ON ELEMENTARY SYMMETRIC FUNCTIONS OF THE ROOTS OF A MULTIVARIATE MATRIX: DISTRIBUTIONS[1]

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1. Introduction and summary. Elementary symmetric functions (e.s.f.'s) of the roots of certain determinantal equations are often associated with test statistics in multivariate analysis. The T^2 of Hotelling (1951) is related to the first e.s.f. of the roots (this is also known as "sum of the roots" or "trace" of its associated matrix). The λ-criterion of Wilks (1932) appropriate to k samples is related to the last e.s.f. of the roots (this is also known as "product of the roots"). Except for these two e.s.f.'s of the roots, very little is known in current literature of the other e.s.f.'s.

Non-symmetric functions, like the largest or smallest root of Roy (1957), also arise in certain situations where their use are preferable to other known statistics. This paper is not concerned with symmetric or non-symmetric functions as test statistics. It is concerned with a unified treatment of the distribution problem of the e.s.f. of the roots. It begins with the derivation of the joint distribution of the e.s.f. of the roots of a multivariate matrix under null hypothesis (Section 2). The next three sections concern with the moment problem of the distribution. Section 3 cites previous material necessary for obtaining determinantal expressions of the moments. Section 4 deals with the proof by construction of obtaining determinantal expressions for moments and product-moments which is applied (Section 6) to the third moment of the second e.s.f. and the product-moments of the first, second and third e.s.f. for illustration purposes. Section 5 gives an evaluating formula for the determinantal expressions.

2. The joint distribution of the e.s.f.'s. Certain determinantal equations of the form

$$|\mathbf{T}_1 - \theta(\mathbf{T}_1 + \mathbf{T}_2)| = 0$$

arises often in multivariate analysis. $\mathbf{T}_1$ and $\mathbf{T}_2$ are independent Wishart matrices with certain degrees of freedom and θ is a root of this determinantal equation. Under null hypothesis, the distribution of the nonzero roots has been obtained independently and almost simultaneously, by Fisher (1939), Girshick (1939) for the case of two roots, Hsu (1939), Mood (1951) and Roy (1939). The standard form of this distribution is

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$$c(s,m,n)\prod_{i=1}^{s}\theta_i^m(1-\theta_i)^n\prod_{i>j}^{s}(\theta_i-\theta_j)\prod_{i=1}^{s}d\theta_i\,,\ 0<\theta_1\leqq\theta_2\leqq\cdots\leqq\theta_s<1, \tag{2.1}$$

where

$$c(s,m,n)=\pi^{s/2}\prod_{i=1}^{s}\frac{\Gamma[\frac{1}{2}(2m+2n+s+i+2)]}{\Gamma[\frac{1}{2}(2m+i+2)]\Gamma[\frac{1}{2}(2n+i+2)]\Gamma(i/2)} \tag{2.2}$$

and m, n and s are functions of sample sizes and the number of variates depending on the null hypothesis (e.g., see Roy, 1957, p. 52). Now define the e.s.f.'s of the roots by

$$\begin{aligned} V_1^{(s)} &= \theta_1+\theta_2+\cdots+\theta_s\,,\\ V_2^{(s)} &= \theta_1\theta_2+\theta_1\theta_3+\cdots+\theta_{s-1}\theta_s\,,\\ &\cdots\\ V_s^{(s)} &= \theta_1\theta_2\cdots\theta_s\,. \end{aligned} \tag{2.3}$$

The $V_j^{(s)}$ notation is the same as that given by an earlier paper (Mijares, 1961, p. 1154). For the purpose of this section, however, if θ_k for $k\leqq s$ is missing, the jth e.s.f. of the remaining $s-1$ roots will be denoted by $V_j(\theta_1, \cdots, \theta_{k-1}, \theta_{k+1}, \cdots, \theta_s)$.

The Jacobian of (2.3) with respect to the θ's is

$$\begin{vmatrix} 1 & 1 & \cdots & 1\\ V_1(\theta_2,\cdots,\theta_s) & V_1(\theta_1,\theta_3,\cdots,\theta_s) & \cdots & V_1(\theta_1,\cdots,\theta_{s-1})\\ & \cdots & & \\ V_{s-1}(\theta_2,\cdots,\theta_s) & V_{s-1}(\theta_1,\theta_3,\cdots,\theta_s) & \cdots & V_{s-1}(\theta_1,\cdots,\theta_{s-1}) \end{vmatrix}. \tag{2.4}$$

It is not difficult to check from (2.4) that the Jacobian $|J|$ with respect to the $V_j^{(s)}$ is

$$|J| = 1/\prod_{i>j}(\theta_i-\theta_j). \tag{2.5}$$

Hence, on substituting (2.3) in (2.1) and making use of (2.5), the joint distribution of the $V_j^{(s)}$ is obtained. Thus

$$\begin{aligned} g(V_1^{(s)},\cdots,V_s^{(s)})\prod_i dV_i^{(s)} &= c(s,m,n)[V_s^{(s)}]^m\\ &\quad\cdot\left[1-\sum_{j=1}^{s}(-1)^{j-1}V_j^{(s)}\right]^n\prod_i dV_i^{(s)}, \end{aligned} \tag{2.6}$$

where $c(s, m, n)$ is given by (2.2). Obtaining the marginal distributions of the $V_j^{(s)}$, $j<s$, from (2.6) appears, however, to be difficult. It is somewhat easier to study the moments of $V_j^{(s)}$ using the distribution (2.1) instead.

3. Notations and previous results needed in this paper. The reader is assumed to be familiar with the results in Mijares (1961). The same notations are followed here generally except for slight modifications introduced for convenience, some of which have already been given in the previous section.

The arbitrary arguments $x_1, \cdots, x_k$ in the elements of the $(k+1)$ ordered triangular matrices (a) and (ϕ) of the former paper will now be replaced by $\theta_1, \cdots, \theta_k$. Denote thus

$$(3.1)\quad (V) = \begin{pmatrix} V_0 & 0 & \cdots & 0 \\ -V_1 & V_0 & \cdots & 0 \\ & \cdots & & \\ (-1)^k V_k & (-1)^{k-1} V_{k-1} & \cdots & V_0 \end{pmatrix}, (\Phi) = \begin{pmatrix} \Phi_0 & 0 & \cdots & 0 \\ \Phi_1 & \Phi_0 & \cdots & 0 \\ & \cdots & & \\ \Phi_k & \Phi_{k-1} & \cdots & \Phi_0 \end{pmatrix},$$

where V_j and Φ_j are understood to be the jth e.s.f. and *complete homogeneous symmetric functions* of degree j, respectively, with complete arguments in $\theta_1, \cdots, \theta_k$. As before, determinants of lower order than $k+1$ formed from (3.1) are to be referred as V-determinants and Φ-determinants, respectively.

Since there is no occasion here to consider the moment generating function of the e.s.f.'s in $\theta_1, \cdots, \theta_s$ the t in $U(q_s, \cdots, q_1; t)$ of the former paper is dropped. The modified notation to be used in this paper now means

$$(3.2)\quad U(q_s, \cdots, q_1) = c(s, m, n) \int \cdots \int \begin{vmatrix} \theta_s^{q_s} \theta_s^{q_s-1} & \cdots & \theta_s^{q_1} \\ & \cdots & \\ \theta_1^{q_s} \theta_1^{q_s-1} & \cdots & \theta_1^{q_1} \end{vmatrix} \cdot \prod_{i=1}^{s} \theta_i^m (1-\theta_i)^n \, d\theta_i$$

where $c(s, m, n)$ is given by (2.2). Note that the determinantal expression of (3.2), by (2.2) of the previous paper, may be reduced into the product of $\prod_{i>j}(\theta_i - \theta_j)$ and an s-ordered Φ-determinant whose elements in the last row are subscripted $q_s, \cdots, q_1$. Denote this s-ordered Φ-determinant by $U'(q_s, \cdots, q_1)$. For instance,

$$\begin{vmatrix} \theta_3^4 & \theta_3^3 & \theta_3^2 \\ \theta_2^4 & \theta_2^3 & \theta_2^2 \\ \theta_1^4 & \theta_1^3 & \theta_1^2 \end{vmatrix} = \prod_{i>j}^{3} (\theta_i - \theta_j) \begin{vmatrix} \Phi_2 & \Phi_1 & \Phi_0 \\ \Phi_3 & \Phi_2 & \Phi_1 \\ \Phi_4 & \Phi_3 & \Phi_2 \end{vmatrix}.$$

The Φ-determinant then is $U'(4, 3, 2)$. Note also that (3.2) above may be expressed as

$$(3.3)\qquad U(q_s, \cdots, q_1) = E[U'(q_s, \cdots, q_1)],$$

where E denotes mathematical expectation.

As a further remark, in connection with formulae (2.2) and (5.8) of the previous paper, the expression

$$|x_{k-i+1}^{d_j-k+j}| / |x_{k-i+1}^{k-j}| = |\phi_{d_j-j+i}| = |a_{\hat{d}_j-j+i}|$$
$$= \{d_1 \cdots d_k\},$$

which in group representation theory is the *character* denoted by $\{d_1 \cdots d_k\}$ of the representation of linear group of signature $d = \langle d_1, \cdots d_k \rangle$, $d_1 \geqq d_2 \geqq \cdots \geqq d_k \geqq 0$. ϕ_i and d_i are respectively the complete homogeneous symmetric function of degree i and the order of the e.s.f., both of arguments $x_1, \cdots, x_k$. [See, e.g., Weyl (1946), formulae (5.15) and (6.5) of Chapter VII and footnote 12, p. 311 of bibliography of the same chapter.] $\hat{d}$ denotes the *conjugate* partition of d; its use will be illustrated later in Section 6. I am indebted to one of the referees for pointing out the relation to group characters.

4. A diagonalization theorem. Consider the first $r \leqq k + 1$ columns of the matrix (V) of (3.1). Suppose we form an r-ordered V-determinant

$$|(-1)^{d_i+i-j} V_{d_i+i-j}|, \ (i, j = 1, \cdots, r) \tag{4.1}$$

from the first r columns of (V), where V_{d_i}, is the ith diagonal element of the determinant and $V_p = 0$ for $p < 0$ from the definition of the e.s.f. The determinant (4.1) can be conveniently identified given the d_i's as that obtained from rows $d_i + i$, $i = 1, \cdots, r$ of (V). The V-determinant can be denoted then by the subscripts of the principal diagonal, viz.,

$$[d_1 \, d_2 \cdots d_r] = |(-1)^{d_i+i-j} V_{d_i+i-j}|. \tag{4.2}$$

The V-determinant (4.2) has an expansion given by

$$\sum \pm V_{d_1+1-j_1} V_{d_2+2-j_2} \cdots V_{d_r+r-j_r}, \tag{4.3}$$

where the summation extends over all permutations $(j_1 \, j_2 \cdots j_r)$ of the integers $(1 \, 2 \cdots r)$ and the sign in each term of the expansion being either positive or negative according as the set $(j_1 \, j_2 \cdots j_r)$ is an even or odd permutation.

If $\sum_{i=1}^{r} d_i = d$, then one may observe that the polynomial expansion given by (4.3) is homogeneous of degree d. It may also be noted that the expansion need not contain $r!$ terms. Denote now the terms of expansion (4.3) also by their subscripts and distinguish them from V-determinants by using the parenthesis $(\cdots)$ instead of the bracket $[\cdots]$, i.e., $[d_1 \, d_2 \cdots d_r]$ is a determinant obtained from the first r columns of (V) with diagonal elements, apart from their signs, consisting of $V_{d_1}, V_{d_2}, \cdots, V_{d_r}$ in a non-decreasing order of subscripts whereas $(d_1 \, d_2 \cdots d_r)$ is a monomial which is a product of factors $V_{d_1}, V_{d_2}, \cdots, V_{d_r}$ in a non-decreasing order of subscripts.

The following diagonalization theorem may now be stated.

THEOREM. *The monomial $(d_1 \cdots d_r)$ can be expressed as a linear combination with integral coefficients of determinants of the type $[d_1' \, d_2' \cdots d_r']$ where the d_i', $i = 1, \cdots, r$, form an r-part partition of d.*

PROOF. This theorem may be proved by showing how, by construction, each V-determinant with diagonal elements having subscripts which form an r-part partition of d may be expanded in a systematic way.

Write the expansion of $[d_1 \, d_2 \cdots d_r]$ in terms of a polynomial in $(\cdots)$. Further, rearrange the terms of the expansion in a lexical manner such that the term

$(d_1'd_2' \cdots d_r')$ precedes the term $(d_1'' d_2'' \cdots d_r'')$ if, for some m such that $1 \leqq m \leqq r$, $d_1' = d_1''$, $\cdots$, $d_{m-1}' = d_{m-1}''$ and $d_m' > d_m''$. Call this the *lexical order*. Then

$$(4.4) \quad [d_1 d_2 \cdots d_r] = (d_1 d_2 \cdots d_r) + \alpha(d_1 \delta_2 \cdots \delta_r) + \beta'(d_1 \delta_2' \cdots \delta_r') + \cdots$$

where α, β', $\cdots$ have values either $+1$ or -1 according as the terms $(d_1 \delta_2 \cdots \delta_r)$, $(d_1 \delta_2' \cdots \delta_r')$, etc. are even or odd permutations of integers $(1\,2 \cdots r)$. Call (4.4) the *lexical expansion* of $[d_1 d_2 \cdots d_r]$. The first term of the lexical expansion resulting from the natural ordered set $(1\,2 \cdots r)$ consists of elements from the principal diagonal of the V-determinant $[d_1 d_2 \cdots d_r]$. The other permutations either give a first d-subscript equal to or less than d_1. If the first d-subscript is less than d_1, then the term in the expansion corresponding to this permutation comes after $(d_1 d_2 \cdots d_r)$ by the lexical expansion. If it is equal to d_1, then the second subscript cannot be greater than d_2 for j_2 in this permutation cannot be equal to integer 1 or 2. Hence, the second subscript δ_2 of the second term in the expansion is either equal to or less than d_2. The argument for the case of the first d-subscript may be repeated for the second subscript and so on. In this way the lexical expansion shows that no term precedes the diagonal term $(d_1 d_2 \cdots d_r)$ in the expansion of $[d_1 d_2 \cdots d_r]$.

Now perform the following:

(a) Take determinants having diagonal elements whose subscripts form an r-part partition of $\sum d_i = d$ and arrange them down the column in the lexical order.

(b) Take the lexical expansion of each determinant $[\cdots]$.

The procedures (a) and (b) may be written in the form of an array as shown in Table 1.

The numbers α, β_1, β_2, γ_1, γ_2, δ $\cdots$ may or may not be zeros depending on whether the corresponding terms are found in the expansion. The sums of all terms in the lexical expansion, except the first term $(d_1 d_2 \cdots d_r)$, can be made equal to 0's by properly multiplying the successive determinants $[\cdots]$ by certain constants; that is,

TABLE 1

Terms of expansion in lexical order

Determinants (lexical order)	$(d_1 d_2 \cdots d_r)$	$(d_1 d_2 \cdots d_{r-1} - 1\, d_r + 1)$	$\cdots$	$(d_1 - 1 d_2 - 1 \cdots d_{r-1} - 1\, d_r + r - 1)$	$\cdots$	$(0 0 \cdots d)$
$[d_1\, d_2 \cdots d_r]$	1	α	$\cdots$	β_1	$\cdots$	γ_1
$[d_1\, d_2 \cdots d_{r-1} - 1\, d_r + 1]$		1	$\cdots$	β_2	$\cdots$	γ_2
$\cdot$			$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$			$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$			$\cdot$	$\cdot$	$\cdot$	$\cdot$
$[d_1 - 1\, d_2 - 1 \cdots d_{r-1} - 1\, d_r + r - 1]$	(Zeros below principal diagonal)			1	$\cdots$	δ
$\cdot$					$\cdot$	
$\cdot$					$\cdot$	
$\cdot$					$\cdot$	
$[0\,0 \cdots 0\, d]$						1

Coefficient of $(d_1\, d_2 \cdots d_{r-1} - 1\; d_r+1)$:
$[d_1\, d_2 \cdots d_r] - \alpha[d_1\, d_2 \cdots d_{r-1}-1\; d_r+1] = 0$, for some α;

$$\cdots$$

Coefficient of $(d_1-1\; d_2-1 \cdots d_{r-1}-1\; d_r+r-1)$:

$$[d_1\, d_2 \cdots d_r] - \alpha\beta_2[d_1\, d_2 \cdots d_{r-1}-1\; d_r+1] + \cdots + \beta[d_1-1\; d_2-1 \cdots d_{r-1}-1\; d_r+r-1] = 0, \quad \text{for some} \quad \beta;$$

$$\cdots$$

Coefficient of $(0\,0 \cdots d)$:

$$[d_1\, d_2 \cdots d_r] - \alpha\gamma_2[d_1\, d_2 \cdots d_{r-1}-1\; d_r+1] + \cdots + \beta\delta[d_1-1\; d_2-1 \cdots d_{r-1}-1\; d_r+r-1] + \cdots + \delta'[0\;0 \cdots d] = 0, \text{ for some } \delta'.$$

Hence

$$(d_1\, d_2 \cdots d_r) = [d_1\, d_2 \cdots d_r] - \alpha[d_1\, d_2 \cdots d_{r-1}-1\; d_r+1] + \cdots + \beta[d_1-1\; d_2-1 \cdots d_{r-1}-1\; d_r+r-1] + \cdots + \delta'[0\,0 \cdots d].$$

This ends the proof of the theorem by construction.

5. An evaluating formula.

THEOREM. *The s-fold integral $U(q_s, \cdots, q_1)$ given by (3.2) may be evaluated using the formula*

$$\begin{aligned} &2\frac{c(s, m, n)}{c(s-2, m, n)(m+n+q_{t'}+1)} \\ (5.1)\quad &\cdot\left[\sum_{t=s,\, t\neq t'}^{1} (\pm)B(2m+q_t+q_{t'}+1, 2n+2)\cdot U(q'_{s-2}, \cdots, q'_1)\right] \\ &+ \frac{m+q_{t'}}{m+n+q_{t'}+1}\, U(q_s, \cdots, q_{t'+1}, q_{t'}-1, q_{t'-1}, \cdots, q_1) \end{aligned}$$

for arbitrarily fixed t', where

(i) *$B(\alpha, \beta)$ is a beta function with parameters α and β given by*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\, dx, \qquad 0 < x < 1,$$

(ii) *the sign $(\pm)$ is used according as the total, less unity, of number of columns preceding those having $q_{t'}$ and q_t is even or odd,*

(iii) *$U(q'_{s-2}, \cdots, q'_1)$ is an $(s-2)$-fold integral with subset $(q'_{s-2}, \cdots, q'_1)$ which is the complement of $(q_t, q_{t'})$ in the set $(q_s, \cdots, q_1)$,* and

(iv) *$c(s, m, n)$ is given by (2.2).*

To illustrate the reduction of $U(6, 3, 1, 0)$, we may choose the first column with $q_{t'} = 6$. Then, by (5.1),

$$U(6,3,1,0) = 2\frac{c(4,m,n)}{c(2,m,n)(m+n+6+1)}$$
$$\cdot[B(2m+3+6+1,2n+2)U(1,0) - B(2m+1+6+1,2n+2)U(3,0)$$
$$+ B(2m+0+6+1,2n+2)U(3,1] + \frac{m+6}{m+n+6+1}U(5,3,1,0).$$

Now put $M = 2m$, $P = 2p$, $p = m + n$. Since

$$\Gamma(y/2) = \pi^{\frac{1}{2}}\cdot 2^{1-y}\Gamma y\{\Gamma[(y+1)/2]\}^{-1},$$

we have

$$\frac{c(4,m,n)}{c(2,m,n)} = \frac{(2p+5)(2p+6)\Gamma(2p+9)}{8\Gamma(2m+4)\Gamma(2n+4)}.$$

The two-fold integrals, by using the expansion (5.1) repeatedly, give

$$U(1,0) = 1,$$
$$U(3,0) = \frac{(2m+3)[(2m+2)(2p+6)+2(2m+6)(2p+5)]}{(2p+5)(2p+6)(2p+8)},$$
$$U(3,1) = \frac{2(2m+2)(2m+3)(2m+5)}{(2p+5)(2p+6)(2p+8)}.$$

Hence,

$$U(6,3,1,0) = \frac{10M(4,5,6)}{P(8,9,10,11,14)}(10m^2 + 10mn + 73m + 27n + 117)$$
$$+ \frac{m+6}{p+7}\cdot U(5,3,1,0)$$

where $M(a, b, \cdots) = (2m + a)(2m + b)\cdots$ and $P(a', b', \cdots) = (2p + a')(2p + b')\cdots$.

This theorem may be regarded as a special case of that given by Roy (1957), p. 201, after a repeated application of the reduction formula and, separately, by Pillai (1956) p. 1110, both of which have been developed for entirely different purposes. If $t' = s$ in the formula here, and in Pillai's formula $x' = 1$ and $t = 0$, the two formulae will be seen to be equivalent. It may be observed that Pillai's formula has a long expansion especially if the difference of exponents in the first two columns of U is large, i.e., if $q_s \gg q_{s-1}$. The evaluation using the formula here shortens the reduction process if t' is properly chosen. The development may be found in Mijares (1962).

6. The moments and product-moments of the e.s.f. Nanda (1950) obtained the first three moments of the sum of two roots under the condition $m = 0$ of the distribution (2.1) for the case of $s = 2$. The moments were obtained by expanding directly the moment generating function of the sum of two roots under the given condition, collecting like terms and integrating term by term.

Pillai (1956) obtained a recursion formula for the sum of s roots in terms of $(s - 2)$ roots using hypergeometric expansions. He also obtained the first four moments of the sum of 2, 3 and 4 roots. Orense (1958) obtained the first three moments of the sum of 5 roots using Pillai's recursion formula. However, since the expansion becomes tedious as the order of the moments or the number of the roots s gets large, the method of differentiation, first used by Mijares (1958), was used for obtaining also the fourth moment of the sum of 5 roots. Mijares obtained the moments of the sum of s roots in determinantal forms and used them to derive a general form of the first four raw moments. These results are given in Mijares (1958) and in Pillai and Mijares (1959). Ting (1959), using these results, transformed the raw moments into central moments. Her results may be found in Pillai (1960). An inverse derivation of these moments (Mijares, 1961) used certain properties of the V's and Φ's which extended the derivation to any moment of any e.s.f. of s roots.

A more systematic procedure for deriving the moments will be illustrated in (a) below for the third moment of the second e.s.f. This procedure will be extended then to the product-moments of the first, second and third e.s.f. It will be shown next in (b) that certain moments using the procedure here are identical with results already well known.

(a) For further simplicity of notation, the superscript (s) of the jth e.s.f. $V_j^{(s)}$ will be dropped whenever there is no danger of confusion. Consider $V_2^3 = V_2 \cdot V_2 \cdot V_2$ for the case of third moment of the second e.s.f of s roots. The sum of subscripts $\sum d_i = 6$. Partition the integer 6 into at most three parts and transform each partition into a determinant by enclosing it with [ ]. Arrange these determinants in lexical order column-wise and take the lexical expansion of each determinant using the diagonalization theorem of Section 4. V_2^3, which is (2 2 2), can then be expanded in terms of the $[\cdots]$ in lexical order by successively multiplying each $[\cdots]$ by proper integers. Thus,

	(222)	(123)	(114)	(033)	(024)	(015)	(006)
[222]	1	−2	1	1	−1		
2 [123]		2	−2	−2		2	
[114]			1		−1	−1	1
[033]				1	−1		
3 [024]					3	−3	
2 [015]						2	−2
[006]							1
	1	0	0	0	0	0	0

That is,

$$(6.1)\quad V_2^3 = (222) = [222] + 2[123] + [114] + [033] + 3[024] + 2[015] + [006].$$

The next step is to transform each V-determinant of (6.1) into their equivalent Φ-determinants using relation (5.8) of the previous paper cited (Mijares, 1961,

p. 1158), then to use relation (2.2) of that same paper to obtain the necessary U'-expressions of the type given by (3.3) of this paper. For instance, to transform the V-determinant [123] to its equivalent Φ-determinant, it should be noted that the determinant has rows 2, 4 and 6 and columns 1, 2 and 3 of (V). Denote [123] by $\sigma(2\,4\,6;\,1\,2\,3;\,V)$ and its algebraic complement in (V) by $\tilde{\sigma}(1\,3\,5\,7\,8\,\cdots\,s+3;\,4\,5\,\cdots\,s+3;\,V)$. The corresponding element in Φ of $\sigma(2\,4\,6;\,1\,2\,3;\,V)$ will have rows and columns labelled by the columns and rows, respectively, of $\tilde{\sigma}(1\,3\,5\,7\,8\,\cdots s+3;\,4\,5\,\cdots\,s+3;\,V)$. Denote this corresponding element by $\sigma(\cdots;\cdots;\Phi)$, then

$$\sigma(2\,4\,6;1\,2\,3;V) = \sigma(4\,5\,\cdots\,s+3;1\,3\,5\,7\,8\,\cdots\,s+3;\Phi)$$

of (Φ) in (3.1) for $k = s + 3$. This means taking the 1st, 3rd, 5th, 7th, 8th, $\cdots$, kth columns of (Φ) for $k = s + 3$. Th subscripts of the Φ's in the last row are $s+2,\ s,\ s-2,\ s-4,\ s-5,\ \cdots,\ 1,\ 0$. Hence,

$$[1\,2\,3] = U'(s+2,\ s,\ s-2,\ s-4,\ s-5,\ \cdots,\ 1,\ 0).$$

By (3.3) the expected value

$$E[1\,2\,3] = U(s+2,\ s,\ s-2,\ s-4,\ s-5,\ \cdots,\ 1,\ 0).$$

Similarly, other V-determinants in (6.1) are transformed into Φ-determinants to obtain the U'-expressions which are easily converted to U-expressions by taking the expected value of U'. Thus, the third moment of V_2^3 is

$$\begin{aligned} E(V_2^3) = {} & U(s+2,\ s+1,\ s-3,\ s-4,\ s-5,\ \cdots,\ 1,\ 0) \\ & + 2U(s+2,\ s,\ s-2,\ s-4,\ s-5,\ \cdots,\ 1,\ 0) \\ & + U(s+2,\ s-1,\ s-2,\ s-3,\ s-5,\ \cdots,\ 1,\ 0) \\ & + U(s+1,\ s,\ s-1,\ s-4,\ s-5,\ \cdots,\ 1,\ 0) \\ & + 3U(s+1,\ s,\ s,\ -\ 2,\ s-3,\ s-5,\ s-6,\ \cdots,\ 1,\ 0) \\ & + 2U(s+1,\ s-1,\ s-2,\ s-3,\ s-4,\ s-6,\ s-7,\ \cdots,\ 1,\ 0) \\ & + U(s,\ s-1,\ s-2,\ s-3,\ s-4,\ s-5,\ s-7,\ s-8,\ \cdots,\ 1,\ 0). \end{aligned} \tag{6.2}$$

By using the evaluating formula of Section 5 the third moment of the second e.s.f. for specific values of s may be obtained in terms of m and n, just as the first and second moments of the second e.s.f. for $s = 2$ in (b) below have been obtained.

The product-moments of the first, second and third e.s.f.'s may be obtained by forming the triangular array starting with [1 2 3] and then successively multiplying each $[\cdots]$ by appropriate numbers to make the coefficients of the corresponding $(\cdots)$ add up to zero. That is,

	(123)	(114)	(033)	(024)	(015)	(006)
[123]	1	−1	−1		1	
[114]		1		−1	−1	1
[033]			1	−1		
2 [024]				2	−2	
2 [015]					2	−2
[006]						1
	1	0	0	0	0	0

Thus

$$(V_1V_2V_3) = [123] + [114] + [033] + 2[024] + 2[015] + [006].$$

Hence, $E(V_1V_2V_3)$ can be obtained easily by transforming each $[\cdots]$ to its equivalent U-expressions which are already given in (6.2).

The U-expressions of (6.2) can be quite readily obtained also by using the idea of *conjugate* partition in group representation theory. If the partition $\langle 2\ 2\ 2\rangle$ is represented by rows of dots of lengths respectively equal to its parts, the following is obtained

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

The *conjugate* partition is now the respective lengths of dot $\langle 3\ 3\rangle$ represented by the columns and on adding

$$\begin{array}{rl} & s-1,\ s-2,\ s-3,\ \cdots,\ 2,\ 1 \\ + & \quad 3, \quad 3, \quad 0,\ \cdots,\ 0,\ 0 \\ \hline & s+2,\ s+1,\ s-3,\ \cdots,\ 2,\ 1 \end{array}$$

Thus $[2\,2\,2] = U'(s + 2, s + 1, s - 3, \cdots, 2, 1)$, the expected value of which is the first U-expression of (6.2).

(b) Some moments and product moments of higher-order e.s.f. may be found in current literature. For testing complete independence, Hotelling (1936) used the distribution of $q = \pm r_1r_2$ for positive values of q and derived two forms of its moments. One form of the moment is given by

$$\mu_{2h} = \left[\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{t-1}{2}+h\right)\Big/\Gamma\left(\frac{t-1}{2}\right)\Gamma\left(\frac{n-1}{2}+h\right)\Gamma h\right] \tag{6.3}$$
$$\cdot(1-\nu)^{n/2}\int_0^1 x^{(n-t)/2}(1-x)^{n-1}\frac{d^h}{dx^h}\left[x^{(t/2)+h-1}(1-\nu x)^{-(n/2)}\right]dx$$

where ν is one of the two correlation coefficients between the two sets of variates which is not zero and t is the number of variates in the second set. Thus for the second moment

$$\mu_2 = \frac{t-1}{n-1}\left\{1 - \frac{n-t}{h}(1-\nu)^{n/2} F\left(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}+1, \nu\right)\right\} \tag{6.4}$$

where F is the ordinary hypergeometric series. For the null hypothesis of complete independence, $\nu = 0$, so that (6.4) reduces to

$$\mu_2 = t(t-1)/n(n-1). \tag{6.5}$$

Using the diagonalization theorem, V_2^2 can be expanded using

	(2 2)	(1 3)	(0 4)
[2 2]	1	−1	
[1 3]		1	−1
[0 4]			1
	1	0	0

Thus $V_2 \cdot V_2 = (2\,2) = [2\,2] + [1\,3] + [0\,4]$. Since V_p, like Φ_p, equals 0 for $p > s = 2$ (the number of variates in this case), the last two V-determinants vanish and hence, $(2\,2) = [2\,2]$. Also, note that $(0\,2) = [0\,2]$. Thus,

$$\mu_2' = E(V_2 \cdot V_2) = E[2\,2] = U(3, 2),$$
$$\mu_1' = E(V_0 V_2) = E[0\,2] = U(2, 1).$$

Use the evaluating formula of Section 5 for $s = 2$ to get

$$\begin{aligned}\mu_2 &= U(3, 2) - \{U(2, 1)\}^2 \\ &= M(2, 3, 4, 5)/P(5, 6, 7, 8) - M(2, 2, 3, 3)/P(5, 5, 6, 6),\end{aligned}$$

where $M(a, b, \cdots)$ and $P(a', b', \cdots)$ are, respectively, $(2m + a)(2m + b) \cdots$ and $(2p + a')(2p + b') \cdots$ with $p = m + n$. Under the null hypothesis of complete independence for $s = 2$ and degree of freedom n' (which is Hotelling's n), substitute

$$m = \tfrac{1}{2}(t-3), \qquad n = \tfrac{1}{2}(n' - t - 2)$$

in μ_2 above to obtain

$$\mu_2 = t(t-1)/n'(n'-1)$$

which conforms to (6.5).

In the paper by Girshick (1939), product-moments have been obtained for $q = r_1 r_2 \cdots r_s$ and $Z = (1 - r_1^2)(1 - r_2^2) \cdots (1 - r_s^2)$ in the case of s variates in the first set and t variates in the second set. From his result

$$E(q^\alpha Z^\beta) = \prod_{i=1}^{s} \frac{\Gamma[\frac{1}{2}(t+\alpha+1-i)]\Gamma[\frac{1}{2}(n-t+2\beta+1-i)]\Gamma[\frac{1}{2}(n+1-i)]}{\Gamma[\frac{1}{2}(t+1-i)]\Gamma[\frac{1}{2}(n-t+1-i)]\Gamma[\frac{1}{2}(n+\alpha+2\beta+1-i)]} \tag{6.6}$$

the αth moment of the sth e.s.f. of s roots can be obtained by putting $\beta = 0$.

Note that the α'th moment of q^2 equals the $2\alpha'$th moment of q. Assuming $t > s$ the notations of Girshick in terms of the distribution (2.1), with i, n replaced by i', n' and under null hypothesis of independence, are equivalent to

$$(6.7)\quad t = 2m + s + 1, \qquad n = 2n' + 2m + 2s + 2, \qquad r_i^2 = \theta_{s+1-i'} .$$

Replace t and n in (6.6) by (6.7), $\alpha/2$ by α' and i by $s + 1 - i'$. This gives the α'th moment of q^2 which is

$$(6.8)\qquad \prod_{i'=1}^{s} \frac{\Gamma[\frac{1}{2}(2m + i' + 1) + \alpha']\Gamma[\frac{1}{2}(2m + 2n + i' + s + 2)]}{\Gamma[\frac{1}{2}(2m + 2n + i' + s + 2) + \alpha']\Gamma[\frac{1}{2}(2m + i' + 1)]} .$$

Alternatively, using the diagonalization theorem the lexical expansion of the product $V_s \cdot V_s \cdots$ to α' factors is simply $(s, \cdots, s) = [s, \cdots, s]$, since all V-determinants in the lexical order after the leading one vanish by noting that $V_{s'} = 0$ for $s' > s$. Hence, applying the method earlier

$$\begin{aligned} \mu'_{\alpha'} &= E\{(q^2)^{\alpha'}\} = E(V_s, \cdots, V_s) \\ &= U(2s - 1, 2s - 2, \cdots, s + 1, s) \\ &= c(s, m, n')/c(s, m + \alpha', n'), \end{aligned}$$

which can be checked easily to be equivalent to (6.8).

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