RANK TESTS FOR PAIRED-COMPARISON EXPERIMENTS INVOLVING SEVERAL TREATMENTS[1]

By K. L. Mehra[2]

University of California, Berkeley

1. Introduction and Summary. Two nonparametric tests presently available for testing the equality of several treatments (varieties, objects etc.) on the basis of paired-comparisons are: (a) the Bradley-Terry [4] likelihood ratio test; (b) Durbin's [7] test (which, in fact, covers the general balanced incomplete block design). These tests, however, were proposed for the case when no meaningful measurements on the quality of treatments are possible. Instead, one can merely decide for each individual comparison which treatment to prefer. Both these tests are, thus, instances where only signs of the comparison differences are involved, and as such can be regarded as generalizations of the Sign-test. The large sample properties of these tests have been discussed by Bradley [3] and by Van Elteren and Noether [17] respectively. As shown in the latter paper, both these tests have asymptotic (Pitman) efficiency equal to $2/\pi$ relative to the F-test (under normality). One can reasonably hope to improve this efficiency by taking into consideration the magnitudes of the observed comparison differences, when they are available.

A test of this nature, based on a generalization of the Wilcoxon-one-sample ranking procedure, is proposed and investigated below for the case when all comparisons are performed under the same experimental conditions (Section 2). The asymptotic distribution of the test statistic proposed is obtained using the results of Godwin and Zaremba [8] and Konijn [12] (Section 3). It is shown that the asymptotic efficiencies of this test relative to the Durbin and the Bradley-Terry tests and the corresponding F-test are independent of the number of treatments involved. These results are also extended to the case of non-uniformity of experimental conditions (Section 4).

(The attention of the reader is also drawn to a rank procedure suggested by Hodges and Lehmann [11] for the general incomplete block design which, in particular, is also applicable to the present problem. However, the efficiency of this procedure has so far not been fully investigated.)

2. Mathematical model and the test. Consider a paired-comparison experiment involving K treatments and suppose that each of the N_{ij} comparisons for a pair (i, j) of treatments $(1 \leqq i < j \leqq K)$ provides an observed comparison difference $Z_{ijl}\,(l = 1, 2, \cdots, N_{ij})$. Let $G_{ij}(z)$ be a c.d.f. denoting their common distribution and assume that G_{ij} is continuous. The hypothesis of no-difference

Received 21 July 1962; revised 22 August 1963.

[1] This investigation was supported (in part) by a research grant (No. RG-3666) from the National Institutes of Health, Public Health Service.

[2] Now at the University of Alberta.

among the treatments can be expressed as $H_0: G_{ij}(z) + G_{ij}(-z) = 1$ and $G_{ij}(z) = G_{i'j'}(z)$ for any two pairs (i, j) and (i', j'). An alternative statement of the null hypothesis would be more appropriate in the following situation: Let F_i $(i = 1, 2, \cdots, K)$ be a continuous c.d.f. denoting the underlying distribution of the ith treatment effect and assume that the random variables X_{il}, X_{jl}, which measure the characteristics corresponding to the (i, j)th pair and lth comparison have distributions F_{il}, F_{jl} satisfying

$$(2.1) \qquad \Omega: F_{il}(x) = F_i(x + \eta_l), \qquad F_{jl}(x) = F_j(x + \eta_l)$$

where η_l is an unknown constant. The hypothesis of no-difference can be stated as $H_0': F_1 = F_2 = \cdots = F_K$. The differences $Z_{ijl} = X_{il} - X_{jl}$ $(l = 1, 2, \cdots, N_{ij})$ for each pair (i, j) have identical distribution, which in case of independence between X_{il}, X_{jl} is given by $G_{ij}(x) = P[Z_{ijl} \leqq x] = \int F_i(x + y)\, dF_j(y)$. Still another form of the null hypothesis would arise when the N_{ij} comparisons for the pair (i, j) of treatments are supposed to provide vector observations; namely, (X_{il}, X_{jl}) $(l = 1, 2, \cdots N_{ij})$ from a certain continuous bivariate distribution $D_{ij}(u, v)$. The hypothesis of interest would then postulate that each distribution $D_{ij}(u, v)$ is symmetrical about the line $u = v$ and that all distributions $D_{ij}(u, v)$ $(1 \leqq i < j \leqq K)$ are identical. The last two problems can be reduced through invariance to that of testing the hypothesis H_0 on the basis of the observed comparison differences Z's by arguments similar to those given in Lehmann ([14], p. 234) for the case $K = 2$. In certain experiments, paired-comparisons become necessary because the experimental units, on which the treatment comparisons might be based, are available only in natural groups of size two. It may be observed that if each pair of experimental units is considered to form a block, the assumptions (2.1) would then correspond to the case of zero interactions between the blocks and the treatments in the usual analysis of variance model.

We now describe the test: Rank the $N = \sum_{i=1}^{K} \sum_{j>i} N_{ij}$ absolute values of the observed comparison differences Z_{ijl} $(1 \leqq i < j \leqq K;\ l = 1, 2, \cdots N_{ij})$. Let r_{ijl} = rank of $|Z_{ijl}|$ if $Z_{ijl} > 0$, otherwise let $r_{ijl} = 0$; similarly let s_{ijl} = rank of $|Z_{ijl}|$ if $Z_{ijl} < 0$, otherwise let $s_{ijl} = 0$. Then $R_N^{(i,j)} = \sum_{l=1}^{N_{ij}} r_{ijl}$ and $S_N^{(i,j)} = \sum_{l=1}^{N_{ij}} s_{ijl}$ are respectively the sums of the ranks of the positive and the negative Z's corresponding to the (i, j)th pair. The test statistic proposed for the hypothesis H_0(or H_0') is

$$(2.2) \qquad L = [6/(N + 1)(2N + 1)K] \sum_{i=1}^{K} \{\sum_{j \neq i} (V_N^{(i,j)}/N_{ij}^{\frac{1}{2}})\}^2$$

where N is the total number of comparisons in the experiment and $V_N^{(i,j)} = R_N^{(i,j)} - S_N^{(i,j)}$, (we observe here that $R_N^{(i,j)} = S_N^{(j,i)}$, so that $V_N^{(i,j)} = -V_N^{(j,i)}$). The test consists in rejecting H_0 at level α if the statistic L exceeds a predetermined number L_α, with $P_{H_0}[L \geqq L_\alpha] = \alpha$. For $K = 2$, this test reduces to the two-sided Wilcoxon paired-comparison test. By following the method of proof of Theorem 3.1 of the next section and using Theorem 4W of [13], it follows (see

Theorem 3.2) that, when H_0 is true, L is asymptotically distributed as a χ^2 variable with $(K - 1)$ degrees of freedom. This provides a large sample approximation of the critical point L_α. The following lemma is useful in connection with the large sample approximation for the distribution of L.

LEMMA 2.1. *When H_0 is true, $E(L) = K - 1 = E\{\chi^2$ variable with $(K - 1)$ degrees of freedom$\}$.*

PROOF. From (2.2) we have

$$E(L) = [6/(N + 1)(2N + 1)K]\sum_{i=1}^{K} \{\sum_{j\neq i} \sigma_{ij}\} \tag{2.3}$$

where σ_{ij} = variance $(V_N^{(i,j)}/N_{ij}^{\frac{1}{2}})$. In the above expression, the product terms vanish since, under the null hypothesis, $E(V_N^{(i,j)}) = 0$ and $V_N^{(i,j)}/N_{ij}^{\frac{1}{2}}$ and $V_N^{(i',j')}/N_{i'j'}^{\frac{1}{2}}$ for any two pairs (i, j) and (i', j') are uncorrelated. Let m_{ij} denote the number of positive Z's corresponding to the (i, j)th pair, and let $\tilde{E}(\cdot)$ and $\tilde{\text{Var}}(\cdot)$ stand, respectively, for the conditional expectation and the conditional variance given m_{ij} $(1 \leqq i < j \leqq K)$. Then

$$\sigma_{ij} = (1/N_{ij})[2 \text{ Var}(R_N^{(i,j)}) + 2 \text{ Var}(S_N^{(i,j)}) - \text{Var}(R_N^{(i,j)} + S_N^{(i,j)})]$$

where

$$\begin{aligned}\text{Var}(R_N^{(i,j)}) &= \text{Var}(\tilde{E}R_N^{(i,j)}) + E(\tilde{\text{Var}}\, R_N^{(i,j)}) \\ &= \text{Var}(m_{ij}(N + 1)/2) + E((N - m_{ij})m_{ij}(N + 1)/12) \\ &= [N_{ij}(N + 1)/48](5N + 2 - N_{ij}), \text{Var}(R_N^{(i,j)}) = \text{Var}(S_N^{(i,j)})\end{aligned}$$

and

$$\text{Var}(R_N^{(i,j)} + S_N^{(i,j)}) = N_{ij}(N - N_{ij})(N + 1)/12.$$

The result now follows by substituting the value of σ_{ij} in (2.3).

Another test statistic for the hypothesis H_0 (suggested by certain related test statistics) could be formed by considering the variables $R_i = \sum_{j\neq i} \{R_N^{(i,j)} - ER_N^{(i,j)}\}$ $(i = 1, 2, \cdots, K)$. By following the method of proof of Theorem 3.1, it can be shown that, under H_0 and for all $N_{ij} = n$, the statistic

$$[192/(5K - 2)K^3(K - 1)^2n^3]\sum_{i=1}^{K} \{\sum_{j\neq i}(R_N^{(i,j)} - ER_N^{(i,j)})\}^2 \tag{2.4}$$

has asymptotically, as $n \rightarrow \infty$, a χ^2 distribution with $(K - 1)$ degrees of freedom. The relative asymptotic efficiency of this statistic with respect to the statistic (2.2) (for shift alternatives K_N of Section 3) was found to be less than 1 for all K and all distributions G. For this reason the statistic (2.4) is not considered further.

3. The asymptotic distribution of L under translation-type alternatives. In this section, we obtain the limiting distribution of the statistic L under a sequence of translation alternatives approaching H_0. This will provide an approximation to the asymptotic power of the test for small translations and also enable us to obtain its asymptotic efficiency relative to other competing tests.

Consider for each fixed N, the alternative hypothesis K_N': $\Omega \cap \{F_i(x) = F(x + \theta_i N^{-\frac{1}{2}})$ for all $i = 1, 2, \cdots, K$ such that for some pair (i, j) $\theta_i \neq \theta_j\}$. It is easily seen that, under K_N', for each pair (i, j) the distribution of the variables Z_{ijl} $(l = 1, 2, \cdots, N_{ij})$ is given by

$$K_N: G_{ij}(x) = G(x + (\theta_i - \theta_j)N^{-\frac{1}{2}})$$

for all (i, j) $(1 \leqq i < j \leqq K)$, where $G(x) = \int F(x + y)\, dF(y)$. We observe that the distribution $G(x)$ is symmetric about the origin, and, thus, in the changed context of the distributions $G_{ij}(x)$, the shift alternatives can be directly defined by K_N, with $G(x)$ as any distribution symmetric about the origin. We now prove

THEOREM 3.1. *Suppose, for each index N, that K_N (or K_N') holds and that the distribution function $G(x)$ satisfies the following*:

(i) *$G(x)$ possesses a continuous derivative $G'(x) = g(x)$ except possibly on a set S with $\int_S dG = 0$.*

(ii) *There exists a function $b(x)$, such that*

$$|[G(x + \theta) - G(x)]/\theta| \leqq b(x) \quad \text{and} \quad \int b(x)\, dG(x) < \infty.$$

(iii) *$g(x)$ is continuous at $x = 0$.*

Assume further that, as $N \to \infty$,

(iv) *$\lim_{N\to\infty} N_{ij}/N = \nu_{ij}$ exists and is positive for all pairs (i, j) $(1 \leqq i < j \leqq K)$. Then the statistic L, defined by (2.2), has a limiting noncentral χ^2 distribution with $(K - 1)$ degrees of freedom and the noncentrality parameter.*

$$\Delta^2 = \frac{12}{K}\left(\int g^2(y)\, dy\right)^2 \sum_{i=1}^{k} \{\sum_{j\neq i} \nu_{ij}^{\frac{1}{2}}(\theta_i - \theta_j)\}^2. \tag{3.1}$$

The proof of this theorem requires two lemmas. To state the first of these we set out some notation: Set $X_{\alpha 1}, X_{\alpha 2}, \cdots X_{\alpha n_\alpha}$ $(\alpha = 1, 2, \cdots C)$ be C independent samples of sizes n_α from distributions F_α $(\alpha = 1, 2, \cdots C)$. Let $\delta(J_\beta, J_\alpha) = 1$ or zero according as $X_{\beta J_\beta} < X_{\alpha J_\alpha}$ or $X_{\beta J_\beta} \geqq X_{\alpha J_\alpha}$ and define

$$U_\alpha = \sum_{\substack{\beta=1\\ \neq\alpha}}^{C} \sum_{J_\alpha=1}^{n_\alpha} \sum_{J_\beta=1}^{n_\beta} \delta(J_\beta, J_\alpha). \tag{3.2}$$

Then if R_α denotes the sum of ranks of X_α's in a combined ranking of all X's

$$R_\alpha = n_\alpha(n_\alpha + 1)/2 + U_\alpha. \tag{3.3}$$

Let $N = \sum_{\alpha=1}^{C} n_\alpha$ and define $W_\alpha = N^{-\frac{3}{2}}\{U_\alpha - E(U_\alpha)\}$.

LEMMA 3.1. *Assume for each index N that $F_\alpha(x) = F(x + \theta_\alpha N^{-\frac{1}{2}})$ and that $\lim_{N\to\infty}(n_\alpha/N) = \nu_\alpha$ exists for each α $(\alpha = 1, 2, \cdots C)$. Then* (i) *the random vector $W = (W_1, W_2, \cdots, W_C)$ has in the limit, as $N \to \infty$, a multivariate normal* *distribution* $N(\mathbf{0}, \mathbf{\Sigma})$ *with* $\mathbf{\Sigma} = \|\sum_{\alpha\beta}\|$ *given by*

$$\sum_{\alpha\beta} = \tfrac{1}{12}(\delta_{\alpha\beta}\nu_\alpha - \nu_\alpha\nu_\beta)$$

where $\delta_{\alpha\beta}$ *is Kronecker's delta and* (ii) *the covariance matrix* $\mathbf{\Sigma}_N$ *of* W *converges to the covariance matrix* $\mathbf{\Sigma}$.

PROOF. The proof of this lemma follows by a direct application of a theorem of Godwin and Zaremba [8] (pp. 683–684) and computations on the same lines as in Andrews [1].

LEMMA 3.2. *Let* $X_{m(N)}$ *be a sequence of random variables* ($N = 1, 2, \cdots$) *such that*

(i) *for given* N, $m(N)$ *is a random variable such that, as* $N \to \infty$, $m(N)/N \to p_0$ *in probability, for some* $0 < p_0 < 1$.

(ii) *For non-random* $m(N)$

$$P\{[X_{m(N)} - \tilde{E}(X_{m(N)})]/\tilde{\sigma}(X_{m(N)}) < t\} \to \Phi(t)$$

as $m(N) \to \infty$, *where* $\tilde{E}(X_{m(N)})$ *and* $\tilde{\sigma}^2(X_{m(N)})$ *denote the conditional expectation and conditional variance of* $X_{m(N)}$, *given* $m(N)$, *respectively and* $\Phi(t)$ *is the standard normal distribution function.*

(iii) $$P\{[\tilde{E}(X_{m(N)}) - E(X_{m(N)})]/\sigma(\tilde{E}X_{m(N)}) < t\} \to \Phi(t)$$

as $N \to \infty$. *Then, as* $N \to \infty$,

$$P\{[X_{m(N)} - E(X_{m(N)})]/\sigma(X_{m(N)}) < t\} \to \Phi(t).$$

PROOF. The proof of this lemma is given by Konijn [12].

Proof of Theorem 3.1. The proof will be carried out in several steps:

(i) Let $C = \frac{1}{2}K(K - 1)$ denote the number of all possible pairs and label them $\alpha = 1, 2, \cdots, C$ in some convenient manner. (If α denotes the ordered pair (i, j), then $R_N^{(i,j)} = R_N^{(\alpha)}$; $V_N^{(i,j)} = V_N^{(\alpha)}$ etc.) Let m_α, n_α ($m_\alpha + n_\alpha = N_\alpha$) be the number of positive and negative Z's respectively for the αth pair and consider first the conditional situation where each m_α ($\alpha = 1, 2, \cdots, C$) is given.

For given m_α, let $Y_{\alpha 1}^+, \cdots, Y_{\alpha, m_\alpha}^+$; $Y_{\alpha 1}^-, \cdots, Y_{\alpha n_\alpha}^-$ denote the absolute values of the positive and the negative Z's respectively for the αth pair. The joint distribution of the Y's is then the average over N_α ! distributions corresponding to the N_α ! permutations of the absolute Z's coinciding with the Y's, so that it follows easily that the marginal distributions for any Y_α^+, Y_α^- are, respectively

$$(3.4)\qquad \begin{aligned} H_{\alpha,N}^+(x) &= \frac{G(x + (\mu_\alpha/N^{\frac{1}{2}})) - G(\mu_\alpha/N^{\frac{1}{2}})}{1 - G(\mu_\alpha/N^{\frac{1}{2}})} && \text{if} \quad x > 0 \\ &= 0 && \text{otherwise} \\ H_{\alpha,N}^-(x) &= \frac{G(\mu_\alpha/N^{\frac{1}{2}}) - G(-x + (\mu_\alpha/N^{\frac{1}{2}}))}{G(\mu_\alpha/N^{\frac{1}{2}})} && \text{if} \quad x > 0 \\ &= 0 && \text{otherwise,} \end{aligned}$$

where $\mu_\alpha = \mu_{ij} = \theta_i - \theta_j$. Thus, in this conditional situation, the absolute values of the positive and negative Z_α's ($\alpha = 1, \cdots, C$) can be regarded as constituting $2C$ independent samples from distributions $H^+_{\alpha,N}$ and $H^-_{\alpha,N}$ ($\alpha = 1, 2, \cdots, C$). Define now U^+_α, U^-_α ($\alpha = 1, 2, \cdots, C$) as in (3.4) and let

$$W_{\alpha,N} = U^+_\alpha - U^-_\alpha . \tag{3.5}$$

Assume further that $\lim_{N\to\infty}(m_\alpha/N) = \nu^+_\alpha$, $\lim_{N\to\infty}(n_\alpha/N) = \nu^-_\alpha$ exist and are positive. Then it follows on account of Lemma 3.1 and a linear transformation that, in this conditional situation, the variables $N^{-\frac{3}{2}}\{W_{\alpha,N} - \tilde{E}(W_{\alpha,N})\}$ ($\alpha = 1, 2, \cdots, C$) have in the limit, as $N \to \infty$, a multivariate normal distribution $N(\mathbf{0}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma} = \| \sum_{\alpha\beta} \|$ given by

$$\textstyle\sum_{\alpha\beta} = [\delta_{\alpha\beta}(\nu^+_\alpha + \nu^-_\alpha) - (\nu^+_\alpha - \nu^-_\alpha)(\nu^+_\beta - \nu^-_\beta)]. \tag{3.6}$$

(The covariance matrix $\boldsymbol{\Sigma}_N$ of the variables $N^{-\frac{3}{2}}\{W_{\alpha,N} - \tilde{E}(W_{\alpha,N})\}$ ($\alpha = 1, 2, \cdots, C$) again converges to the matrix $\boldsymbol{\Sigma}$)

(ii) We shall now apply Lemma 3.2 and the above result to the variables $V_N^{(\alpha)}$ ($\alpha = 1, 2, \cdots, C$). The conditioning with respect to the random variables m_α ($\alpha = 1, 2, \cdots, C$) is now dropped. We note from (3.3) and (3.5) that

$$V_N^{(\alpha)} = R_N^{(\alpha)} - S_N^{(\alpha)} = (m_\alpha - N_\alpha/2)(N_\alpha + 1) + W_{\alpha,N} \tag{3.7}$$

so that

$$\begin{aligned}\tilde{E}(V_N^{(\alpha)}) - E(V_N^{(\alpha)}) \\ = (m_\alpha - p_{\alpha,N}\cdot N_\alpha)(N_\alpha + 1) + \tilde{E}(W_{\alpha,N}) - E(W_{\alpha,N})\end{aligned} \tag{3.7a}$$

where $p_{\alpha,N} = G(-\mu_\alpha/N^{\frac{1}{2}})$. Now by setting $p_{\beta^+,\alpha^+} = P[Y^+_\beta < Y^+_\alpha]$, $p_{\beta^-,\alpha^+} = P[Y^-_\beta < Y^+_\alpha]$ etc., we can write

$$\begin{aligned}\tilde{E}(W_{\alpha,N}) &= m_\alpha \sum_{\substack{\beta=1 \\ \neq\alpha}}^{C} (m_\beta p_{\beta^+,\alpha^+} + n_\beta p_{\beta^-,\alpha^+}) \\ &\quad - n_\alpha \sum_{\substack{\beta=1 \\ \neq\alpha}}^{C} (m_\beta p_{\beta^+,\alpha^-} + n_\beta p_{\beta^-,\alpha^-}) + m_\alpha \cdot n_\alpha(p_{\alpha^-,\alpha^+} - p_{\alpha^+,\alpha^-})\end{aligned} \tag{3.8}$$

so that on account of $(m_\alpha/N_\alpha) - E(m_\alpha/N_\alpha) = o_p(1)$, $(m_\alpha/N_\alpha)^2 - E(m_\alpha/N_\alpha)^2 = o_p(1)$ and that $p_{\beta^+,\alpha^+} = \frac{1}{2} + O(N^{-\frac{1}{2}})$ we have

$$\tilde{E}(W_{\alpha,N}) - E(W_{\alpha,N}) = (m_\alpha - p_{\alpha,N}N_\alpha)(N - N_\alpha) + o_p(N^{\frac{3}{2}}); \tag{3.8a}$$

thus, from (3.7a) it follows that

$$\tilde{E}(V_N^{(\alpha)}) - E(V_N^{(\alpha)}) = (m_\alpha - p_{\alpha,N}N_\alpha)(N + 1) + o_p(N^{\frac{3}{2}}). \tag{3.9}$$

Since m_α is a binomial $(N_\alpha, p_{\alpha,N})$ random variable for each fixed N, it follows that $(\tilde{E}V_N^{(\alpha)} - EV_N^{(\alpha)})/\sigma(\tilde{E}V_N^{(\alpha)})$ converges in distribution to $N(0, 1)$ variable. Further, under the conditional situation given m_α ($\alpha = 1, 2, \cdots, C$) the variable

$$[V_N^{(\alpha)} - \tilde{E}(V_N^{(\alpha)})]/\tilde{\sigma}(V_N^{(\alpha)}) = [W_{\alpha,N} - \tilde{E}(W_{\alpha,N})]/\tilde{\sigma}(W_{\alpha,N})$$

converges in distribution to $N(0, 1)$ on account of (3.6). From these and a repeated application of Lemma 3.2, it follows that the random variable

$$3^{\frac{1}{2}}\{V_N^{(\alpha)} - EV_N^{(\alpha)}\}/NN_\alpha^{\frac{1}{2}} \tag{3.10}$$

is asymptotically a $N(0, 1)$ variable ($\alpha = 1, 2, \cdots, C$). (In (3.10) we have replaced $\sigma^2(V_N^{(\alpha)}) = E\{\tilde{\sigma}^2 V_N^{(\alpha)}\} + \sigma^2\{\tilde{E}V_N^{(\alpha)}\}$ by $N^2N_\alpha/3$, since $\lim_{N\to\infty}[\sigma^2(V_N^{(\alpha)})/(N^2N_\alpha/3)] = 1$.) By following exactly the same reasoning the asymptotic normality of an arbitrary linear combination of the variables (3.10) can now be easily proved, using (3.9) and the conditional joint normality of the variable $N^{-\frac{3}{2}}\{W_{\alpha,N} - \tilde{E}(W_{\alpha,N})\}$ ($\alpha = 1, 2, \cdots, C$). The joint normality of the variables (3.10) now follows by using the same argument as in Section 7 of Wald and Wolfowitz [18].

(iii) We will now show that the random variables (3.10) are uncorrelated in the limit. This coupled with the joint normality of these variables establishes their asymptotic independence. To prove the last statement, consider

$$\begin{aligned} &E\left\{\frac{W_{\alpha,N} - \tilde{E}(W_{\alpha,N}) + (m_\alpha - p_{\alpha,N}\cdot N_\alpha)(N+1)}{NN_\alpha^{\frac{1}{2}}}\right. \\ &\qquad \left.\cdot\frac{W_{\beta,N} - \tilde{E}(W_{\beta,N}) + (m_\beta - p_{\beta,N}\cdot N_\beta)(N+1)}{NN_\beta^{\frac{1}{2}}}\right\} \\ &\qquad\qquad = E\tilde{E}\left\{\frac{W_{\alpha,N} - \tilde{E}W_{\alpha,N}}{NN_\alpha^{\frac{1}{2}}}\cdot\frac{W_{\beta,N} - \tilde{E}(W_{\beta,N})}{NN_\beta^{\frac{1}{2}}}\right\} \end{aligned} \tag{3.11}$$

(all other terms vanish since m_α and m_β are independent). From (3.6), it follows that for large N

$$\begin{aligned} &\tilde{E}\left\{\frac{W_{\alpha,N} - \tilde{E}(W_{\alpha,N})}{NN_\alpha^{\frac{1}{2}}}\cdot\frac{W_{\beta,N} - \tilde{E}(W_{\beta,N})}{NN_\beta^{\frac{1}{2}}}\right\} \\ &\qquad = \frac{-\frac{1}{3}(m_\alpha/N_\alpha - \frac{1}{2})(m_\beta/N_\beta - \frac{1}{2})N_\alpha^{\frac{1}{2}}N_\beta^{\frac{1}{2}}}{N} + o_p(1). \end{aligned}$$

Observing that, as $N \to \infty$, the expression on the right converges in probability to zero and that the random variables involved are uniformly bounded, an application of the dominated convergence theorem shows that (3.11) converges to zero. On account of (3.7) and (3.9), this implies that the covariance of any two of the variables (3.10) converges to zero, as $N \to \infty$. Since the variance of each of these variables converges to the variance of the limiting distribution, it follows that the limiting multivariate distribution has zero covariances. Thus the $C = K(K-1)/2$ random variables (3.10) are asymptotically independent $N(0, 1)$ variables.

(iv) Reverting back to the original notation, it now follows that the variables $\{T_{N,i} - E(T_{N,i})\}$ ($i = 1, 2, \cdots, K$), where

$$T_{N,i} = [3^{\frac{1}{2}}\sum_{j\neq i}(V_N^{(i,j)}/N_{ij}^{\frac{1}{2}})]/NK^{\frac{1}{2}}$$

have, in the limit, a multivariate normal distribution $N(\mathbf{0}, \mathbf{\Lambda})$ where $\mathbf{\Lambda} = \|\delta_{ii'} - 1/K\|$. Observing that the covariance matrix $\mathbf{\Lambda}_N^*$ of the vector $\mathbf{T}_N = (T_{N,1}, \cdots, T_{N,K-1})'$ is non-singular and converges to the corresponding submatrix of $\mathbf{\Lambda}$ and that $\sum_{i=1}^K T_{N,i} = 0$, it is easily seen that the statistic L is asymptotically equivalent to the statistic $\mathbf{T}_N'\mathbf{\Lambda}_N^{*-1}\mathbf{T}_N$ and, consequently, is distributed in the limit as a non-central χ^2 variable with $(K - 1)$ degrees of freedom and the noncentrality parameter

$$\Delta^2 = (3/K)\sum_{i=1}^{K} \{\sum_{j\neq i} \lim_{N\to\infty} E(V_N^{(i,j)}/NN_{ij}^{\frac{1}{2}})\}^2. \tag{3.12}$$

(v) It remains to compute the noncentrality parameter, which is accomplished below: From (3.7) and (3.8), we have

$$\begin{aligned}\lim_{N\to\infty} E\left(\frac{V_N^{(\alpha)}}{NN_\alpha^{\frac{1}{2}}}\right) &= \tfrac{1}{2}\nu_\alpha^{\frac{1}{2}} \sum_{\substack{\beta=1\\ \neq\alpha}}^{C} \nu_\beta \lim_{N\to\infty} N^{\frac{1}{2}}\{p_{\alpha,N}\, p_{\beta^+,\alpha^+} - (1 - p_{\alpha,N})p_{\beta^+,\alpha^-}\}\\ &+ \tfrac{1}{2}\nu_\alpha^{\frac{1}{2}} \sum_{\substack{\beta=1\\ \neq\alpha}}^{C} \nu_\beta \lim_{N\to\infty} N^{\frac{1}{2}}\{p_{\alpha,N}\, p_{\beta^-,\alpha^+} - (1 - p_{\alpha,N})p_{\beta^-,\alpha^-}\}\\ &+ \tfrac{1}{4}\nu_\alpha^{\frac{3}{2}} \lim_{N\to\infty} N^{\frac{1}{2}}(2p_{\alpha^-,\alpha^+} - 1) + \nu_\alpha^{\frac{1}{2}} \lim_{N\to\infty} N^{\frac{1}{2}}(p_{\alpha,N} - \tfrac{1}{2}).\end{aligned} \tag{3.13}$$

On account of Assumption (iii), for sufficiently large N, we have for some $0 < \delta < 1$

$$N^{\frac{1}{2}}(p_{\alpha,N} - \tfrac{1}{2}) = -\mu_\alpha g(\delta\mu_\alpha/N^{\frac{1}{2}}) \to -\mu_\alpha g(0). \tag{3.14}$$

Further

$$\begin{aligned}\lim_{N\to\infty} N^{\frac{1}{2}}\{p_{\alpha,N}\, p_{\beta^+,\alpha^+} - (1 - p_{\alpha,N})p_{\beta^+,\alpha^-}\}\\ &= \tfrac{1}{2} \lim_{N\to\infty} N^{\frac{1}{2}} \int_{-\infty}^{\infty} \{H_{\alpha,N}^-(y) - H_{\alpha,N}^+(y)\}\, dH_{\beta,N}^+(y) - \mu_\alpha g(0)\\ &= -4\mu_\alpha \int_0^{\infty} g^2(y)\, dy.\end{aligned}$$

The last equality follows since limit and integration are interchangeable on account of Assumptions (i) and (ii). Similarly

$$\begin{aligned}&\lim_{N\to\infty} N^{\frac{1}{2}}\{p_{\alpha,N}\, p_{\beta^-,\alpha^+} - (1 - p_{\alpha,N})p_{\beta^-,\alpha^-}\}\\ &\quad= \lim_{N\to\infty} N^{\frac{1}{2}}\{p_{\alpha,N}\, p_{\beta^+,\alpha^+} - (1 - p_{\alpha,N})p_{\beta^+,\alpha^-}\} = -4\mu_\alpha \int_0^{\infty} g^2(y)\, dy\end{aligned} \tag{3.15}$$

and

$$\lim_{N\to\infty} N^{\frac{1}{2}}(2p_{\alpha^-,\alpha^+} - 1) = 16\mu_\alpha \left\{\tfrac{1}{4}g(0) - \int_0^{\infty} g^2(y)\, dy\right\}. \tag{3.16}$$

Substituting (3.14), (3.15), and (3.16) in (3.13), we get

$$\lim_{N\to\infty} E\left(\frac{V_N^{(i,j)}}{NN_{ij}^{\frac{1}{2}}}\right) = -4\mu_{ij}\,\nu_{ij}^{\frac{1}{2}} \int_0^{\infty} g^2(y)\, dy$$

so that (3.12) reduces to (3.1). The proof is complete.

For the special case when all $N_{ij}(1 \leqq i < j \leqq K)$ are equal, the noncentrality parameter reduces to

$$\Delta^2 = \frac{24}{K-1}\left(\int g^2(y)\,dy\right)^2 \sum_{i=1}^{K} (\theta_i - \bar{\theta})^2$$

where $\bar{\theta} = \sum_{i=1}^{K} \theta_i/K$.

As an immediate consequence of Theorem 3.1, it follows by letting $\theta_1 = \theta_2 = \cdots = \theta_K$ in (3.1) that under the null hypothesis H_0 (or H_1) the statistic L has limiting central χ^2 distribution with $(K - 1)$ degrees of freedom provided the common distribution function $G(x)$ satisfies the conditions (i)-(iv) of Theorem 3.1. However, these conditions are not necessary for the statistic L to possess limiting central χ^2 distribution, which result holds under much weaker conditions. By following the method of proof of Theorem 3.1 and using Theorem $4W$ of [13], one can prove

THEOREM 3.2. *Suppose that the hypothesis H_0 (or H_1) is true and that for each pair (i, j) the ratio (N_{ij}/N) remains bounded away from zero and one as $N \to \infty$. Then the statistic L, defined by (2.2), has asymptotically, as $N \to \infty$, a χ^2 distribution with $(K - 1)$ degrees of freedom.*

4. Extension: the case of non-uniformity of experimental conditions. In experiments where the experimental conditions are not uniform throughout, it would be unreasonable to assume all differences (Z's) to have identical distributions, even under the hypothesis of no-difference among the treatment effects. For example, different sets of comparisons might be performed in different laboratories. The term "blocks" in this model will refer to different experimental conditions. In such a case one way of constructing a test statistic would be to compute the statistic L for each block separately and add them all. Asymptotically, the new statistic will again have a χ^2 distribution with degrees of freedom equal to the sum of the degrees of freedom of the individual terms. However, this would be a test against inhomogeneity in each of the blocks separately; whereas for the present problem an appropriate test should be directed against concordance between rankings in different blocks, caused by a common trend of the underlying distributions of the treatment effects. We will now develop and investigate such a test.

Let b denote the total number of blocks and N_{ijt} be the number of comparisons for the (i, j)th pair and tth block $(1 \leqq i < j \leqq K; t = 1, 2, \cdots, b)$. Let $N_t = \sum_{i=1}^{K} \sum_{j>i} N_{ijt}$ and $N = \sum_{t=1}^{b} N_t$. Let X_{it} be the random variable denoting the characteristic of the ith treatment in the tth block and let $F_{it}(x)$ denote the distribution corresponding to it. If $G_{ijt}(x)$, a continuous c.d.f., denotes the distribution of the difference random variable $Z_{ijt} = X_{it} - X_{jt}$, then the hypothesis of no-difference among the treatments can be expressed as H_0^* : For each combination (i, j, t), $G_{ijt}(x) + G_{ijt}(-x) = 1$ and $G_{ijt}(x) = G_{i'j't}(x)$

for any two pairs (i, j) and (i', j') and the same t $(1 \leqq i < j \leqq K;\ t = 1, 2, \cdots, b)$.

We now describe the test: Rank the absolute values of all Z's separately for each block t $(t = 1, 2, \cdots, b)$. Let R_{ijt} and S_{ijt} denote the sum of ranks of positive and negative Z's respectively for the combination (i, j, t) and let

$$V_{ijt} = R_{ijt} - S_{ijt}\,; \qquad V_{ij} = \sum_{t=1}^{b} V_{ijt}\,; \qquad V_i = \sum_{j \neq i} V_{ij}\,.$$

The covariance matrix $\mathbf{M} = \|\tau_{ii'}\|$ of the variables V_i $(i = 1, 2, \cdots, K)$ is given by

$$(4.1) \qquad \begin{aligned} \tau_{ii} &= \tfrac{1}{6} \sum_{t=1}^{b} \sum_{j \neq i}^{K} N_{ijt}(N_t + 1)(2N_t + 1) \\ \tau_{ii'} &= -\tfrac{1}{6} \sum_{t=1}^{b} N_{ii't}(N_t + 1)(2N_t + 1). \end{aligned}$$

Suppose that Rank $\mathbf{M} = K - 1$ (for Rank $\mathbf{M} < K - 1$ see remarks below) and let

$$\mathbf{M}_V = \begin{Vmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1K} & V_1 \\ & & \cdots & & \\ & & \cdots & & \\ \tau_{K1} & \tau_{K2} & \cdots & \tau_{KK} & V_K \\ V_1 & V_2 & \cdots & V_K & 0 \end{Vmatrix}.$$

Let Δ_V and Δ be the matrices obtained from M_V and M respectively, by omitting the ith row and ith column, for an arbitrary i, $(1 \leqq i \leqq K)$, but not $i = K + 1$. Let $|\Delta_V|$ and $|\Delta|$ denote their determinants; then the test statistic proposed for the hypothesis H_0^* is

$$(4.2) \qquad L^* = |\Delta_V| / |\Delta|$$

with large values of L^* constituting the critical region. It is easily verified that L^* does not depend on the choice of rows and columns omitted in the matrices M_V and M, since in both these matrices each row (column), except for the last one in M_V, is a linear combination of the other rows (columns). In representing L^* we have used the same technique as Benard and van Elteren [2]. We observe here that L^* is just a convenient representation of the quadratic form

$$(4.3) \qquad \mathbf{V}'\mathbf{\Delta}^{-1}\mathbf{V}$$

where $\mathbf{V} = (V_1, \cdots, V_{i-1}, V_{i+1}, \cdots, V_K)'$. Since (4.3) is positive definite it can be transformed into a quadratic form of the type $\sum C_i u_i^2$, with $C_i > 0$ $(i = 1, \cdots, K - 1)$ and $\sum_i u_i^2 = \sum_i V_i^2$. A large difference among the treatment effects will tend to make $\sum_i V_i^2$, and therefore L^*, large and consequently lead to the rejection of the null hypothesis.

Theorems 4.1 and 4.2 below will enable us to approximate the critical points for the large numbers of comparisons:

THEOREM 4.1. *Suppose that the hypothesis H_0^* is true and that for each pair (i, j) the ratio (N_{ijt}/N) is bounded away from zero and one for some t, as $N \to \infty$ (where b need not tend to ∞). Then the statistic L^*, defined by (4.2), has asymptotically, as $N \to \infty$, a χ^2 distribution with $(K - 1)$ degrees of freedom.*

PROOF. The proof of this theorem follows on the same lines as Theorem 3.1 and from the definition of the statistic L^*.

THEOREM 4.2. *Suppose that the hypothesis H_0^* is true and that the total number of blocks $b \to \infty$ [the N_{ijt} may be finite for some or all combinations (i, j, t)] such that*

(i) *For each $i = 1, 2, \cdots, K$,*

$$\lim_{b\to\infty} \sum_t \{E|\sum_{j\neq i} V_{ijt}|^3/\tau_{ii}^{3/2}\} = 0$$

(ii) *the matrix $\mathfrak{P} = \|\rho_{ii'}\|$ has rank $(K - 1)$, where $\rho_{ii'} = \lim_{b\to\infty}[\tau_{ii'}/(\tau_{ii}\tau_{i'i'})^{1/2}]$, then the statistic L^*, defined by (4.2), has asymptotically, as $b \to \infty$, a χ^2 distribution with $(K - 1)$ degrees of freedom.*

PROOF. The proof of this theorem follows from the central limit theorem for random vectors (see Uspensky [16], p. 318) and the definition of the statistic L^*.

LEMMA 4.1. *If $N_{ijt} < \eta$ (a constant) for all combinations (i, j, t) $(1 \leqq i < j \leqq K$; $t = 1, \cdots, b)$, then the Conditions* (i) *and* (ii) *of Theorem 4.2 are implied by the following*:

(i)′ *For each $i = 1, 2, \cdots, K$, $\lim_{b\to\infty}(\sum_{j\neq i} b_{ij}/b) > 0$ where, for the pair (i, j), b_{ij} is the number of blocks t for which $N_{ijt} > 0$ $(t = 1, 2, \cdots, b)$.*

(ii)′ *The matrix $\mathbf{M}^* = \|\tau_{ii'}^*\|$, where $\tau_{ii'}^* = \lim_{b\to\infty} -(b_{ii'}/b)$ and $\tau_{ii}^* = \lim_{b\to\infty}(\sum_{j\neq i} b_{ij}/b)$ is not of the type*

$$\begin{Vmatrix} \mathbf{P} & \mathbf{0}' \\ \mathbf{0} & \mathbf{Q} \end{Vmatrix}$$

(see definition below).

PROOF. The method of proof of this lemma is similar to that of Theorem V of Benard and van Elteren [2]. Since $N_{ijt} < \eta$ for each combination (i, j, t), $\{\sum_t E|\sum_{j\neq i} V_{ijt}|^3/b]$ remains bounded as $b \to \infty$. Also

$$\begin{aligned} \lim_{b\to\infty} \tau_{ii}/b &= \lim_{b\to\infty}(1/6b)\sum_{t=1}^{b} \sum_{j\neq i}^{K} N_{ijt}(N_t + 1)(2N_t + 1) \\ &\geqq \lim_{b\to\infty} \sum_{j\neq i} b_{ij}/b > 0 \end{aligned} \tag{4.4}$$

so that the Condition (i) is satisfied. Further on account of (4.4), it also follows that $\rho_{ii'} = 0$ if and only if $\lim_{b\to\infty}(b_{ii'}/b) = 0$ $(i \neq i')$.

Now consider the following definition: Let us say that a $K \times K$ matrix $\mathbf{M}$ is of type

$$\begin{Vmatrix} \mathbf{P} & \mathbf{U} \\ \mathbf{0} & \mathbf{Q} \end{Vmatrix},$$

where $\mathbf{P}$ and $\mathbf{Q}$ are square matrices and $\mathbf{0}$ consists of zeros, if and only if it can be transformed into a matrix of the type by the same permutation of rows and columns.

From this definition, the preceding result and Condition (ii)′ of this theorem it follows that the matrix Σ of Theorem 4.2 is not of this type. Consequently, the matrix $\Sigma_{\nu\nu}$ obtained from Σ by omitting the νth row and νth column is also not of this type.

Tausski [15] has proved the following result: Let $\|a_{ii'}\|$ be a $K \times K$ matrix with complex elements such that

$$|a_{ii}| \geqq \sum_{\substack{i'=1 \\ \neq i}} |a_{ii'}| \qquad (i = 1, 2, \cdots, K) \tag{4.5}$$

with equality in at most $(K - 1)$ cases. Then the determinant $(a_{ii'}) \neq 0$ if and only if this matrix is not of the type

$$\left\|\begin{matrix} \mathbf{P} & \mathbf{U} \\ \mathbf{0} & \mathbf{Q} \end{matrix}\right\|.$$

The matrix $\Sigma_{\nu\nu}$ defined above satisfied the Condition (4.5), so that its rank is $(K - 1)$. Since evidently the rank of Σ is at most $(K - 1)$, it is, therefore, exactly $(K - 1)$, and the proof is complete.

It may happen that there are two or more—say $q > 1$— "noncompared" subsets of objects; i.e., no comparison is made between objects belonging to different mutually exclusive subsets. Benard and van Elteren [2] have proved a theorem (whose scope also covers the present setup) which states:

The rank of the matrix $\mathbf{M}$ [see (4.1)] is $(K - q)$ if and only if there are more than q non-compared subsets of objects.

In view of this theorem, the statistic L^* can be defined as above if and only if $q = 1$. Further, it also follows that the Condition (ii)′ of Theorem 4.3 will be violated if and only if there are more than one "non-compared" subsets of objects in the "asymptotic sense" i.e. as $b \to \infty$, the proportions of comparisons between different subsets of objects tend to zero. If $q > 1$, then one can compute L^* statistic for each of the q subsets and add them all; the new statistic will have asymptotically a χ^2 distribution with $(K - q)$ degrees of freedom.

The last remark also applies to the statistic L of Section 2. Further, in this model if some $N_{ij} = 0$ (without partitioning the set of K treatments into disjoint "non-compared" subsets), a statistic similar to L^*, but with over-all ranking of absolute differences Z's, can be defined.

5. The asymptotic distribution of L^* under K_N. Consider the following sequence $\{N = 1, 2, \cdots\}$ of translation alternatives: K_N^*: $F_{it}(x) = F(x + \xi_t + \theta_i N^{-\frac{1}{2}})$ for all $i = 1, 2, \cdots, K$ and $t = 1, 2, \cdots, b$, where ξ_t is a constant and not all θ_i are equal. In the changed context of distributions of Z's, the alternative K_N^* reduces to K_N (see Section 3), so that the shift alternatives may again be directly defined by K_N.

The asymptotic distribution of the statistic L^* of Section 4 will be derived for the following "balanced" design only: Let

$$(5.1)\quad \begin{aligned} N_{ijt} &= n, \text{ for all combinations } (i, j, t),\\ \lambda &= \text{the number of treatments compared in each block } t = 1, 2, \cdots, b,\\ \mu &= \text{the number of blocks in which each pair } (i, j)\ (1 \leqq i < j \leqq K) \text{ of treatments is compared, and}\\ l &= \text{the number of blocks in which each individual treatment } i \text{ appears } (i = 1, 2, \cdots, K). \end{aligned}$$

Then, $\mu = \lambda(\lambda - 1)b/K(K - 1)$ and $l = (K - 1)\mu/(\lambda - 1)$. For this case, (4.3) and therefore the statistic L^* is easily seen to reduce to the form

$$L^* = [6/n(N' + 1)(2N' + 1)K\cdot\mu] \sum_{i=1}^{K} \{\sum_t \sum_{j\neq i} V_{ijt}\}^2$$

where $N' = \frac{1}{2}\lambda(\lambda - 1)n$.

We now state, the following two theorems, which are counterparts of Theorems 4.1 and 4.2 of Section 4, respectively:

THEOREM 5.1. *For each index N, let the hypothesis K_N be valid. Then under Assumptions* (i), (ii) *and* (iii) *of Theorem* 3.1 *and the Assumption* (5.1) *above, the statistic L^*, defined by* (4.2), *has asymptotically, as $n \to \infty$, a noncentral χ^2-distribution with $(K - 1)$ degrees of freedom and the noncentrality parameter*

$$(5.2)\qquad \Delta_1^* = \frac{24}{K - 1}\left\{\int g^2(y)\, dy\right\}^2 \sum_{i=1}^{K} (\theta_i - \bar{\theta})^2$$

where $\bar{\theta} = \sum_{i=1}^{K} \theta_i/K$.

PROOF. The proof of this theorem is accomplished on the same lines as that of Theorem 3.1.

THEOREM 5.2. *For each index N, let the hypothesis K_N be valid. Then, under the assumptions of Theorem* 5.1, *the statistic L^*, defined by* (4.2), *has asymptotically as $b \to \infty$, a noncentral χ^2-distribution with $(K - 1)$ degrees of freedom and the noncentrality parameter*

$$(5.3)\qquad \Delta_2^{*2} = \frac{48}{(N' + 1)(2N' + 1)} \cdot \left[(N' - 1)\int g^2(y)\, dy + g(0)\right]^2 \frac{1}{K - 1}\sum_{i=1}^{K} (\theta_i - \bar{\theta})^2$$

where $\bar{\theta} = \sum_{i=1}^{K} \theta_i/K$ and $N' = n\binom{\lambda}{2}$.

PROOF. The proof of this theorem follows from the central limit theorem for random vectors (see Uspenski ([16], p. 318) with the computations of the noncentrality parameter as in Theorem 3.1.

6. Asymptotic relative efficiency. The asymptotic efficiencies of the L and L^* tests relative to other competing tests will be obtained only for the "balanced" case (5.1). It follows from van Elteren and Noether [17] that for this case, Durbin's χ_D^2-statistic and the corresponding F-statistic have, under K_N, noncentral χ^2-distributions with $(K - 1)$ degrees of freedom and noncentrality parameters Δ_D^2 and Δ_F^2 respectively, with

$$(6.1)\qquad \begin{aligned} \Delta_D^2 &= \frac{8}{K-1} g^2(0) \sum_{i=1}^{K} (\theta_i - \bar{\theta})^2 \\ \Delta_F^2 &= \frac{2}{K-1} \left\{ \sum_{i=1}^{K} (\theta_i - \bar{\theta})^2 / \sigma_G^2 \right\} \end{aligned}$$

where σ_G^2 is the variance of G and $\bar{\theta} = \sum_{i=1}^{K} \theta_i / K$. For such cases, the asymptotic efficiency e_{S,S^*} of a statistic S relative to another statistic S^* is given by the ratio of their respective noncentrality parameters (see [1], [9]). Let L_1^* and L_2^* stand for the asymptotic test L^* in the two situations corresponding to Theorems 4.1 and 4.2, respectively. From (3.17), (5.2), (5.3) and (6.1), we thus have

$$(6.2)\qquad \begin{aligned} e_{L,D} &= e_{L_1^*,D} = 3\left[\int g^2(y)\, dy\right]^2 \Big/ g^2(0) \\ e_{L_2^*,D} &= \frac{6}{(N'+1)(2N'+1)} \left[\left\{N' - 1\right) \int g^2(y)\, dy + g(0)\right\} \Big/ g(0)\right]^2. \end{aligned}$$

Similarly

$$(6.3)\qquad \begin{aligned} e_{L,F} &= e_{L_1^*,F} = 12\sigma_G^2 \left[\int g^2(y)\, dy\right]^2 \\ e_{L_2^*,F} &= \frac{24}{(N'+1)(2N'+1)} \sigma_G^2 \left[(N' - 1) \int g^2(y)\, dy + g(0)\right]^2. \end{aligned}$$

Further, van Elteren and Noether [17] have pointed out that the Bradley-Terry and the Durbin paired-comparison tests are asymptotically equivalent. Thus the efficiencies of the L and L^* tests relative to the Bradley-Terry test are the same as (6.2). We observe here that these efficiency expressions (6.2) and (6.3) are the same as those of the Wilcoxon-one-sample test relative to the sign-test and the t-test respectively. For the particular case, when G is $N(0, \sigma^2)$

$$(6.4)\qquad e_{L,F} = e_{L_1^*,F} = \frac{3}{\pi} \qquad e_{L_2^*,F} = \frac{3}{\pi} \frac{2(N' - 1 + 2^{\frac{1}{2}})^2}{(N'+1)(2N'+1)}.$$

We note that, as $n \to \infty$ $[N' = \frac{1}{2}\lambda(\lambda - 1)n]$, $e_{L_2^*,F}$ of (6.3) tends to the value $12\,\sigma^2(\int g^2(y)\, dy)^2$ which is the same as $e_{L_1^*,F}$. For the normal case (6.4), $e_{L_2^*,F}$ increases monotonically to the value $e_{L_1^*,F} = 3/\pi$, as $n \to \infty$. Further, consider the case such that in each "block" only comparisons for a single pair of treatments is made, i.e., $\lambda = 2$ and $N' = n$. For this case $e_{L_2^*,F}$ of (6.3) and (6.4) assume respectively the values $4\,\sigma^2 g^2(0)$ and $2/\pi$ when $n = 1$. This must be so, since in this case L^* statistic reduces to the Durbin χ_D^2-statistic.

7. Discussion and concluding remarks. It is evident that under the assumptions of Section 2, where it is possible to rank the absolute Z's in a combined sample, one can also rank the absolute Z's corresponding to each pair (i, j) of treatments $(1 \leqq i < j \leqq K)$ separately and use the L^*-statistic of Section 4, instead of the statistic L based on the "joint ranking" procedure. By looking at the asymptotic relative efficiency expressions (6.2) and (6.3), one might question the need of a "joint ranking" procedure and the L-test when a more convenient "separate ranking" L^*-test is available, which has the same asymptotic Pitman efficiency as the L-test. However, it must be remembered that Pitman efficiency is just a limiting number and provides a comparison only for the local power and for large samples. Hodges and Lehmann [10] have emphasized (see also [5] and [6]) that a single number cannot provide a comprehensive efficiency comparison of two tests.

That the L-test based on the "joint-ranking" procedure should provide better local power against all alternatives of shift in location with the underlying distribution as one for which $e_{L,D} = e_{L_1^*,D} > 1$ is suggested by considering the two tests L and L^* when $N_{ij} = 1$ for each pair (i, j) $(1 \leqq i \leqq j \leqq K)$. For then the "separate ranking" L^*-test reduces to Durbin's paired-comparison test, whereas the L-test still utilizes the magnitudes of the observed comparison differences Z's. This suggests—roughly speaking—that the L-test should be preferred to the L^*-test for the alternatives for which it is preferred to the Bradley-Terry or Durbin tests. (Normal shift alternatives is one such instance). The author has carried out computations for comparing the local power of the above two tests for finite number of comparisons $N_{ij} = n$ for each pair (i, j) and large number of treatments. The results supported the above contention. These results will be presented in a subsequent paper.

Acknowledgment. I wish to acknowledge my indebtedness to Professor E. L. Lehmann for proposing this investigation and his constant encouragement and many valuable suggestions during its progress.

REFERENCES

[1] ANDREWS, F. C. (1954). Asymptotic behavior of some rank tests for analysis of variance. *Ann. Math. Statist.* **25** 724–735.

[2] BENARD, A. and VAN ELTEREN, PH. (1953). A generalization of the method of m rankings. *Proc. Kon. Ned. Ak. van Wet. A 56, Indag. Math.* **15** 358–369.

[3] BRADLEY, R. A. (1955). Rank analysis of incomplete block designs, III. *Biometrika* **42** 450–470.

[4] BRADLEY, R. A. and TERRY, M. E. (1962). Rank analysis of incomplete block designs, I. *Biometrika* **39** 324–345.

[5] DIXON, W. J. (1953). Power functions of the sign test and power efficiency for normal alternatives. *Ann. Math. Statist.* **24** 467–473.

[6] DIXON, W. J. (1954). Power under normality of several non-parametric tests. *Ann. Math. Statist.* **25** 610–614.

[7] DURBIN, J. (1951). Incomplete blocks in ranking experiments. *British J. Psychology.* **4** 85–90.

[8] GODWIN, J. J. and ZAREMBA, S. K. (1961). A central limit theorem for partly dependent variables. *Ann. Math. Statist.* **32** 677–686.

[9] HANNAN, E. J. (1956). The asymptotic powers of certain tests based on multiple correlations. *J. Roy. Statist. Soc. Ser. B.* **18** 227–233.

[10] HODGES, J. L., JR. and LEHMANN, E. L. (1956). The efficiency of some non-parametric competitors of the *t*-test. *Ann. Math. Statist.* **27** 324–335.

[11] HODGES, J. L., JR. and LEHMANN, E. L. (1962). Rank methods for combination of independent experiments in the analysis of variance. *Ann. Math. Statist.* **33** 482–497.

[12] KONIJN, H. S. (1957). Some non-parametric tests for treatment effects in paired replication. *J. Indian Soc. Agric. Statist.* **9** 145–167.

[13] KRUSKAL, WILLIAM H. (1952). A non-parametric test for the several sample problem. *Ann. Math. Statist.* **23** 525–540.

[14] LEHMANN, E. L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.

[15] TAUSSKI, O. (1949). A recurring theorem on determinants. *Amer. Math. Monthly* **56** 672–676.

[16] USPENSKY, J. V. (1937). *Introduction to Mathematical Probability*. McGraw-Hill, New York.

[17] VAN ELTEREN, PH. and NOETHER, G. E. (1959). The asymptotic efficiency χ_r^2-test for a balanced incomplete block design. *Biometrika* **46** 475–477.

[18] WALD, A. and WOLFOWITZ, J. (1944). Statistical tests based on permutations of the observations. *Ann. Math. Statist.* **15** 358–372.