BOUNDS FOR DISTRIBUTIONS WITH MONOTONE HAZARD RATE, II

By Richard E. Barlow[1] and Albert W. Marshall

University of California, Berkeley, and Boeing Scientific Research Laboratories

1. Introduction. In the preceding paper, which we will refer to as "I", we have derived improvements of Markov's inequality under the condition that F has a monotone hazard rate. These results are based on the assumption that for some monotone function ζ, $\int_{0-}^{\infty} \zeta(x)\, dF(x) = \nu$ is known (typically, $\zeta(x) = x^r$, in which case we denote ν by μ_r). In Section 3 of the present paper, we derive similar inequalities for the case that μ_1 and μ_2 are both known. These results may be regarded as improvements, made possible by the assumption of a monotone hazard rate, of the following inequality given by Chebyshev (1874): If F is a probability distribution such that $F(0-) = 0$ and $\int_0^\infty x^r\, dF(x) = \mu_r$, $r = 1, 2$ and $\mu_1 = 1$, then

$$(1.1)\qquad \begin{aligned} 1 - F(t) &\leqq 1, && 0 \leqq t \leqq 1 \\ &\leqq t^{-1}, && 1 < t < \mu_2 \\ &\leqq (\mu_2 - 1)/[\mu_2 - 1 + (t-1)^2], && t \geqq \mu_2 ; \end{aligned}$$

$$(1.2)\qquad \begin{aligned} 1 - F(t) &\geqq (1-t)^2/[\mu_2 - 1 + (1-t)^2], && 0 \leqq t \leqq 1 \\ &\geqq 0, && t \geqq 1. \end{aligned}$$

Improvements of (1.1) and (1.2) have also been obtained by Royden (1953), who assumed that F is concave on $[0, \infty)$ (see (6.1) and (6.2)).

The method used in this paper differs from those of I and can be utilized to provide alternate proofs of the results given there. However, we have not been able to obtain the results of this paper by the more straightforward methods of I.

In Section 5, we again consider the problem discussed in I of improving Markov's inequality, but in this paper we assume that F has a density f which is a Pólya frequency function of order 2 (PF_2). Again, the methods of I do not seem to be useful, but the result is obtained by a method similar to that used in Section 3.

Throughout this paper we assume unless otherwise stated that distribution functions are left continuous.

2. The method of proof. Let $\mathfrak{F}$ be a family of probability distributions. Call $\mathfrak{G} \subset \mathfrak{F}$ extremal for $\mathfrak{F}$ on T if for each $t \in T$ and $F \in \mathfrak{F}$, there exists $G \in \mathfrak{G}$ such that $F(t) = G(t)$. If $\mathfrak{G}$ is extremal for $\mathfrak{F}$ and $F \in \mathfrak{F}$, then clearly

$$\inf_{\mathfrak{G}} G(t) \leqq F(t) \leqq \sup_{\mathfrak{G}} G(t).$$

Received 20 June 1963; revised 30 April 1964.

[1] Work done under the auspices of Boeing Scientific Research Laboratories.

If the family $\mathcal{G}$ is sufficiently small, the bound may not be difficult to obtain. Similar methods have been used by Royden (1953) and Mallows (1956).

Our proofs that $\mathcal{G}$ is extremal involve a parameterization of $\mathcal{G}$: $\mathcal{G} = \{G_\alpha : \alpha \varepsilon I\}$. We single out a crossing of F and G_α, and show that this crossing must occur at each $t \varepsilon T$ as α ranges over I. Although this is conceptually simple, it is usually difficult to rigorize.

Example 2.1. Let $\mathfrak{F}$ be the class of distributions F where F is convex on its interval of support and satisfies $F(0) = 0$, $\int_0^\infty x\, dF(x) = \mu_1$. Let $\mathcal{G} = \{G_\alpha : 0 \leqq \alpha \leqq \mu_1\}$, where

$$\begin{aligned} G_\alpha(x) &= 0, & x &< \alpha \\ &= (x - \alpha)/2(\mu_1 - \alpha), & \alpha \leqq x &\leqq 2\mu_1 - \alpha \\ &= 1, & x &\geqq 2\mu_1 - \alpha. \end{aligned}$$

Suppose that $F \varepsilon \mathfrak{F}$. Then F and G_α have at most two crossings. If there is a crossing of F by G_α from below, denote this crossing point by u_α; otherwise,

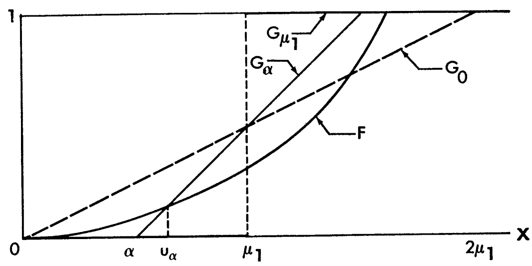

FIG. 2.1

let $u_\alpha = \alpha$. Then since G_0 is the uniform distribution on $[0, 2\mu_1]$, $u_0 = 0$, and since G_{μ_1} is degenerate at μ_1, $u_{\mu_1} = \mu_1$. It is clear that u_α is continuous in α (we make no attempt at a rigorous proof) so the crossing u_α must range over the interval $T = [0, \mu_1]$ as α ranges over the same interval. For $0 < t \leqq \mu_1$, we compute $\sup_{0 \leqq \alpha \leqq \mu_1} G_\alpha(t) = G_0(t) = t/2\mu_1$ and conclude that

$$F(t) \leqq t/2\mu_1, \qquad t \leqq \mu_1. \tag{2.1}$$

Example 2.2. Let $\mathfrak{F}$ be the class of IHR distributions satisfying $F(0) = 0$, and $\int_0^\infty \zeta(x)\, dF(x) = \nu$, where $\zeta(x)$ is increasing on $[0, \infty)$.

Let $\mathcal{G}_1 = \{G_w : 0 \leqq w \leqq \zeta^{-1}(\nu)\}$, where

$$\begin{aligned} 1 - G_w(x) &= 1, & x &\leqq w \\ &= e^{-a(x-w)}, & x &\geqq w \end{aligned} \tag{2.2}$$

and a is determined by

$$\int_0^\infty \zeta(x)\, dG_w(x) = \nu. \tag{2.3}$$

Let $\mathcal{G}_2 = \{G_w : \zeta^{-1}(\nu) \leqq w\}$, where

$$(2.4)\qquad \begin{aligned} 1 - G_w(x) &= e^{-bx} & 0 \leqq x \leqq w \\ &= 0, & x \geqq w, \end{aligned}$$

and b is also determined by the moment condition (2.3). Then $\mathcal{G}_i \subset \mathcal{F}$, and we wish to show that $\mathcal{G}_1 \cup \mathcal{G}_2$ is extremal for $\mathcal{F}$.

Let $F \varepsilon \mathcal{F} - (\mathcal{G}_1 \cup \mathcal{G}_2)$. Since $\int_0^\infty \zeta(x)\, dF(x) = \int_0^\infty \zeta(x)\, dG_w(x)$, F and G_w cross at least once; since F is IHR, they cross at most twice ($1 - F$ is log concave, and $1 - G_w$ is essentially log linear). Let v_w be the crossing from above of $1 - G_w(x)$ by $1 - F(x)$ if such a crossing exists, and otherwise, let $v_w = \infty$.

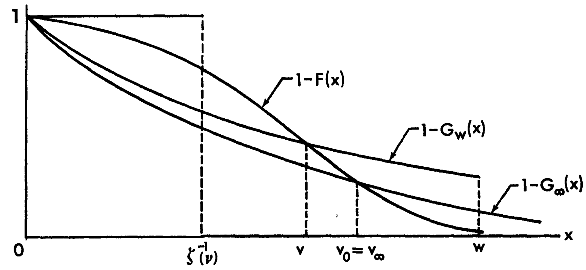

FIG. 2.2

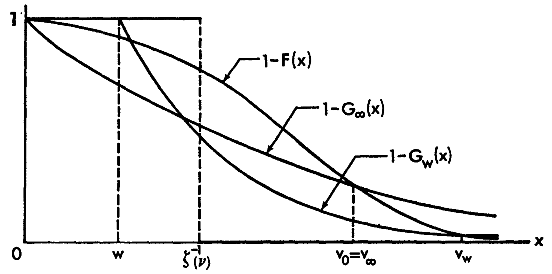

FIG. 2.3

Note that $G_{\zeta^{-1}(\nu)}$ is degenerate at $\zeta^{-1}(\nu)$, and $G_0 = G_\infty$ is exponential.

If we decrease w from ∞ to $\zeta^{-1}(\nu)$, then v_w decreases from $v_0 = v_\infty$ to zero (see Figure 2.2). If we then decrease w from $\zeta^{-1}(\nu)$ to zero, v_w decreases from $v_{\zeta^{-1}(\nu)} = \infty$ to $v_0 = v_\infty$ (see Figure 2.3).

A proof that v_w is continuous in w (which we do not include) would complete the argument that $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ is extremal for $\mathcal{F}$, so that

$$\inf_{\mathcal{G}} G(t) \leqq F(t) \leqq \sup_{\mathcal{G}} G(t).$$

Further pursuit of this leads to Lemmas 3.1 and 3.4 of I.

In case $\zeta(x) = x$, we obtain from (2.3) with $\mu_1 = 1$ that $a = (1 - w)^{-1}$.

If $t < 1$, $\inf_{G \varepsilon \mathcal{G}_1 \cup \mathcal{G}_2} [1 - G(t)] = \min_{0 \leqq w \leqq 1} \exp[-(t - w)/(1 - w)] = e^{-t}$, which is (3.8) of I with $r = 1$.

Note that for G_w in $\mathcal{G}_1$,

$$\mu_2 = 2 \int_0^\infty x [1 - G_w(x)] \, dx = 1 + (1 - w)^2,$$

and as w ranges over [0, 1], μ_2 ranges over [1, 2]. But for any IHR distribution, $\mu_1^2 \leqq \mu_2 \leqq 2\mu_1^2$, and since $1 - G_w(1) = e^{-1}$ for all w, we see that e^{-1} is a sharp lower bound for $1 - F(\mu_1)$ whenever F is IHR with mean μ_1, regardless of the specified value of μ_2.

3. Bounds for $1 - F$ when F is IHR. In this section, we obtain upper and lower bounds for $1 - F(t)$ in terms of μ_1 and μ_2. For convenience, take $\mu_1 = 1$, and fix μ_2. Recall that F IHR implies $\mu_1^2 \leqq \mu_2 \leqq 2\mu_1^2$ (Barlow, Marshall and Proschan (1963)).

In order to define the extremal distributions for this problem, let $T_0 = 1 - (\mu_2 - 1)^{\frac{1}{2}}$ and let

$$\begin{aligned} 1 - G_{T_1}(x) &= e^{-a_0 t}, & t &\leqq T_1, \\ &= 0, & t &> T_1, \end{aligned}$$

where a_0 and T_1 are chosen so that G_{T_1} satisfies the moment conditions. These conditions clearly determine a_0 and T_1 uniquely, and yield $T_1 = -a_0^{-1}\log(1 - a_0)$ where a_0 in [0, 1] satisfies

$$\frac{2}{a_0}\left[1 + \frac{1 - a_0}{a_0} \log (1 - a_0)\right] = \mu_2 .$$

Let $\mathcal{G}_3 = \{G_T : T \geqq T_1\}$ and $\mathcal{G}_4 = \{G_T : T_0 \leqq T \leqq T_1\}$, where

$$\begin{aligned} 1 - G_T(x) &= 1, & x &< \Delta \\ &= e^{-a(x-\Delta)}, & \Delta \leqq x &\leqq T \quad T \geqq T_1, \\ &= 0, & x &> T \end{aligned} \tag{3.1}$$

$$\begin{aligned} 1 - G_T(x) &= e^{-a_1 x}, & x &\leqq T \\ &= e^{-a_1 T - a_2(x-T)}, & x &\geqq T \end{aligned} \quad T_0 \leqq T \leqq T_1, \tag{3.2}$$

and (a, Δ), (a_1, a_2) are determined by the moment conditions

$$1 = \int_0^\infty [1 - G_T(x)] \, dx, \tag{3.3}$$

$$\frac{\mu_2}{2} = \int_0^\infty x [1 - G_T(x)] \, dx. \tag{3.4}$$

We defer the proof that solutions of (3.3) and (3.4) uniquely exist. These conditions guarantee that $\mathcal{G}_3 \cup \mathcal{G}_4 \, \varepsilon \, \mathfrak{F}_2$, where $\mathfrak{F}_2$ is the class of IHR distributions F such that $F(0) = 0$, $\int_0^\infty x \, dF(x) = 1$, and $\int_0^\infty x^2 \, dF(x) = \mu_2$.

A principal result of this section is

THEOREM 3.1. *$\{(x, 1 - F(x)) : F \varepsilon \mathfrak{F}_2\} = \{(x, 1 - G(x)) : G \varepsilon \mathcal{G}_3 \cup \mathcal{G}_4\}$, and hence*

$$\inf [1 - G(t)] \leqq 1 - F(t) \leqq \sup [1 - G(t)] \tag{3.5}$$

where the extremums are taken over $\mathcal{G}_3 \cup \mathcal{G}_4$.

Since $\mathcal{G}_3 \cup \mathcal{G}_4 \subset \mathfrak{F}_2$, it is clear that (3.5) is sharp, although it is not clear that equality is attainable.

We defer the proof of Theorem 3.1.

REMARK. We use repeatedly the fact that the functions $c_1 e^{-b_1 x}$, $c_2 e^{-b_2 x}$ are identical or have at most a single crossing (simple intersection).

COROLLARY 3.2. *Let F be* IHR, *$F(0) = 0$, and let F have first and second moments $\mu_1 = 1$, and μ_2. Then*

$$1 - F(t) \geqq \inf_{G \varepsilon \mathcal{G}_3} [1 - G(t)] = \inf_{T \geqq T_1} e^{-a(t - \Delta)}, \qquad t < 1 \tag{3.6}$$

where a and Δ are determined by (3.3) and (3.4) as functions of T; $1 - F(t) \geqq e^{-1}$, $t = 1$,

$$1 - F(t) \geqq \inf_{G \varepsilon \mathcal{G}_4} [1 - G(t)] = \inf_{T_0 \leqq T \leqq t} e^{-a_1 T - a_2(t - T)}, \qquad 1 < t < T_1, \tag{3.7}$$

where a_1 and a_2 are determined by (3.3) and (3.4) as functions of T; $1 - F(t) \geqq 0$, $t \geqq T_1$. The bounds are sharp.

PROOF. For $T_0 < T < T_1$, and $x \leqq T$, $1 - G_T(x) \geqq 1 - G_{T_1}(x)$, since otherwise G_T and G_{T_1}, cannot cross twice. If $T < 1$, then since $1 - G_T(x)$ and $1 - G_{T_0}(x)$ must cross twice, $1 - G_T(T) \geqq 1 - G_{T_0}(T)$. This together with $1 - G_T(1) \geqq e^{-1} = 1 - G_{T_0}(1)$ (Theorem 3.8, I) implies $1 - G_T(x) \geqq 1 - G_{T_0}(x)$, $T \leqq x \leqq 1$. But $G_{T_0} = G_\infty$, and thus (3.6) follows from (3.5).

If $T_1 < T < \infty$, then G_T and $G_{T_0} = G_\infty$ can cross only once in $(0, T)$. Since $1 - G_T(1) \geqq 1 - G_{T_0}(1) = e^{-1}$, and since $1 - G_T(T_0) < 1 - G_{T_0}(T_0) = 1$, this crossing must occur in $(T_0, 1]$. Hence $1 - G_T(x) > 1 - G_{T_0}(x)$ for $1 < x < t$, and we conclude from (3.5) that $1 - F(t) \geqq \inf_{\mathcal{G}_4} [1 - G(t)]$, $T_1 \geqq t > 1$. The remainder of (3.7) follows from the fact that for $x < T < T_1$, $1 - G_T(x) > 1 - G_{T_1}(x)$ (otherwise G_T and G_{T_1} cannot cross twice).||

THEOREM 3.3. *Let F be* IHR, *$F(0) = 0$, and let F have first and second moments $\mu_1 = 1$ and μ_2. Then*

$$1 - F(t) \leqq 1, \quad 0 \leqq t \leqq T_0 = 1 - (\mu_2 - 1)^{\frac{1}{2}}; \tag{3.8}$$

$$1 - F(t) \leqq e^{-a_1 t}, \qquad T_0 < t \leqq T_1, \tag{3.9}$$

where a_1 is determined by (3.3) and (3.4) with $T = t$;

$$1 - F(t) \leqq e^{-a(t - \Delta)}, \qquad t \geqq T_1, \tag{3.10}$$

where a and Δ are determined by (3.3) and (3.4) with $T = t$. These bounds are sharp.

PROOF. Let us first assume that (3.3) and (3.4) have the required solutions. It is easily verified from (3.3) and (3.4) with $G \varepsilon \mathcal{G}_3$ that $\lim_{T \to \infty} \Delta = 1 -$

$(\mu_2 - 1)^{\frac{1}{2}}$, and sharpness of (3.8) follows. Let $T_0 < t \leqq T_1$ and suppose $1 - F(t) > e^{-a_1 t}$; then $1 - F(x) > e^{-a_1 x}$, $0 < x \leqq t$. Since F and G_t cross at least twice, this would force $1 - F(t)$ and $\exp[-a_1 t - a_2(x - t)]$ to intersect three times which is impossible. If $t \geqq T_1$, then $1 - F(t) > 1 - G_t(t)$ together with the fact that F and G_t cross at least twice would force $F(x)$ and $e^{-a(x-\Delta)}$ to cross three times and again we obtain a contradiction.||

Theorem 3.3 also follows as a corollary of Theorem 3.1, since from Theorem 3.1, we need only show that $1 - G_t(t) \geqq 1 - G_s(t)$ for all $s \neq t$; but this follows from the fact that G_t and G_s must cross twice.

To complete the proof of Theorem 3.3, it is necessary to show that (3.3) and (3.4) have the required solutions. This proof is given in

LEMMA 3.4. *For every $T \geqq T_0$, there is a unique solution of* (3.3) *and* (3.4). *Furthermore, these solutions are continuous in T.*

PROOF. Consider first the case that $T > T_1$; fix $T > T_1$, $\Delta \varepsilon [0, 1]$, and let

$$\alpha(a, T, \Delta) = a^{-1}(1 - e^{-a(T-\Delta)}) + \Delta - 1.$$

Then $\lim_{a\to\infty} \alpha(a, T, \Delta) = \Delta - 1$, $\lim_{a\to 0} \alpha(a, T, \Delta) = T - 1 \geqq 0$ $(T_1 \geqq 1)$ and $\partial\alpha(a, T, \Delta)/\partial a \leqq 0$ for all a. Therefore $\alpha(a, T, \Delta) = 0$, i.e., (3.3) with $T \geqq T_1$ has a unique solution $a = a(T, \Delta)$ for each fixed Δ and T; furthermore $\alpha(a, T, \Delta) < 0$ (>0) for $a > a(T, \Delta)$ $(<a(T, \Delta))$. Let $\delta > 0$. Then $\alpha(a(T, \Delta) - \delta, T, \Delta) > 0$, $\alpha(a(T, \Delta) + \delta, T, \Delta) < 0$. By continuity of α, there exists $\epsilon_1 > 0$, $\epsilon_2 > 0$ (possibly depending on a, δ, T and Δ) such that $|T - T'| < \epsilon_1$, $|\Delta - \Delta'| < \epsilon_2$ implies $\alpha(a(T, \Delta) - \delta, T', \Delta') > 0$, $\alpha(a(T, \Delta) + \delta, T', \Delta') < 0$. Hence there exists $a(T', \Delta')$, $a(T, \Delta) - \delta < a(T', \Delta') < a(T, \Delta) + \delta$, such that $\alpha(a(T', \Delta'), T', \Delta') = 0$. This proves that $a(T, \Delta)$ is continuous in T and Δ.

Let

$$K(\Delta, T) = \Delta^2 - 2a^{-2}(aT + 1)e^{-a(T-\Delta)} + 2a^{-2}(a\Delta + 1)$$

where $a = a(T, \Delta)$ is determined by (3.3). We want to show that $K(\Delta, T) = \mu_2$, i.e. (3.4), has a unique solution $\Delta(T)$ continuous in T. If $\Delta = 0$, (3.3) implies $e^{-aT} = 1 - a$, so that $K(0, T) = 2a^{-1}(1 - Te^{-aT})$, and

$$\partial K(0, T)/\partial T = 2a^{-2}[(1 - T)\partial a/\partial T - a(1 - a)] > 0$$

where $\partial a/\partial T = ae^{-aT}(1 - Te^{-aT})^{-1} = a(1 - a)(1 - Te^{-aT})^{-1}$ if

$$(1 - T)a(1 - a)(1 - Te^{-aT})^{-1} \geqq a(1 - a),$$

which is clear if $0 \leqq a \leqq 1$. But this follows from $e^{-aT} = 1 - a$ and $T \geqq 1$. Therefore $T \geqq T_1$ implies

$$K(0, T) \geqq K(0, T_1) = \mu_2 \geqq 1 = \lim_{\Delta\to 1} K(\Delta, T).$$

This implies that $K(\Delta, T) = \mu_2$ has a solution $\Delta(T)$. Uniqueness of $\Delta(T)$ follows from the fact that for given T, there is at most one element of $\mathcal{G}_3$; two distributions in $\mathcal{G}_3$ are identical, or cross exactly twice, and the latter is impossible if they correspond to the same T.

Continuity of $\Delta(T)$ follows in the same manner as continuity of $a(T, \Delta)$. This completes the proof of Lemma 3.4 in case $T > T_1$.

Let $T_0 \leqq T \leqq T_1$. Solving (3.3) for a_2 as a function of a_1 and T, we obtain

$$a_2(a_1, T) = a_1 e^{-a_1 T}(a_1 - 1 + e^{-a_1 T - 1}),$$

and substitution in (3.4) yields

$$h(a_1, T) = \{e^{-a_1 T}[1 + (2 + a_1 T)(a_1 - 1)] + (a_1 - 1)^2\}/a_1^2 e^{-a_1 T} = \mu_2/2.$$

It is easily verified that $h(1, T) = 1$ and $\lim_{a_1 \to 0} h(a_1, T) = (1 + (1 - T)^2)/2$. Now $T \geqq T_0 = 1 - (\mu_2 - 1)^{1/2}$ implies $(1 + (1 - T)^2)/2 \leqq \mu_2/2 \leqq 1$. Since h is continuous, there exists $a_1 = a_1(T)$ satisfying $h(a_1, T) = \mu_2/2$.Furthermore. by arguments previously used, it can be shown that $a_1(T)$ is unique and continuous.||

If F is IHR, there can be at most two crossings of $1 - F$ and an exponential. Furthermore, the crossing points must be well defined, since if $1 - F(x)$ and ce^{-bx} coincide for all x in some interval, then $1 - F(x) \leqq ce^{-bx}$ for all x, and there can be no crossing. This is a simple consequence of the log concavity of $1 - F$.

Proof of Theorem 3.1. Let $F \varepsilon \mathfrak{F}_2 - (\mathcal{G}_3 \cup \mathcal{G}_4)$. For $T \geqq T_1$, let $r(T)$ be the point in (Δ, T) that $1 - F$ crosses $1 - G_T$ from below if such a crossing exists; otherwise, let $r(T) = \Delta$. Let $s(T)$ be the crossing in (Δ, T) from above of $1 - G_T$ by $1 - F$ if such a crossing exists; otherwise, let $s(T) = T$. Note that $r(T) \leqq s(T)$.

For $T_0 \leqq T \leqq T_1$, let $u(T)$ be the crossing in (T, ∞) from below of $1 - G_T$ by $1 - F$; $u(T)$ always exists. Let $v(T)$ be the crossing in (T, ∞) from above of $1 - G_T$ by $1 - F$ if such a crossing exists; otherwise let $v(T)$ be the right-hand endpoint of the support of F.

In order to show that r, s, u, and v are continuous in the interior of their range suppose that F and G_T cross at $x = x_0$ (in case $T \geqq T_1$, let $x_0 \neq T$). Choose $\epsilon > 0$ sufficiently small that $[G_T(x_0 - \epsilon) - F(x_0 - \epsilon)][G_T(x_0 + \epsilon) - F(x_0 + \epsilon)] < 0$ (and $x_0 + \epsilon < T$ when $T \geqq T_1$). By Lemma 3.4, $G_T(x)$ is con-

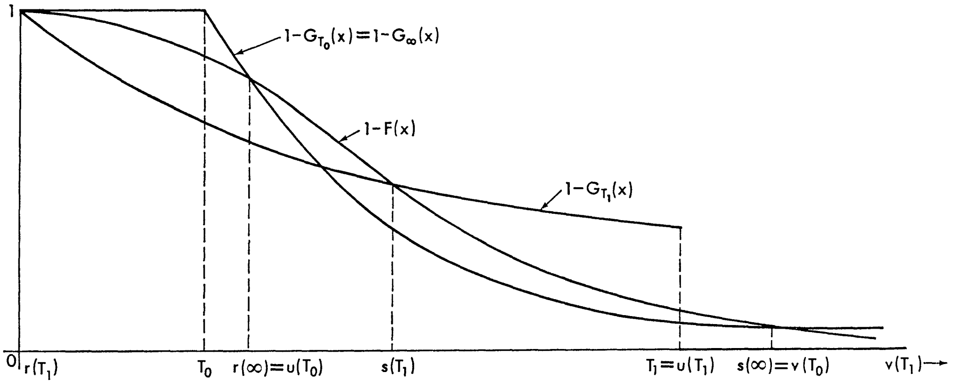

Fig. 3.1

tinuous in T for all x ($x < T$ in case $T \geqq T_1$). Hence there exists $\delta > 0$ such that $|T' - T| < \delta$ implies $[G_{T'}(x_0 - \epsilon) - F(x_0 - \epsilon)][G_{T'}(x_0 + \epsilon) - F(x_0 + \epsilon)] < 0$. This means that $G_{T'}$ and F cross in the interval $(x_0 - \epsilon, x_0 + \epsilon)$.

To show that for all $x < T_1$, there exists T such that F and G_T cross at x, it suffices to show that $\lim_{T \downarrow T_1} r(T) = 0$, $\lim_{T \to \infty} r(T) = u(T_0)$, and $\lim_{T \to T_1} u(T) = T_1$. The second two limits are clear from the definitions. Proof that $\lim_{T \downarrow T_1} r(T) = 0$ is similar to the proof of continuity.

To show that for all $x \geqq T_1$, there exists T such that F and G_T cross at x, we note that $s(T_1) < T_1$, $\lim_{T \to \infty} s(T) = v(T_0)$, $\lim_{T \to T_1} v(T) =$ right-hand endpoint of the support of F.||

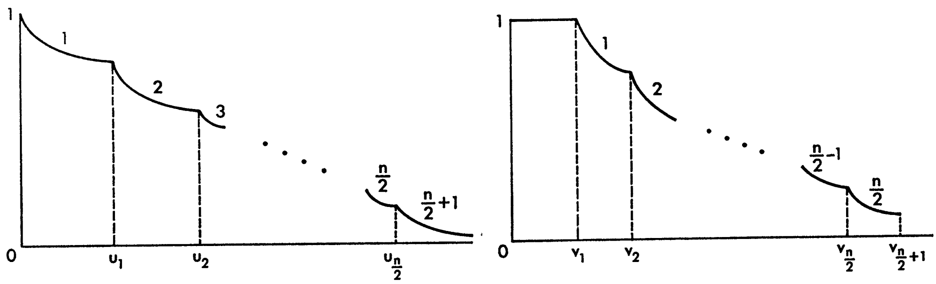

FIG. 3.2. Extremal distributions, n even

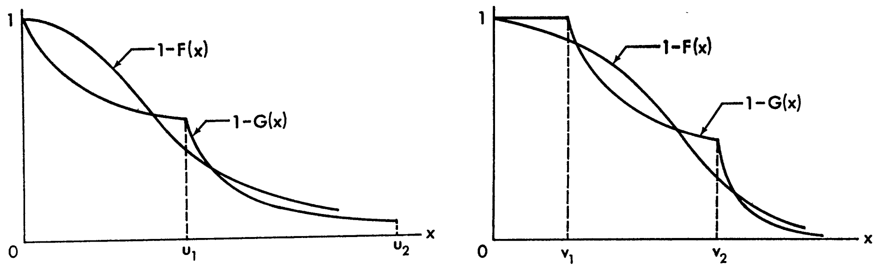

FIG. 3.3. Extremal distributions for $n = 3$

REMARKS ON GENERALIZATIONS. Let $(\mu_1, \mu_2, \cdots, \mu_n)$ be the first n moments from an IHR distribution F. We conjecture that there exists a family of extremal distributions G with these same moments that are piecewise exponential, with at most $[(n + 3)/2]$ pieces and the possibility of truncation on the right. The slopes of the logarithms of the exponential segments together with the endpoints of the segments produce $n + 1$ parameters. The extremal family is generated by varying the single parameter undetermined by the moment conditions.

We conjecture that the u_i and v_i shown in Figure 3.2 satisfy $v_1 \leqq u_1 \leqq v_2 \leqq \cdots \leqq v_{(n/2)+1}$, which we have shown to be true for $n = 2$.

With this conjecture, it is easily seen that at any endpoint w of an exponential segment, $1 - G(w) \geqq 1 - F(w)$ with strict inequality providing $F \not\equiv G$ and

$1 - G(w) < 1$. Otherwise, F and G cannot have the required number of sign changes (recall that if F and G have n equal moments, they must cross at least n times).

4. Bounds for $1 - F$ when F is DHR. Let

$$(4.1) \qquad \begin{aligned} 1 - G_{T;\alpha}(x) &= \alpha e^{-a_1 x}, \qquad 0 < x \leqq T \\ &= \alpha e^{-a_2 x + (a_2 - a_1) T}, \quad x \geqq T, \end{aligned} \qquad 0 \leqq T \leqq \infty$$

where

$$(4.2) \qquad 1 = \int_0^\infty [1 - G_{T;\alpha}(x)]\, dx,$$

$$(4.3) \qquad \mu_2/2 = \int_0^\infty x\,[1 - G_{T,\alpha}(x)]\, dx.$$

Following the proof of Lemma 3.4, we conclude that for every $T \geqq 0$ and every α, $2\mu_2^{-1} \leqq \alpha \leqq 1$, there exists a_1 and a_2 satisfying (4.2) and (4.3). Note that

$$1 - G_{\infty; 2\mu_2^{-1}}(x) = 2\mu_2^{-1} e^{-2x/\mu_2}, \qquad x > 0.$$

Note that $G_{T;\alpha}$ is DHR ($a_1 \geqq a_2$) if $\mu_2 \geqq 2$.

THEOREM 4.1. *If F is* DHR, $F(0) = 0$ *and if F has first and second moments* $\mu_1 = 1$ *and* μ_2, *then*

$$(4.4) \qquad \begin{aligned} 1 - F(t+) &\geqq 2\mu_2^{-1}, \qquad && t = 0 \\ &\geqq e^{-a_1 t}, && t > 0, \end{aligned}$$

where a_1 is determined by (4.2) *and* (4.3) *with* $\alpha = 1$ *and* $T = t$. *The bound is sharp.*

Note that since F is DHR, $\mu_2 \geqq 2$.

PROOF. Since F and $G_{\infty; 2\mu_2^{-1}}$ have the same first two moments, they cross at least twice. Since F is DHR there are exactly two crossings, and the first crossing of $1 - G_{\infty;2\mu_2^{-1}}$ by $1 - F$ must be from above. Hence $1 - F(0+) \geqq 2\mu_2^{-1}$. Now let $t > 0$ and suppose that $1 - F(t) < 1 - G_{t;1}(t)$. Then since $1 - F(0+) \leqq 1 - G_{t;1}(0)$, F and $G_{t;1}$ can cross at most once in $[0, t]$, and it follows from $1 - F(t) < 1 - G_{t;1}(t)$ that there are no crossings in $[0, t]$. Since $1 - F(t) < 1 - G_{t;1}(t)$, there can be at most one crossing of F and $G_{t;1}$ in (t, ∞). Hence F and $G_{t;1}$ cross at most once in $[0, \infty)$, contradicting the assumption that they have the same first two moments.||

A sketch of $\log 1 - F$, $\log 1 - G_{t;1}$ and $\log 1 - G_{\infty;2\mu_2^{-1}}$ should make the above proof clear.

THEOREM 4.2. *If F is* DHR, $F(0) = 0$ *and F has first and second moments* $\mu_1 = 1$ *and* μ_2, *then*

$$
(4.5)\qquad
\begin{aligned}
1 - F(t) &\leqq e^{-t}, && 0 \leqq t \leqq 1\\
&\leqq (te)^{-1}, && 1 \leqq t \leqq \mu_2/2\\
&\leqq 2\mu_2^{-1}e^{-2t/\mu_2}, && \mu_2/2 \leqq t \leqq \mu_2\\
&\leqq \sup_{0<T<t} 1 - G_{T;1}(t), && t > \mu_2 .
\end{aligned}
$$

These bounds are sharp.

PROOF. Recall from (3.13) of I that

$$
\begin{aligned}
1 - F(t) &\leqq e^{-t}, && t \leqq 1\\
&\leqq (te)^{-1}, && t \geqq 1.
\end{aligned}
$$

We wish to show these bounds are sharp for $t \leqq \mu_2/2$. Let a_1, a_2 and α be determined by (4.2) and (4.3), and assume that $\mu_2 > 2$ (so that F is not exponential). By (4.3), $\lim_{T\to\infty} Ta_2^{-1}e^{-a_1T} < \mu_2/2\alpha < \infty$, so that $\lim_{T\to\infty} a_2^{-1}e^{-a_1T} = 0$. Hence from (4.2),

$$1 = \lim_{T\to\infty}\left[\alpha\int_0^T e^{a_1x}\,dx + a_2^{-1}e^{-a_1T}\right]$$

which implies $\lim_{T\to\infty} a_1 = \alpha$. This means that $\lim_{T\to\infty} 1 - G_{T;\alpha}(x) = \alpha e^{-\alpha x}$. Since $\lim_{T\to\infty} 1 - G_{T;1}(t) = e^{-t}$, (4.5) is sharp for $t \leqq 1$. Since $\lim_{T\to\infty} 1 - G_{T;1/t}(t) = (te)^{-1}$, (4.5) is sharp for $1 \leqq t \leqq \mu_2/2$. Note that for $t = \mu_2/2$, equality is attained by the distribution $1 - G_{\infty;2\mu_2^{-1}}$.

Next, recall from (3.13) of I that

$$1 - F(t) \leqq 2e^{-2}\mu_2/t^2, \qquad t \geqq 2\mu_2^{\frac{1}{2}};$$

this proves (4.5) for $t = \mu_2$. Equality is attained in (4.5) for $t = \mu_2$ again by the distribution $1 - G_{\infty;2\mu_2^{-1}}$.

We have shown that $1 - F(t) \leqq 1 - G_{\infty;2\mu_2^{-1}}(t)$ for $t = \mu_2/2$ and $t = \mu_2$. Since F is DHR, this implies $1 - F(t) \leqq 1 - G_{\infty;2\mu_2^{-1}}(t)$ for all t in $[\mu_2/2, \mu_2]$, so that (4.5) holds for $\mu_2/2 \leqq t \leqq \mu_2$.

Finally, we consider the case $t > \mu_2$. Since $1 - F(0) \leqq 1 - G_{T;1}(0)$, there is at most one crossing of $1 - G_{T;1}$ by $1 - F$ in $(0, T]$, and hence there is a first crossing $u(T)$ to the right of T. Since $1 - G_{T;1}(T) \leqq 1 - F(T)$ by (4.4), this crossing is from above. Since $u(T) > T$, $\lim_{T\to\infty} u(T) = \infty$. Since $\lim_{T\to 0} 1 - G_{T;1}(x) = 1 - G_{\infty;2\mu_2^{-1}}(x)$, $\lim_{T\to 0} 1 - G_{T;1}(\mu_2) = 2\mu_2^{-1}e^{-2} \geqq 1 - F(\mu_2)$, and hence $\lim_{T\to 0} u(T) \leqq \mu_2$. By arguments similar to those of Section 3, it follows that $u(T)$ is continuous in T, so that for every $t > \mu_2$, there exists $T < t$ such that $u(T) = t$, that is, $1 - G_{T;1}(t) = 1 - F(t)$.||

We remark that the bounds of (4.5) are sharp with the additional assumption that $F(0+) = 0$. However, it may be that the bound can only be approximated, and equality is unattainable.

5. Upper bounds for $1 - F$ **when** f **is** PF_2. In this section, we obtain a sharp upper bound for $1 - F(t)$, given a single expectation $\int_0^\infty \zeta(x)\, dF(x) = \nu$ (ζ monotone), when $F(0) = 0$ and F has a density f that is a Pólya frequency function of order 2 (PF_2). Briefly, f is PF_2 if $\log f(x)$ is concave on the support of F, an interval (see Schoenberg (1951) for a precise definition). The condition that f is PF_2 implies that F is IHR (Barlow, Marshall, Proschan (1963)), so that the result here is a sharpening of inequality (3.5) of I.

Under the condition that f is PF_2, no sharpening of (3.1) of I is possible, since the extremal distributions there are exponential, and therefore have PF_2 (indeed, PF_∞) densities.

Let

$$G_m(x; b) = (1 - e^{-bx})/(1 - e^{-bm}), \qquad 0 \leqq x \leqq m$$
$$= 1, \qquad x > m,$$

for $m > 0$ and $b \neq 0$; let $G_m(x; 0) = \lim_{b \to 0} G_m(x; b)$. This distribution has a density

$$g_m(x; b) = be^{-bx}/(1 - e^{-bm}), \qquad 0 \leqq x \leqq m$$
$$= 0, \qquad \text{elsewhere},$$

which is obtained by truncating an exponential density. Hence g_m is PF_2.

THEOREM 5.1. *Let f be a PF_2 density such that $f(x) = 0$ for $x < 0$. Let ζ be a function continuous and strictly increasing on $[0, \infty)$ such that $\int_0^\infty \zeta(x)\, dF(x) = \nu$ exists finitely. Then for each $m > \zeta^{-1}(\nu)$, there exists a unique b_m satisfying*

$$\int_0^\infty \zeta(x)\, dG_m(x; b_m) = \nu. \tag{5.1}$$

Furthermore, for all $t > 0$,

$$1 - F(t) \leqq 1, \qquad t < \zeta^{-1}(\nu)$$
$$\leqq \sup_{m \geqq t} [1 - G_m(t; b_m)], \qquad t \geqq \zeta^{-1}(\nu). \tag{5.2}$$

In the case that $\zeta(x) \equiv x$, this bound has been computed numerically, and is graphed in Figure 6.1 of I. Here we have the explicit bound $1 - F(\mu_1) \leqq 1 - e^{-1}$. Since f is PF_2, F is also log concave and this result follows from Jensen's inequality (see the remark following Theorem 3.8 of I).

Before proving Theorem 5.1, we prove some useful lemmas.

LEMMA 5.2. *$\int_0^\infty \zeta(x)\, dG_m(x; b) \equiv \Phi(b, m)$ is continuous in b for fixed m and continuous in m for fixed b.*

PROOF. Since $\lim_{b \to b^*} G_m(x; b) = G_m(x, b^*)$, and $\lim_{m \to m^*} G_m(x; b) = G_{m^*}(x; b)$ for all $m^* > 0$ and all b^*, the theorem follows from the Helly-Bray lemma (Loève (1960) p. 182).||

LEMMA 5.3. *For all $m > 0$ and $\nu \in [\zeta(0), \zeta(m)]$ there exists a unique b_m satisfying (5.1).*

PROOF. We first show that $G_m(x; b)$ is strictly increasing in b for each $x < m$.

If $b \neq 0$, $\partial G_m(x; b)/\partial b > 0$ if and only if $\varphi(x) > \varphi(m)$ where $\varphi(z) = ze^{-bz}/(1 - e^{-bz})$. But $\varphi'(z) = e^{-bz}(1 - bz - e^{-bz})/(1 - e^{-bz})^2 < 0$ for all $bz \neq 0$. Hence for $x < m$ and $b \neq 0$, $\varphi(x) > \varphi(m)$. If $b = 0$, then $\partial G_m(x; b)/\partial b \,|_{b=0} = x(m - x)/2m > 0$ for $x < m$. Thus $G_m(x; b)$ is strictly increasing in b for each $x < m$, and hence $\Phi(b, m) \equiv \int \zeta(x)\, dG_m(x, b)$ is strictly *decreasing* in b (since ζ is increasing). Since $\Phi(b, m)$ is continuous in b by Lemma 5.2, it remains only to show that $\lim_{b\to\infty} \int \zeta(x)\, dG_m(x; b) = \zeta(0)$ and $\lim_{b\to-\infty} \int \zeta(x)\, dG_m(x; b) = \zeta(m)$. But this follows by the Helly-Bray lemma, since

$$\lim_{b\to\infty} G_m(x; b) = 0, \quad x \leqq 0 \quad \text{and} \quad \lim_{b\to-\infty} G_m(x; b) = 0, \quad x < m$$
$$= 1, \quad x > 0, \qquad\qquad\qquad\qquad = 1, \quad x \geqq m. \|$$

For convenience, we introduce the notation $g_m(x) \equiv g_m(x; b_m)$.

LEMMA 5.4. *$g_m(t)$ is continuous in $m \geqq t$.*

PROOF. It is sufficient to show that b_m is continuous in m, where b_m is determined by (5.1). Let $\epsilon > 0$ and fix m. Since $\Phi(b, m)$ is decreasing in b (see the proof of Lemma 5.3), and since Φ is continuous in b (Lemma 5.2), there exists $\eta > 0$ such that $\Phi(b_m + \epsilon, m) > \nu - \eta$ and $\Phi(b_m - \epsilon, m) < \nu + \eta$. Now since Φ is continuous in m, there exists $\delta > 0$ such that $|m - m'| < \delta$ implies $\Phi(b_m + \epsilon, m') > \nu - 2\eta$, $\Phi(b_m - \epsilon, m') < \nu + 2\eta$. Then by monotonicity and continuity of Φ, $b_{m'} \in (b_m - \epsilon, b_m + \epsilon)$. That is, $|b_m - b_{m'}| < \epsilon$ whenever $|m - m'| < \delta. \|$

Suppose that for all m, $f \neq g_m$. Then g_m crosses f exactly once from below; since $\log f(x)$ is concave and $\log g_m(x)$ is linear in $x \in [0, m)$, there is at most one such crossing (see Karlin, Proschan and Barlow (1961)). By (5.1), F and G_m must cross at least once (f and g_m must cross twice) so that there exists at least one such crossing. Denote the unique such crossing point by $x^*(m)$.

LEMMA 5.5. *$x^*(m)$ is continuous in m.*

PROOF. Fix m and let $x^*(m) = x^*$. Since g_m crosses f from below at x^*, there exists $\epsilon > 0$ such that

$$f(x) - g_m(x) > 0, \qquad x^* - 2\epsilon < x < x^*$$
$$<0, \qquad x^* < x < x^* + 2\epsilon.$$

Let $2\eta = \min \{[f(x^* - \epsilon) + g_m(x^* - \epsilon)], [g_m(x^* + \epsilon) - f(x^* + \epsilon)]\}$. Since $g_m(x)$ is continuous in $m > x$ for fixed x, there exists $\delta > 0$ such that $|m - m'| < \delta$ implies $|g_m(x) - g_{m'}(x)| < \eta$ for $x = x^* \pm \epsilon$. Then

$$|f(x^* + \epsilon) - g_{m'}(x^* + \epsilon)|$$
$$\geqq |f(x^* + \epsilon) - g_m(x^* + \epsilon)| - |g_m(x^* + \epsilon) - g_{m'}(x^* + \epsilon)| > \eta,$$
$$|f(x^* - \epsilon) - g_{m'}(x^* - \epsilon)|$$
$$\geqq |f(x^* - \epsilon) - g_m(x^* - \epsilon)| - |g_m(x^* - \epsilon) - g_{m'}(x^* - \epsilon)| > \eta.$$

By continuity (in x) of $f(x)$ and $g_{m'}(x)$, $x^*(m') \in (x^*(m) - \epsilon, x^*(m) + \epsilon)$; i.e., $|m - m'| < \delta$ implies $|x^*(m) - x^*(m')| < \epsilon. \|$

PROOF OF THEOREM 5.1. We suppose without loss of generality that $f \neq g_m$ for all $m > 0$ and consider the case that $t \geqq \zeta^{-1}(\nu)$. Since $\int \zeta(x)\, dF(x) = \nu$, it follows that $\zeta(0) \leqq \nu$, and for $m \geqq t \geqq \zeta^{-1}(\nu)$, $\nu \leqq \zeta(m)$. Thus b_m satisfying (5.1) exists uniquely by Lemma 5.3 for all $m \geqq t$. Now assume that $t < x^*(\infty)$ (otherwise the theorem is obvious). Clearly $x^*(t) < t$. Hence by Lemma 5.5 there exists m_0 such that $x^*(m_0) = t$. Since g_{m_0} is logarithmically linear in $x < m_0$, f and g_{m_0} can cross at most twice in $(0, m_0)$. If there are two such crossings, then since g_{m_0} crosses f from below at t, the other crossing point x_1 satisfies $x_1 < t$. If there is only one such crossing (at t by choice of m_0), let $x_1 = 0$. Then in either case,

$$
(5.3) \qquad \begin{aligned} f(x) &< g_{m_0}(x), \quad 0 < x < x_1 \quad \text{or} \quad t < x < m_0 \\ &\geqq g_{m_0}(x), \quad x_1 < x < t \quad \text{or} \quad x > m_0 . \end{aligned}
$$

Let $l(x) = \alpha + \beta\zeta(x)$, where $\beta = [\zeta(m_0) - \zeta(x_1)]^{-1}$ and $\alpha = -\zeta(x_1)\beta$. Then $\chi_{[t,\infty)}(x) - l(x)$ changes sign with $f(x) - g_{m_0}(x)$($\chi_{[t,\infty)}$ is the indicator function of $[t, \infty)$), and consequently

$$
(5.4) \qquad [\chi_{[t,\infty)}(x) - l(x)][f(x) - g_{m_0}(x)] \leqq 0.
$$

Integration on x from 0 to ∞ yields $\int_t^\infty f(x)\, dx \leqq \int_t^\infty g_{m_0}(x)\, dx$.||

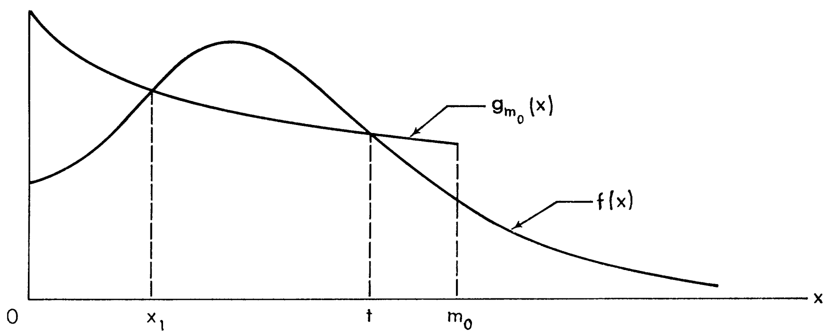

FIG. 5.1

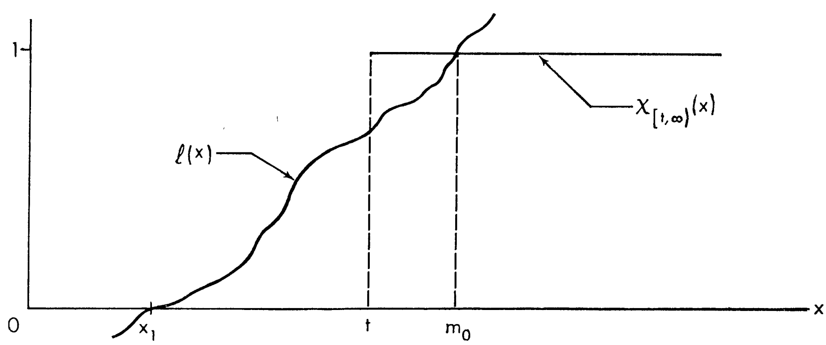

FIG. 5.2

THEOREM 5.5. *Inequality* (5.2) *is sharp.*

PROOF. In case $\nu \leqq \zeta(t)$ the theorem is obvious; in case $\nu > \zeta(t)$, equality is attained by the distribution degenerate at ν. This degenerate distribution can occur in many ways as a limit of distributions with PF_2 densities.||

COROLLARY 5.6. *Let f be a PF_2 density such that $f(x) = 0$, $x < 0$, and such that $\int_0^t f(x)\,dx = p$. If ζ is a function continuous and strictly increasing on $[0, \infty)$, then*

$$\int_0^\infty \zeta(x)f(x)\,dx \geqq \inf_{m>t} \int_0^\infty \zeta(x)\, g_m(x; c_m)\,dx, \tag{5.5}$$

where for each $m > t$, c_m is uniquely determined by

$$\int_0^t g_m(x; c_m)\,dx = p. \tag{5.6}$$

PROOF. $G_m(t; c)$ is strictly increasing in c (see the proof of Lemma 5.3), $\lim_{c\to\infty} G_m(t; c) = 1$ and for $t < m$, $\lim_{c\to-\infty} G_m(t; c) = 0$. Hence (5.6) has a unique solution c_m for each $m > t$ and $p \in (0, 1)$.

Consider now the case that $\zeta(t) \geqq \nu = \int_0^\infty \zeta(x)f(x)\,dx$. Let m_0 be as defined in the proof of Theorem 5.1. Then by (5.4), $G_{m_0}(t; b_{m_0}) \leqq F(t) = G_{m_0}(t; c_{m_0})$ so that $G_{m_0}(x; b_{m_0}) \leqq G_{m_0}(x; c_{m_0})$ for all x. This together with monotonicity of ζ yields

$$\int_0^\infty \zeta(x)f(x)\,dx = \int_0^\infty \zeta(x)\, g_{m_0}(x; b_{m_0})\,dx \geqq \int_0^\infty \zeta(x)\, g_{m_0}(x; c_{m_0})\,dx$$

$$\geqq \inf_{m>t} \int_0^\infty g_m(x; c_m)\,dx.$$

Next suppose that $\zeta(t) < \nu$. Since, for fixed c, $\lim_{m\downarrow t} G_m(t; c) = 1$, it follows that $\lim_{m\downarrow t} c_m = -\infty$. Since ζ is continuous at t, $\lim_{m\downarrow t} \int_0^\infty \zeta(x)g_m(x; c_m)\,dx = \zeta(t) < \int_0^\infty \zeta(x)f(x)\,dx$.||

Theorem 5.1 remains true if ζ is strictly decreasing rather than increasing. In this case, the statements of the lemmas remain unchanged and the proofs require only minor modifications. Inequality (5.4) is replaced by

$$[\chi_{[0,t]}(x) - l(x)][f(x) - g_{m_0}(x)] \geqq 0,$$

where $l(x) = \alpha + \beta\zeta(x)$ and $\beta = [\zeta(x_1) - \zeta(m_0)]^{-1}$, $\alpha = -\zeta(m_0)\beta$. If ζ is decreasing rather than increasing, the direction of inequality (5.5) is reversed, and the infimum is replaced by supremum.

For Theorem 5.1, the continuity of ζ was used only for the applications of the Helly-Bray lemma in Lemmas 5.2 and 5.3. This condition can be relaxed, as can the condition that ζ be *strictly* monotone. In particular, (5.2) holds if for some $s > t$, $\zeta(x) = \chi_{[s,\infty)}(x)$ (i.e., if ν is a percentile).

We remark that for $t \geqq \nu$,

$$f(t) \geqq g_t(t). \tag{5.7}$$

This inequality follows from arguments similar to those advanced in the discussion preceding Lemma 5.5. Further bounds for densities will appear in a forthcoming paper by the authors.

Note that g_m is *not* PF_3. This means that in case f is PF_3, inequality (5.2) is not sharp, but can be improved.

6. Some numerical comparisons. Extensive tables of the various bounds obtained in this paper are given by Barlow and Marshall (1963). We present here

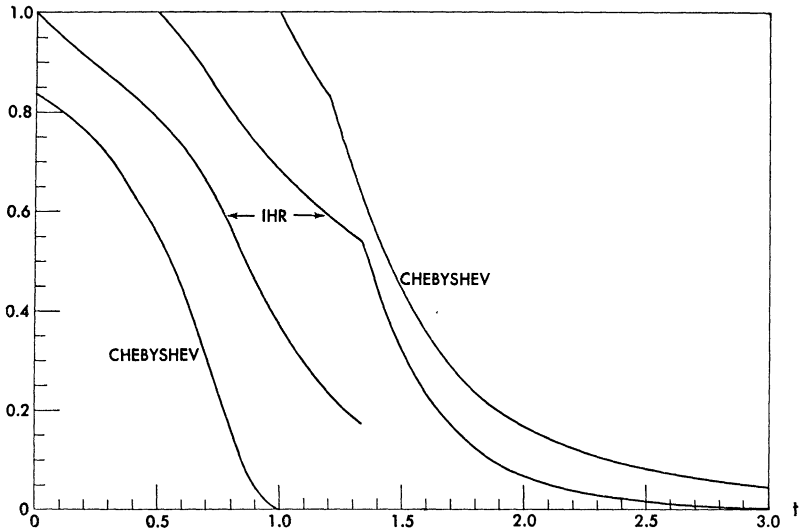

FIG. 6.1. Upper and lower bounds for $1 - F(t)$, $\mu_1 = 1$, $\mu_2 = 1.2$

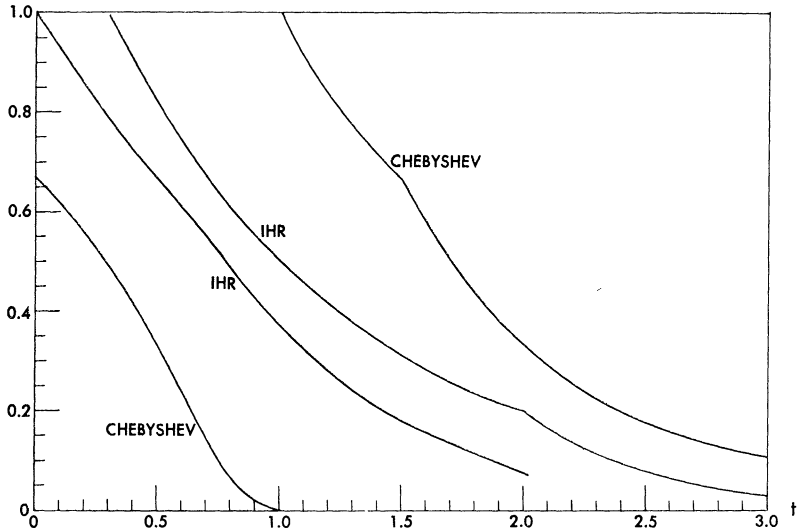

FIG. 6.2. Upper and lower bounds for $1 - F(t)$, $\mu_1 = 1$, $\mu_2 = 1.5$

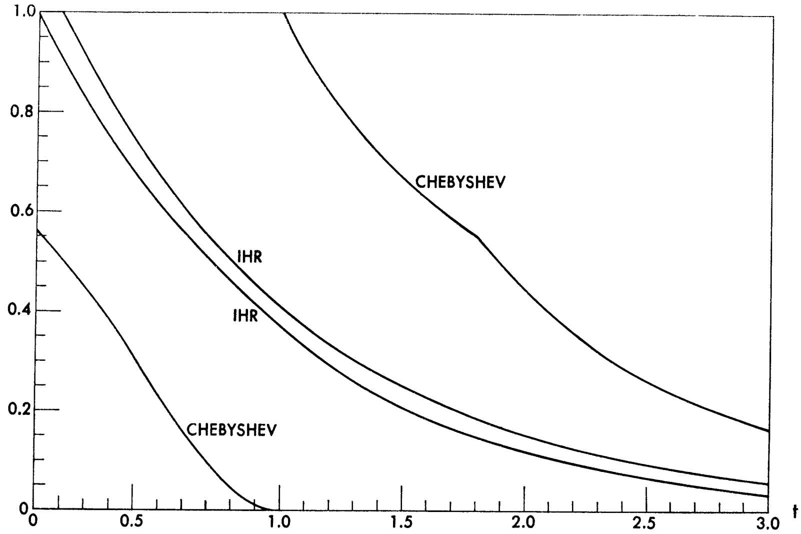

FIG. 6.3. Upper and lower bounds for $1 - F(t)$, $\mu_1 = 1$, $\mu_2 = 1.8$

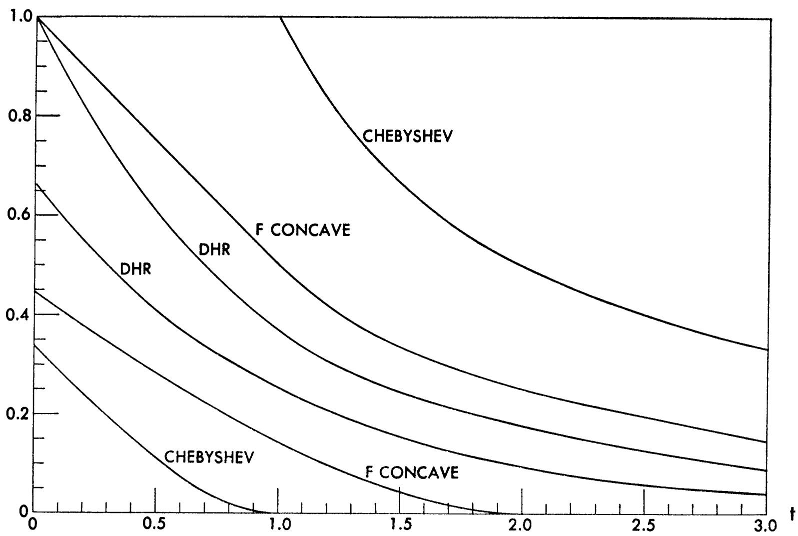

FIG. 6.4. Upper and lower bounds for $1 - F(t)$, $\mu_1 = 1$, $\mu_2 = 3$

some numerical results in the form of graphs, and make some comparisons with the bounds of Chebyshev (1874) and Royden (1953).

The Chebyshev bounds have been given in (1.1) and (1.2). Royden's improvement for the case that F is concave on $[0, \infty)$ are:

$$
(6.1)\quad
\begin{aligned}
1 - F(t) &\leqq 1 - t/2, && 0 \leqq t \leqq 1\\
&\leqq (2t)^{-1} && 1 \leqq t \leqq 3\mu_2/4\\
&\leqq 4(3\mu_2 - 2t)/9\mu_2^2, && 3\mu_2/4 \leqq t \leqq \mu_2\\
&\leqq (3\mu_2 - 4)/[4(3\alpha^2 - 4\alpha) + 3\mu_2], && t \geqq \mu_2,
\end{aligned}
$$

where α is the unique root $\geqq t/2$ of $t = 16\alpha^2(\alpha - 1)/[4(3\alpha^2 - 4\alpha) + 3\mu_2]$;

$$
(6.2)\quad
\begin{aligned}
1 - F(t) &\geqq (2 - t)^2/(3\mu_2 - 2t), && 0 \leqq t \leqq 2\\
&\geqq 0, && t > 2.
\end{aligned}
$$

Figures 6.1, 6.2, and 6.3 show the upper and lower bounds of Chebyshev (1.1), (1.2) together with their IHR improvements given in Corollary 3.2 and Theorem 3.3. The striking improvement in the IHR case with $\mu_2 = 1.8$ is partially explained by the fact that if F is IHR with $\mu_1 = 1$ and $\mu_2 = 2$, then F is exponential.

Figure 6.4 for $\mu_1 = 1$, $\mu_2 = 3$ shows the sharp upper and lower bounds of Chebyshev ((1.1), (1.2)), their improvements in case f is decreasing (F is concave) on $[0, \infty)$, ((6.1), (6.2)), and their further improvements in case F is DHR, given in Theorems 4.5 and 4.6 (recall that F DHR implies F concave).

In case f is PF_2, $\zeta(x) = x$ and $\mu_1 = 1$ the bound given in (5.2) has been graphed in Figure 6.1 of I.

REFERENCES

BARLOW, R. E., MARSHALL, A. W. and PROSCHAN, F. (1963). Properties of distributions with monotone hazard rate. *Ann. Math. Statist.* **34** 375–389.

BARLOW, R. E. and MARSHALL, A. W. (1963). Tables of bounds for distributions with monotone hazard rate. Boeing Scientific Research Laboratories Document Dl-82-0249.

BARLOW, R. E. and MARSHALL, A. W. (1964). Bounds for distributions with monotone hazard rate, I. *Ann. Math. Statist.* **35** 1234–1257.

CHEBYSHEV, P. L. (1874). Sur les valeurs limités des intégrales. *J. Math. Pures Appl.* (2) **19** 157–160.

KARLIN, S., PROSCHAN, F. and BARLOW, R. E. (1961). Moment inequalities of Pólya frequency functions. *Pacific J. Math.* **11** 1023–1033.

LOÈVE, M. (1960). *Probability Theory*, (2nd ed.). Van Nostrand, Princeton.

MALLOWS, C. L. (1956). Generalizations of Tchebycheff's inequalities. *J. Roy. Statist. Soc. Ser. B* **18** 139–176.

ROYDEN, H. L. (1953). Bounds on a distribution function when its first n moments are given. *Ann. Math. Statist.* **24** 361–376.

SCHOENBERG, I. J. (1951). On Pólya frequency functions. *J. Analyse Math.* **1** 331–374.