{-\rho X} - \alpha_1)^r.$$

Then the last limit must be zero, and, by Lemma 3, this implies that $EX^q < \infty$ for every $q < \frac{1}{2}r + 1$, in particular for $q = \frac{1}{2}(r+1)$. The theorem is thus proved.

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