A MONOTONICITY PROPERTY OF THE POWER FUNCTIONS OF SOME TESTS OF THE EQUALITY OF TWO COVARIANCE MATRICES[1]

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1. Summary. Invariant tests of the hypothesis that $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2$ are based on the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$, say $c_1 \geqq c_2 \geqq \cdots \geqq c_p$, where $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ and $\mathbf{S}_1$ and $\mathbf{S}_2$ are the population and sample covariance matrices, respectively, of two multivariate normal populations; the power of such a test depends on the characteristic roots of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$. It is shown that the power function is an increasing function of each ordered root of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ if the acceptance region of the test has the property that if $(c_1, \cdots, c_p)$ is in the region then any point with coordinates not greater than these, respectively, is also in the region. Examples of such acceptance regions are given. For testing the hypothesis that $\mathbf{\Sigma} = \mathbf{I}$, a similar sufficient condition is derived for a test depending on the roots of a sample covariance matrix $\mathbf{S}$, based on observations from a normal distribution with covariance matrix $\mathbf{\Sigma}$, to have the power function monotonically increasing in each root of $\mathbf{\Sigma}$.

2. Tests of the equality of two covariance matrices. Samples of size N_1 and N_2 are drawn from $N(\boldsymbol{\mu}^{(1)}, \mathbf{\Sigma}_1)$ and $N(\boldsymbol{\mu}^{(2)}, \mathbf{\Sigma}_2)$, respectively, where $N(\boldsymbol{\mu}^{(j)}, \mathbf{\Sigma}_j)$ denotes a p-variate nonsingular normal distribution with mean vector $\boldsymbol{\mu}^{(j)}$ and covariance matrix $\mathbf{\Sigma}_j$. On the basis of these data we wish to test the null hypothesis

$$H_0: \mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 .$$

Since the null hypothesis is invariant with regard to transformations

$$\mathbf{X}^{(1)} \rightarrow \mathbf{A}\mathbf{X}^{(1)} + \mathbf{b}^{(1)}, \qquad \mathbf{X}^{(2)} \rightarrow \mathbf{A}\mathbf{X}^{(2)} + \mathbf{b}^{(2)}, \tag{2.1}$$

where $\mathbf{X}^{(j)}$ is distributed according to $N(\boldsymbol{\mu}^{(j)}, \mathbf{\Sigma}_j)$ and $\mathbf{A}$ is any nonsingular matrix and $\mathbf{b}^{(1)}$ and $\mathbf{b}^{(2)}$ are any vectors, we ask for test procedures that are invariant with regard to transformations (2.1). A minimal sufficient set of statistics consists of the sample mean vectors and the sample covariance matrices $\bar{\mathbf{x}}^{(1)}$, $\mathbf{S}_1$ and $\bar{\mathbf{x}}^{(2)}$, $\mathbf{S}_2$ based on the samples from $N(\boldsymbol{\mu}^{(1)}, \mathbf{\Sigma}_1)$ and $N(\boldsymbol{\mu}^{(2)}, \mathbf{\Sigma}_2)$, respectively. Any invariant test depending on this sufficient set of statistics depends only on the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$.

The power of any invariant test depends on the parameters only through the characteristic roots of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$, say $\gamma_1 \geqq \gamma_2 \geqq \cdots \geqq \gamma_p$, which are the roots of $|\mathbf{\Sigma}_1 - \gamma\mathbf{\Sigma}_2| = 0$. The null hypothesis can then be stated as

$$H_0: \gamma_1 = \cdots = \gamma_p = 1.$$

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In this paper we are interested in tests against alternatives

$$H_1: \gamma_i \geqq 1, \quad i = 1, \cdots, p; \qquad \sum_{i=1}^{p} \gamma_i > p,$$

or

$$H_2: \gamma_i \leqq 1, \quad i = 1, \cdots, p; \qquad \sum_{i=1}^{p} \gamma_i < p.$$

The alternatives H_1, for example, can also be defined as pairs of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ such that $\mathbf{a}'\mathbf{\Sigma}_1\mathbf{a}/\mathbf{a}'\mathbf{\Sigma}_2\mathbf{a} \geqq 1$ for every nontrivial p-vector $\mathbf{a}$, or equivalently pairs of normal distributions such that the variance of $\mathbf{a}'\mathbf{X}^{(1)}$ is at least as great as the variance of $\mathbf{a}'\mathbf{X}^{(2)}$ for every vector $\mathbf{a}$.

3. The monotonicity property. We first consider the problem of testing H_0 against H_1. The nature of the alternative hypotheses suggests that a good test procedure is one for which the null hypothesis is rejected if the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ are large. The tests we consider have acceptance regions such that if a pair of $\mathbf{S}_1$ and $\mathbf{S}_2$ leads to acceptance of H_0 so does another pair with each characteristic root no greater than the corresponding root of the first pair. The main result of this paper is that such a test has a power that increases as each ordered root of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ increases. For the dual problem of testing H_0 against H_2 we consider the test procedures having the above acceptance regions as rejection regions; then the power of such a test will decrease as each ordered root of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ increases.

Let $\mathrm{ch}_1(\mathbf{A}) \geqq \cdots \geqq \mathrm{ch}_p(\mathbf{A})$ denote the characteristic roots of the $p \times p$ matrix $\mathbf{A}$, and let $\mathrm{ch}(\mathbf{A}) = [\mathrm{ch}_1(\mathbf{A}), \cdots, \mathrm{ch}_p(\mathbf{A})]$. The main result of this paper is proved using the following theorem:

THEOREM 1. *Let* $\mathbf{X}: p \times n (n \geqq p)$ *be a random matrix having density*

$$f(\mathbf{X}; \mathbf{\Sigma}, n) = (2\pi)^{-\frac{1}{2}np} |\mathbf{\Sigma}|^{-\frac{1}{2}n} \exp\left[-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}^{-1} \mathbf{X}\mathbf{X}'\right], \tag{2.2}$$

where $\mathbf{\Sigma}$ *is positive definite. Let* $c_1 \geqq c_2 \geqq \cdots \geqq c_p$ *be the characteristic roots of* $\mathbf{X}\mathbf{X}'$ *and* ω *be a set in the space of* $c_1, \cdots, c_p$ *such that when a point* $(c_1, \cdots, c_p)$ *is in* ω *so is every point* $(\bar{c_1}, \cdots, \bar{c_p})$ *for* $\bar{c_i} \leqq c_i$ $(i = 1, \cdots, p)$. *Then the probability of the set* ω *depends on* $\mathbf{\Sigma}$ *only through* $\mathrm{ch}(\mathbf{\Sigma})$ *and is a monotonically decreasing function of each of the characteristic roots of* $\mathbf{\Sigma}$.

The proof of the theorem involves the following two lemmas:

LEMMA 1. *The distribution of* $\mathrm{ch}(\mathbf{X}\mathbf{X}')$, *where* $\mathbf{X}$ *has the density* $f(\mathbf{X}; \mathbf{\Sigma}, n)$, *is the same as the distribution of* $\mathrm{ch}\,[(\mathbf{\Delta}\mathbf{Y})(\mathbf{\Delta}\mathbf{Y})']$, *where* $\mathbf{Y}$ *has the density* $f(\mathbf{Y}; \mathbf{I}_p, n)$ *and* $\mathbf{\Delta}$ *is the diagonal matrix having* $\mathrm{ch}_1^{\frac{1}{2}}(\mathbf{\Sigma}), \cdots, \mathrm{ch}_p^{\frac{1}{2}}(\mathbf{\Sigma})$ *as its diagonal elements.*

PROOF. There exists a $p \times p$ orthogonal matrix $\mathbf{L}$ such that $\mathbf{\Sigma} = \mathbf{L}\mathbf{\Delta}^2\mathbf{L}'$. It can be seen that the density of $\mathbf{Y} = \mathbf{\Delta}^{-1}\mathbf{L}'\mathbf{X}$ is $f(\mathbf{Y}; \mathbf{I}_p, n)$, and

$$\mathrm{ch}(\mathbf{X}\mathbf{X}') = \mathrm{ch}\,[(\mathbf{L}\mathbf{\Delta}\mathbf{Y})(\mathbf{L}\mathbf{\Delta}\mathbf{Y})'] = \mathrm{ch}\,[(\mathbf{\Delta}\mathbf{Y})(\mathbf{\Delta}\mathbf{Y})'].$$

LEMMA 2. *Let* $\mathbf{A}$ *be a positive definite matrix of order* p, *and* $\mathbf{D}$ *and* $\mathbf{D}^*$ *be two diagonal matrices of order* p *such that* $\mathbf{D}^* - \mathbf{D}$ *is positive semidefinite, and* $\mathbf{D}$ *is* *positive definite. Then*

$$\mathrm{ch}_i\,(\mathbf{DAD}) \leqq \mathrm{ch}_i\,(\mathbf{D}^*\mathbf{AD}^*), \qquad i = 1, 2, \cdots, p.$$

The above lemma follows from Corollary 2.2.1 of [2].

PROOF OF THEOREM 1. Let $\mathbf{\Sigma}^*$ be a positive definite matrix of order p such that $\mathrm{ch}_i\,(\mathbf{\Sigma}^*) \geqq \mathrm{ch}_i\,(\mathbf{\Sigma})$, $i = 1, \cdots p$, and let $\mathbf{\Delta}^*$ denote the diagonal matrix having $\mathrm{ch}_1^{\frac{1}{2}}\,(\mathbf{\Sigma}^*), \cdots, \mathrm{ch}_p^{\frac{1}{2}}\,(\mathbf{\Sigma}^*)$ as its diagonal elements. Let $A(\mathbf{X})$ denote the region $[\mathbf{X}:\mathrm{ch}(\mathbf{XX}')\ \varepsilon\ \omega]$. It follows from Lemma 1 that

$$\Pr(\omega; \mathbf{\Sigma}) = \int_{A(\mathbf{X})} f(\mathbf{X}; \mathbf{\Sigma}, n)\, d\mathbf{X} = \int_{B(\mathbf{Y},\mathbf{\Delta})} f(\mathbf{Y}; \mathbf{I}_p, n)\, d\mathbf{Y},$$

where $B(\mathbf{Y}, \mathbf{\Delta})$ is the set $[\mathbf{Y}:\mathrm{ch}(\mathbf{\Delta YY}'\mathbf{\Delta})\ \varepsilon\ \omega]$. From Lemma 2, we have for any $i(i = 1, \cdots, p)$

$$\mathrm{ch}_i\,[\mathbf{\Delta}(\mathbf{YY}')\mathbf{\Delta}] \leqq \mathrm{ch}_i\,[\mathbf{\Delta}^*(\mathbf{YY}')\mathbf{\Delta}^*].$$

If we write $c_i = \mathrm{ch}_i\,[(\mathbf{\Delta}^*\mathbf{Y})(\mathbf{\Delta}^*\mathbf{Y})']$ and $c_i^- = \mathrm{ch}_i\,[(\mathbf{\Delta Y})(\mathbf{\Delta Y})']$, then it follows from the structure of the set ω that $B(\mathbf{Y}, \mathbf{\Delta}) \supset B(\mathbf{Y}, \mathbf{\Delta}^*)$. Hence $\Pr\,(\omega; \mathbf{\Sigma}) \geqq \Pr\,(\omega; \mathbf{\Sigma}^*)$.

Consider random matrices $\mathbf{U}_1:p \times n_1$ and $\mathbf{U}_2:p \times n_2$ which are independently distributed with densities $f(\mathbf{U}_1\,;\,\mathbf{\Sigma}_1\,,\,n_1)$ and $f(\mathbf{U}_2\,;\,\mathbf{\Sigma}_2\,,\,n_2)$, respectively. The sample covariance matrices $\mathbf{S}_1$ and $\mathbf{S}_2$, as defined in Section 2, can be written as $n_i\mathbf{S}_i = \mathbf{U}_i\mathbf{U}_i'$ $(i = 1, 2)$ with normal densities. Thus any invariant test depends only on the characteristic roots of $(\mathbf{U}_1\mathbf{U}_1')(\mathbf{U}_2\mathbf{U}_2')^{-1}$, say $c_1 \geqq c_2 \geqq \cdots \geqq c_p$. We use $n_i = N_i - 1$ $(i = 1, 2)$.

THEOREM 2. *Let ω be a set in the space of the characteristic roots of* $(\mathbf{U}_1\mathbf{U}_1')(\mathbf{U}_2\mathbf{U}_2')^{-1}$ *satisfying the condition stated in Theorem 1. Then the probability of the set ω depends on $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ only through* $\mathrm{ch}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})$ *and is a monotonically decreasing function of each of the characteristic roots of* $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$.

First we prove the following lemma:

LEMMA 3. *The distribution of the characteristic roots of* $(\mathbf{U}_1\mathbf{U}_1')(\mathbf{U}_2\mathbf{U}_2')^{-1}$ *when* $\mathbf{U}_1$ *and* $\mathbf{U}_2$ *are independently distributed with densities* $f(\mathbf{U}_1\,;\,\mathbf{\Sigma}_1\,,\,n_1)$ *and* $f(\mathbf{U}_2\,;\,\mathbf{\Sigma}_2\,,\,n_2)$, *respectively, is the same as the distribution of the characteristic roots of* $(\mathbf{V}_1\mathbf{V}_1')(\mathbf{V}_2\mathbf{V}_2')^{-1}$ *when* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are independently distributed with densities* $f(\mathbf{V}_1\,;\,\mathbf{\Gamma}, n_1)$ *and* $f(\mathbf{V}_2\,;\,\mathbf{I}_p\,,\,n_2)$, *respectively, where $\mathbf{\Gamma}$ is the diagonal matrix with* $\mathrm{ch}_i\,(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})$ *as its ith diagonal element.*

PROOF. There exists a $p \times p$ nonsingular matrix $\mathbf{L}$ such that $\mathbf{\Sigma}_1 = \mathbf{L\Gamma L}'$, $\mathbf{\Sigma}_2 = \mathbf{LL}'$. It can be seen that $\mathbf{V}_1 = \mathbf{L}^{-1}\mathbf{U}_1$ and $\mathbf{V}_2 = \mathbf{L}^{-1}\mathbf{U}_2$ are independently distributed with densities $f(\mathbf{V}_1\,;\,\mathbf{\Gamma}, n_1)$ and $f(\mathbf{V}_2\,;\,\mathbf{I}_p\,,\,n_2)$, respectively, and

$$\mathrm{ch}\,[(\mathbf{U}_1\mathbf{U}_1')(\mathbf{U}_2\mathbf{U}_2')^{-1}] = \mathrm{ch}\,[(\mathbf{V}_1\mathbf{V}_1')(\mathbf{V}_2\mathbf{V}_2')^{-1}].$$

PROOF OF THEOREM 2. It follows from Lemma 3 that

$$\int_{Q(\mathbf{U}_1,\mathbf{U}_2)} f(\mathbf{U}_1; \mathbf{\Sigma}_1, n_1) f(\mathbf{U}_2; \mathbf{\Sigma}_2, n_2)\, d\mathbf{U}_1\, d\mathbf{U}_2$$
$$= \int_{Q(\mathbf{V}_1,\mathbf{V}_2)} f(\mathbf{V}_1; \mathbf{\Gamma}, n_1) f(\mathbf{V}_2; \mathbf{I}_p, n_2)\, d\mathbf{V}_1\, d\mathbf{V}_2 = \Pr(\omega; \mathbf{\Gamma}),$$

say, where $Q(\mathbf{U}_1, \mathbf{U}_2)$ denotes the region $[\mathbf{U}_1, \mathbf{U}_2 : \text{ch}\{(\mathbf{U}_1\mathbf{U}_1')(\mathbf{U}_2\mathbf{U}_2')^{-1}\} \, \varepsilon \, \omega]$. Consider $\mathbf{V}_2$ as fixed, and let $(\mathbf{V}_2\mathbf{V}_2')^{-1} = \mathbf{T}'\mathbf{T}$, where $\mathbf{T}$ is nonsingular. Then the density of $\mathbf{W} = \mathbf{T}\mathbf{V}_1$ is $f(\mathbf{W}; \mathbf{T\Gamma T}', n)$, and $\text{ch}\,\{(\mathbf{V}_1\mathbf{V}_1')(\mathbf{V}_2\mathbf{V}_2')^{-1}\} = \text{ch}\,(\mathbf{WW}')$. Thus for any fixed $\mathbf{V}_2$, we have

$$\int_{R(\mathbf{V}_1)} f(\mathbf{V}_1; \mathbf{\Gamma}, n_1)\, d\mathbf{V}_1 = \int_{A(\mathbf{W})} f(\mathbf{W}; \mathbf{T\Gamma T}', n_1)\, d\mathbf{W}, \tag{2.3}$$

where $R(\mathbf{V}_1)$ denotes the region $[\mathbf{V}_1 ; \text{ch}\{(\mathbf{V}_1\mathbf{V}_1')(\mathbf{V}_2\mathbf{V}_2')^{-1}\} \, \varepsilon \, \omega]$.

Let $\mathbf{\Gamma}^*$ be a diagonal matrix such that $\mathbf{\Gamma}^* - \mathbf{\Gamma}$ is positive semidefinite. It follows from Lemma 2 that

$$\text{ch}_i\,[\mathbf{T\Gamma}^*\mathbf{T}'] = \text{ch}_i\,[\mathbf{\Gamma}^{*\frac{1}{2}}(\mathbf{T}'\mathbf{T})\mathbf{\Gamma}^{*\frac{1}{2}}] \geqq \text{ch}_i\,[\mathbf{\Gamma}^{\frac{1}{2}}(\mathbf{T}'\mathbf{T})\mathbf{\Gamma}^{\frac{1}{2}}] = \text{ch}_i\,[\mathbf{T\Gamma T}'].$$

From Theorem 1 and (2.3), we have

$$\int_{R(\mathbf{V}_1)} f(\mathbf{V}_1; \mathbf{\Gamma}, n_1)\, d\mathbf{V}_1 \geqq \int_{R(\mathbf{V}_1)} f(\mathbf{V}_1; \mathbf{\Gamma}^*, n_1)\, d\mathbf{V}_1,$$

for any fixed $\mathbf{V}_2$. Multiplying both the sides of the above inequality by $f(\mathbf{V}_2 ; \mathbf{I}_p, n_2)$ and integrating with respect to $\mathbf{V}_2$, we have $\Pr(\omega; \mathbf{\Gamma}) \geqq \Pr(\omega; \mathbf{\Gamma}^*)$.

COROLLARY 1. *If an invariant test has an acceptance region such that if* $(c_1, \cdots, c_p)$ *is in the region, so is* $(c_1^-, \cdots, c_p^-)$ *for* $c_i^- \leqq c_i$, *then the power of the test is a monotonically increasing function of each* γ_i.

COROLLARY 2. *The cumulative distribution function of* $c_{i_1}, c_{i_2}, \cdots, c_{i_k}$, *where* $(i_1, \cdots, i_k)$ *is a subset of* $(1, 2, \cdots, p)$, *is a monotonically decreasing function of each* γ_i.

COROLLARY 3. *If* $g(c_1, \cdots, c_p)$ *is monotonically increasing in each of its arguments, a test with acceptance region* $g(c_1, \cdots, c_p) \leqq \mu$ *has a monotonically increasing power function in each* γ_i.

In particular, Corollary 3 includes tests with acceptance regions $\sum_{i=1}^p d_i T_i \leqq a$, where $d_i \geqq 0$ and T_i is the sum of all different products of $c_1, \cdots, c_p$ taken i at a time. Special cases of the above regions are

$$(n_2/n_1)^p \prod_{i=1}^p c_i = |\mathbf{S}_1|/|\mathbf{S}_2| \leqq a,$$

and

$$(n_2/n_1) \sum_{i=1}^p c_i = \text{tr}\,(\mathbf{S}_1\mathbf{S}_2^{-1}) \leqq a.$$

It can also be seen that Corollary 3 includes tests with acceptance regions $\sum_{i,j=1}^p a_{ij} W_{ij} \leqq \mu$, where $a_{ij} \geqq 0$, and $W_{ij} = T_i/T_j$ $(i > j)$. The two tests proposed by Roy [4] having acceptance regions $c_1 \leqq a_1$ and $c_p \leqq a_p$, respectively, are also special cases of Corollary 3. The monotonicity properties of these two tests were announced by Mikhail [3], but his proof is not complete.

4. Discussions. If $\mathbf{X}^{(1)*}$ is distributed as $\mathbf{X}^{(1)} + \mathbf{Y}$, where $\mathbf{Y}$ is normally distributed, independently of $\mathbf{X}^{(1)}$, with covariance matrix $\mathbf{\Psi}$, then $\mathbf{\Sigma}_1^* = \mathbf{\Sigma}_1 + \mathbf{\Psi}$

and $\text{ch}_i(\mathbf{\Sigma}_1^*\mathbf{\Sigma}_2^{-1}) \geqq \text{ch}_i(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})$, [2]. The implication is that if the first distribution becomes more spread out, there is greater probability of discovering it by a test considered in Section 3.

The tests treated here are analogues of one-sided tests of the univariate problem $\sigma_1^2 = \sigma_2^2$. It is much more difficult to investigate tests with reasonable power against all alternatives. A modification of the likelihood-ratio test (by replacing N_i by $n_i = N_i - 1$) has the acceptance region

$$\frac{|n_1\mathbf{S}_1|^{n_1}\,|n_2\mathbf{S}_2|^{n_2}}{|n_1\mathbf{S}_1 + n_2\mathbf{S}_2|^{n_1+n_2}} \geqq k,$$

a constant ([1], p. 249). It is difficult even to determine whether this test is unbiased.

5. Tests that a covariance matrix is a specified matrix. The development of Section 3 can also be applied to testing the hypothesis $\mathbf{\Sigma} = \mathbf{I}$ when a sample of size $n + 1$ is drawn from $N(\boldsymbol{\mu}, \mathbf{\Sigma})$. (The hypothesis $\mathbf{\Sigma} = \mathbf{\Sigma}_0$ can be transformed into this.) Since the hypothesis is invariant under orthogonal transformations and changes of location, we ask for procedures that are invariant under these transformations. Such procedures are based on $\text{ch}(\mathbf{S})$, where $\mathbf{S}$ is the sample covariance matrix. The matrix $n\mathbf{S}$ is distributed as $\mathbf{XX}'$ in Theorem 1. It follows from Theorem 1 that any invariant test with acceptance region such that if $(c_1, \cdots, c_p)$ is in the region so also is $(c_1^-, \cdots, c_p^-)$ with $c_i^- \leqq c_i$ has a power function that is an increasing function of each characteristic root of $\mathbf{\Sigma}$.

Remark. It can be seen from the proofs of Lemma 1 and Lemma 3 that we did not use the explicit form (2.2) of $f(\mathbf{X}; \mathbf{\Sigma}, n)$, but we have only used the property of $f(\mathbf{X}; \mathbf{\Sigma}, n)$ that if $\mathbf{X}$ has the density $f(\mathbf{X}; \mathbf{\Sigma}, n)$ then $\mathbf{Y} = \mathbf{AX}$ has the density of $f(\mathbf{Y}; \mathbf{A\Sigma A}', n)$ for any nonsingular matrix $\mathbf{A}$. Thus instead of assuming (2.2), if we assume the above property of $f(\mathbf{X}; \mathbf{\Sigma}, n)$, then Theorem 1 and Theorem 2 will still hold. (That the probability of the set ω depends on $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ only through $\text{ch}(\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1})$ was indicated by Anderson, [1], p. 259, and Roy, [4], p. 188, in the case of normality.) If the column vectors of $\mathbf{X}$ are assumed to be independently and identically distributed, then a necessary and sufficient condition for $f(\mathbf{X}; \mathbf{\Sigma}, n)$ to have the above property is that the common density function of the column vectors of $\mathbf{X}$ can be expressed as

$$p(\mathbf{x}; \mathbf{\Sigma}) = |\mathbf{\Sigma}|^{-\frac{1}{2}} g(\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x}).$$

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