ON THE LIKELIHOOD RATIO TEST OF A NORMAL MULTIVARIATE TESTING PROBLEM[1]

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0. Introduction and summary. Let the random vector $X = (X_1 \cdots X_p)'$ have a multivariate normal distribution with unknown mean $\xi = (\xi_1 \cdots \xi_p)'$ and unknown nonsingular covariance matrix Σ. Write $\Sigma^{-1}\xi = \Gamma = (\Gamma_1 \cdots \Gamma_p)'$. The problem considered here is that of testing the hypothesis $H_0 : \Gamma_{q+1} = \cdots = \Gamma_p = 0$ against the alternative $H_1 : \Gamma_{p'+1} = \cdots = \Gamma_p = 0$ when $p \geqq p' > q$ and ξ, Σ are both unknown. This problem arises in discriminating between two multivariate normal populations with the same unknown covariance matrix when one is interested to test whether the variables $X_{q+1} \cdots X_{p'}$ contribute significantly to the discrimination. For a comprehensive treatment of this subject, the reader is referred to Rao (1952), Chapter 7.

In this paper we will find the likelihood ratio test of H_0 against H_1 and show that this test is uniformly most powerful similar invariant. The problem of testing H_0 against H_1 remains invariant under the groups G_1 and G_2 where G_1 is the group of $p' \times p'$ non-singular matrices

$$g = \begin{pmatrix} g_{11} & 0 \\ g_{22} & g_{22} \end{pmatrix}$$

(with g_{11} a $q \times q$ matrix) which transform the coordinates $X_1 \cdots X_{p'}$ of X and G_2 is the group of translations of the coordinates $X_{p'+1} \cdots X_p$ of X. We may restrict our attention to the space of the sufficient statistic $(\bar{X}, S)$ of (ξ, Σ). A maximal invariant under G_1 and G_2 in the space of $(\bar{X}, S)$ is $R = (R_1, R_2)'$, and a corresponding maximal invariant in the parametric space of (ξ, Σ) is $\delta = (\delta_1, \delta_2)'$, where $R_i \geqq 0$, $\delta_i \geqq 0$ are defined in Section 2. In Section 1, we will find the likelihood ratio test of H_0 against H_1 in the usual way. The likelihood ratio test is invariant under all transformations which keep the problem invariant, and hence is a function only of R. In Section 2, we will find the joint density of R_1 and R_2 under the hypothesis and under the alternatives and then follow Neyman's approach of invariant similar regions to show that the likelihood ratio test in this case is uniformly most powerful similar invariant.

In terms of maximal invariants, the above problem reduces to that of testing $H_0 : \delta_2 = 0, \delta_1 > 0$ against the alternative $H_1 : \delta_2 > 0, \delta_1 > 0$. According to a Fisherian philosophy of statistical inference applied to invariant procedures, it is reasonable to think of R_1 as giving information about the discriminating

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ability of the set of variables $(X_1 \cdots X_q)$, but no information about parameters governing additional discriminating ability from variables $X_{q+1} \cdots X_{p'}$. Thus Fisher might call R_1 ancillary for the problem at hand and condition on it. We are not concerned here with the philosophical issues of statistical inference; instead, we will find (in Section 3) the distribution of the likelihood ratio conditional on R_1 which forms the basis of inference in a Fisherian approach. It will be shown that in this conditional situation, the likelihood ratio test is uniformly most powerful invariant.

A more general statement of this same problem is to find the likelihood ratio test of the hypothesis H_0': $\Gamma \varepsilon \mathcal{Z}'$ that Γ belongs to $\mathcal{Z}'$ against the alternative H_1': $\Gamma \varepsilon \mathcal{Y}'$ that Γ belongs to $\mathcal{Y}'$, when ξ, Σ are both unknown and $\mathcal{Z}' \subset \mathcal{Y}'$ are linear sub-spaces of the adjoint space $\mathcal{X}'$ of the space of X's, and are of dimensions q and p' respectively. This problem can be easily reduced to that above by a proper choice of coordinate system, depending on the particular forms of $\mathcal{Z}'$ and $\mathcal{Y}'$. One could have worked with this general formulation instead of that above but the author did not find it convenient for computational purposes.

As a corollary, if $q = 0$ then H_0 falls back to the usual null hypothesis of multivariate analysis of variance. It is easy to see that the likelihood ratio test for $q = 0$ reduces to the usual Hotelling's T^2 test which is uniformly most powerful invariant (Lehmann (1959)).

Fisher (1938) has dealt with a particular case of the general formulation where $\mathcal{Z}'$ is the one-dimensional linear sub-space of $\mathcal{X}'$, and a test based on a discriminant function was suggested by him. The problem of testing H_0 against H_1 has been dealt with by Rao (1949) and a test depending on the ratio of Mahalanobis' D^2 statistics based on the first q and p' components of X (which is related to Fisher's discriminant function in a simple manner) was suggested by him. It will be seen that both the tests are the likelihood ratio test.

1. Likelihood ratio test of H_0 against H_1. Let $X^\alpha = (X_1^\alpha \cdots X_p^\alpha)'$, $\alpha = 1 \cdots N (N > p)$ be independently identically distributed normal p-vectors with unknown mean ξ and unknown non-singular covariance matrix Σ. The likelihood of the observation $X^1 \cdots X^N$ is

$$\begin{aligned} L(\xi, \Sigma \mid X^1 \cdots X^N) &= (2\pi)^{-\frac{1}{2}Np}(\det \Sigma)^{-\frac{1}{2}N} \\ &\quad \exp\left[-\frac{1}{2}\sum_{\alpha=1}^{N}(X^\alpha - \xi)'\Sigma^{-1}(X^\alpha - \xi)\right] \\ &= (2\pi)^{-\frac{1}{2}Np}(\det \Sigma)^{-\frac{1}{2}N} \\ &\quad \cdot \exp\left[-\tfrac{1}{2}\operatorname{tr}\{\Sigma^{-1}S^* - 2N\Gamma\bar{X}' + N\Sigma\Gamma\Gamma'\}\right] \end{aligned} \tag{1.1}$$

where $S^* = S + N\bar{X}\bar{X}'$, $N\bar{X} = \sum_{\alpha=1}^N X^\alpha$ and $S = \sum_{\alpha=1}^N (X^\alpha - \bar{X})(X^\alpha - \bar{X})'$.

Given the observations, L is a function of ξ, and Σ only, and we will denote it by $L(\xi, \Sigma)$. The likelihood ratio criterion for testing H_0 against H_1 is

$$\lambda = \max_{H_0} L(\xi, \Sigma)/\max_{H_1} L(\xi, \Sigma). \tag{1.2}$$

Now,

$$\max_{H_0} L(\xi, \Sigma) = \max_{\Gamma_{[1]},\Sigma}(2\pi)^{-\frac{1}{2}Np}(\det \Sigma)^{-\frac{1}{2}N} \cdot \exp\left[-\tfrac{1}{2}\mathrm{tr}\{\Sigma^{-1}S^* - 2N\Gamma_{[1]}\bar{X}'_{[1]} + N\Sigma_{11}\Gamma_{[1]}\Gamma'_{[1]}\}\right]. \tag{1.3}$$
$$= \max_{\Sigma}(2\pi)^{-\frac{1}{2}Np}(\det\Sigma)^{-\frac{1}{2}N}\exp\left[-\tfrac{1}{2}\mathrm{tr}\{\Sigma^{-1}S^* - N\Sigma_{11}^{-1}\bar{X}_{[1]}\bar{X}'_{[1]}\}\right]$$

where $\Gamma_{[1]} = (\Gamma_1 \cdots \Gamma_q)'$, $X_{[1]} = (X_1 \cdots X_q)'$, and Σ_{11} is the upper left-hand $q \times q$ submatrix of Σ. Since Σ and S^* are positive definite, there exist non-singular $p \times p$ upper triangular matrices K and T such that $\Sigma = KK'$ and $S^* = TT'$. Partition K and T as

$$K = \begin{pmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{pmatrix}, \qquad T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}, \tag{1.4}$$

where K_{11} and T_{11} are $q \times q$ submatrices. It is easily verified that

$$K^{-1} = \begin{pmatrix} K_{11}^{-1} & -(K_{11}^{-1}K_{12}K_{22}^{-1}) \\ 0 & K_{22}^{-1} \end{pmatrix}, \qquad T^{-1} = \begin{pmatrix} T_{11}^{-1} & -(T_{11}^{-1}T_{12}T_{22}^{-1}) \\ 0 & T_{22}^{-1} \end{pmatrix} \tag{1.5}$$

and $K_{11}K'_{11} = \Sigma_{11}$, $T_{11}T'_{11} = S^*_{11}$, where S^*_{11} is the upper left-hand $q \times q$ submatrix of S^*. Let $L = T^{-1}K$, $\Sigma^* = LL'$ and let us partition L and Σ^* similar to K into submatrices L_{ij} and Σ^*_{ij} $(i, j = 1,2)$ respectively. It is easy to check that $K_{11} = T_{11}L_{11}$. Writing $Z_{[1]} = T_{11}^{-1}\bar{X}_{[1]}$, we get from (1.5)

$$\max_{H_0} L(\Gamma, \Sigma) = \max_K(2\pi)^{-\frac{1}{2}Np}(\det K)^{-N} \cdot \exp\left[-\tfrac{1}{2}\mathrm{tr}\{T'K'^{-1}K^{-1}T - N(K_{11}K'_{11})^{-1}\bar{X}_{[1]}\bar{X}'_{[1]}\}\right] \tag{1.6}$$
$$= \max_K (2\pi)^{-\frac{1}{2}Np}(\det S^*)^{-\frac{1}{2}N}(\det \Sigma^*)^{-\frac{1}{2}N} \times \exp\left[-\tfrac{1}{2}\mathrm{tr}\{\Sigma^{*-1} - NZ'_{[1]}\Sigma_{11}^{*-1}Z_{[1]}\}\right].$$

Further, let $\Sigma^{*-1} = \Lambda$, and Λ be partitioned into submatrices Λ_{ij} similar to Σ^*. From (1.6),

$$\max_{H_0} L(\Gamma, \Sigma) = \max(2\pi)^{-\frac{1}{2}Np}(\det S^*)^{-\frac{1}{2}N}(\det \Lambda_{22})^{-\frac{1}{2}N}(\det \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21}))^{\frac{1}{2}N}$$
$$\times \exp\left[-\tfrac{1}{2}\mathrm{tr}\{\Lambda_{11} + \Lambda_{22} - (\Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21})(N^{\frac{1}{2}}Z_{[1]})(N^{\frac{1}{2}}Z_{[1]})'\}\right]$$
$$= (2N\pi)^{-\frac{1}{2}Np}(\det S^*)^{-\frac{1}{2}N}(\det(I - NZ_{[1]}Z'_{[1]}))^{-\frac{1}{2}N} \cdot \exp[-\tfrac{1}{2}Np] \tag{1.7}$$
$$= (2N\pi)^{-\frac{1}{2}Np}(\det S^*)^{-\frac{1}{2}N} \cdot (1 - N\bar{X}'_{[1]}(S_{11} + N\bar{X}_{[1]}\bar{X}'_{[1]})^{-1}\bar{X}_{[1]})^{-\frac{1}{2}N}\exp[-\tfrac{1}{2}Np],$$

which follows from the fact that the maximum likelihood estimates of Λ_{12}, Λ_{22} and Λ_{11} are 0, I/N and $(I - NZ_{[1]}Z'_{[1]})/N$ respectively. In a similar way, one can get

$$\text{(1.8)}\quad \begin{aligned}&\max\nolimits_{H_1} L(\Gamma, \Sigma)\\ &= (2N\pi)^{-\frac{1}{2}Np}(\det S^*)^{-\frac{1}{2}N}(1 - N\bar{X}'_{[2]}(S_{22} + N\bar{X}_{[2]}\bar{X}'_{[2]})^{-1}\bar{X}_{[2]})^{-\frac{1}{2}N} \exp[-\tfrac{1}{2}Np],\end{aligned}$$

where S_{22} is the upper left-hand $p' \times p'$ submatrix of S and $X_{[2]} = (X_1 \cdots X_{p'})'$. Hence, from (1.7) and (1.8) we get

$$\text{(1.9)}\qquad \lambda^{2/N} = \frac{1 - N\bar{X}'_{[2]}(S_{22} + N\bar{X}_{[2]}\bar{X}'_{[2]})^{-1}\bar{X}_{[2]}}{1 - N\bar{X}_{[1]}(S_{11} + N\bar{X}_{[1]}\bar{X}'_{[1]})^{-1}\bar{X}_{[1]}} = Z \quad \text{(say)}.$$

Thus we have the following theorem:

THEOREM 1. *On the basis of observation X^α, $\alpha = 1 \cdots N$, the likelihood ratio test of $H_0 : \Gamma_{q+1} = \cdots = \Gamma_p = 0$ against $H_1 : \Gamma_{p'+1} = \cdots = \Gamma_p = 0$, when ξ, Σ are both unknown, is given by $Z \leqq Z_0$, where Z is defined in* (1.9) *and Z_0 is determined in such a way that the probability that $Z \leqq Z_0$ is equal to the chosen level of significance.*

Z has central beta-distribution with parameters $\frac{1}{2}(N - p')$, $\frac{1}{2}(p' - q)$. This follows trivially from (2.14). It is also given by Rao (1952).

REMARK 1. The marginal probability density of $X_{[2]}$ is normal with mean $\xi_{[2]} = (\xi_1 \cdots \xi_{p'})'$ and covariance matrix $\Sigma_{[22]}$ (= the upper left-hand $p' \times p'$ submatrix of Σ). Since we are only interested in probabilities of $Z \leqq Z_0$ under H_0 and H_1, we can take $p' = p$, i.e. $X_{[2]} = X$. This, in no way, restricts our original formulation of H_0 and H_1 with $p' \leqq p$.

2. The uniformly most powerful invariant similar test of H_0 against H_1. Since much of the development in this section proceeds along standard lines, we shall omit some of the routine details. The reader may consult Lehmann (1959) for nomenclature and a treatment of the theory of invariance and similar regions in hypothesis testing.

We have obtained the likelihood ratio test of H_0 against H_1. In this section, we want to show that the likelihood ratio test is uniformly most powerful invariant similar for testing H_0 against H_1. It may be verified that the problem of testing H_0 against H_1 remains invariant under the groups of transformations G_1 and G_2 and hence depends only on the maximal invariants under G_1 and G_2.

We need only consider test functions which depend on the sufficient statistic $(\bar{X}, S)$, the Lebesgue density of which is

$$\text{(2.1)}\quad \begin{aligned}f_{\xi\Sigma}(\bar{x}, s) = C(\det \Sigma)^{-\frac{1}{2}(N+p-1)}(\det s)^{\frac{1}{2}(N-p-2)}\\ \times \exp[-\tfrac{1}{2}\operatorname{tr}\{\Sigma^{-1}(s + N(\bar{x} - \xi)(\bar{x} - \xi)'\}],\end{aligned}$$

where

$$C = N^{\frac{1}{2}p}2^{\frac{1}{2}Np}\pi^{\frac{1}{4}p(p+1)} \prod_{i=1}^{p} \Gamma[\tfrac{1}{2}(N - i)].$$

Furthermore, it is easy to see that the action of the group G_2 on X is to eliminate the components $X_{p'+1} \cdots X_p$ from consideration to compute the maximal invariants. We, therefore, take $p' = p$ and consider the group G_1 only for the invariance of the problem, and compute a maximal invariant of $(\bar{X}, S)$ under the action of the group G_1 which leaves the problem invariant in the usual fashion: If a function ϕ is invariant, then $\phi(\bar{X}, S) = \phi(g\bar{X}, gSg')$ for all g, $\bar{X}$, and S. We may consider the domain of S to be positive definite symmetric matrices which have probability one for all ξ, Σ. Since S is positive definite, there exists an F in G_1 such that $S = FF'$ and $S_{11} = F_{11}F'_{11}$. If $\phi(\bar{X}, S) = \phi(g\bar{X}, gSg')$, $g \varepsilon G_1$, then it may be verified that ϕ is a function only of the vector $Z = (Z_1, Z_2)'$, where $Z_1 = \bar{X}'_{[1]}F'^{-1}_{11}F^{-1}_{11}\bar{X}_{[1]} = \bar{X}'_{[1]}S^{-1}_{11}\bar{X}_{[1]}$ and $Z_2 = \bar{X}'F'^{-1}F^{-1}\bar{X} = \bar{X}'S^{-1}\bar{X}$. The vector Z is thus a maximal invariant in $(\bar{X}, S)$, if it is invariant under G_1 which is easily seen to be the latter. Z_1, Z_2 are essentially Hotelling's statistics computed from the first q and p coordinates of X respectively. We shall find it more convenient to work with the equivalent statistic $R = (R_1, R_2)$, where

$$R_i = NZi/(1 + NZ_i) - NZ_{i-1}/(1 + NZ_{i-1}), \qquad i = 1, 2, \tag{2.2}$$

where $Z_{-1} = 0$ by definition. It is easily verified that $R_i \geqq 0$, $\sum_{i=1}^{2} Ri \leqq 1$. A corresponding maximal invariant $\Delta = (\delta_1, \delta_2)$ in the parametric space of (ξ, Σ) under G_1, when H_1 is true, is easily seen to be given by

$$\begin{aligned} \delta_1 + \delta_2 &= N\xi'\Sigma^{-1}\xi = N\Gamma'\Sigma\Gamma, \quad \text{and} \\ \delta_2 &= N\xi'\Sigma^{-1}\xi - N\xi'_{[1]}\Sigma^{-1}_{11}\xi_{[1]} = N\Gamma'_{[2]}(\Sigma^{22})^{-1}\Gamma_{[2]} \end{aligned} \tag{2.3}$$

where $(\Sigma^{22})^{-1} = (\Sigma_{22} - \Sigma_{21}\Sigma^{-1}_{11}\Sigma_{12})$, $\xi_{[1]} = (\xi_1 \cdots \xi_q)'$, $\Gamma_{[2]} = (\Gamma_{q+1} \cdots \Gamma_p)'$ and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Here, $\delta_i \geqq 0$. The corresponding maximal invariant under H_0 takes on the value $(\delta_1, 0)$. The Lebesgue density function f^*_Δ of R, depends on Δ under H_1 and on δ_1 under H_0.

In terms of the maximal invariants, the likelihood ratio test is given by $(1 - R_1 - R_2)/(1 - R_1) \leqq Z_0$. Now, to show that the likelihood ratio test is uniformly most powerful invariant similar, we must compute f^*_Δ and $f^*_{\delta_1}$. Since f^* depends on (ξ, Σ) only through Δ, we may put $\Sigma = I$ and redefine $N^{\frac{1}{2}}\xi = N^{\frac{1}{2}}\rho$ such that $N\rho'_{[1]}\rho_{[1]} = \delta_1$, $N\rho'_{[2]}\rho_{[2]} = \delta_2$ where $\rho_{[1]} = (\rho_1 \cdots \rho_q)'$ and $\rho_{[2]} = (\rho_{q+1} \cdots \rho_p)'$. We will use the method of Stein (1956) for deriving the probability ratio $f^*_\Delta/f^*_{\delta_1}$ of R, based on some simple considerations concerning invariant measure. The reason of computing the ratio, instead of f^*_Δ and $f^*_{\delta_1}$, will be clear in the later part of this section.

Let G be a group operating (not necessarily transitively) on the topological space $\mathcal{Z}$ and let λ be a left invariant measure under G in $\mathcal{Z}$. Assume there are given two probability densities p_1 and p_2 with respect to λ, i.e. $P_1(S) = \int_S p_1(Z)\, d\lambda(Z)$, $P_2(S) = \int_S p_2(Z)\, d\lambda(Z)$, $S \subset \mathcal{Z}$ and p_1, p_2 vanish simultaneously. Let $f(Z)$ be a maximal invariant under G. Furthermore, let P^*_i be the distribution of $f(Z)$, when Z has the distribution P_i $(i = 1, 2)$. Then under certain conditions which are satisfied in this problem we have

$$dP_2^*(f)/dP_1^*(f) = \int_G P_2(gZ)\, d\mu(g) \Big/ \int_G p_1(gZ)\, d\mu(g), \tag{2.4}$$

where μ is left invariant Haar measure in G.

For our problem, $f(Z)$ is the pair (R_1, R_2) and Z is the pair $(N^{\frac{1}{2}}\bar{X}, S)$. A left invariant measure in Z under G_1 is $d(N^{\frac{1}{2}}\bar{X}, S) = (\det S)^{-\frac{1}{2}(p+2)}\, d(N^{\frac{1}{2}}\bar{X})\, dS$. The proof of this follows from the fact that the Jacobian of the transformation $(\bar{X}, S) \to (g\bar{X}, gSg')$ is $|\det g|^{p+2}$. Since, for g, h belonging to G_1, $\partial gh/\partial h = |\det g_{11}|^{-q}|\det g_{22}|^{-p}$, the invariant Haar measure in G_2 is $d\mu(g) = |\det g_{11}|^{-q} \cdot |\det g_{22}|^{-p}\, dg$. Now

$$\frac{f_\Delta^*(R_1, R_2)}{f_{\delta_1}^*(R_1, R_2)} = \frac{\int_{G_1} p_1(gX, gSg')\, |\det g_{11}|^{-q}\, |\det g_{22}|^{-p}\, dg}{\int_{G_1} p_0(gX, gSg')\, |\det g_{11}|^{-q}\, |\det g_{22}|^{-p}\, dg} = \frac{I_1}{I_0}, \tag{2.5}$$

where $p_i(\bar{X}, S) = (\det S)^{\frac{1}{2}(p+2)} \times$ density of $(\bar{X}, S)$ under H_i. With $\Sigma = I$,

$$\begin{aligned} I_1 &= C \exp[-\tfrac{1}{2}(\delta_1 + \delta_2)](\det S)^{\frac{1}{2}N}|\det g_{11}|^{N-q}|\det g_{22}|^{N-p} \\ &\quad \exp[-\tfrac{1}{2}\operatorname{tr}\{g(S + N\bar{X}\bar{X}')g' - 2\rho\bar{X}'g'\}]\, dg, \end{aligned} \tag{2.6}$$

the range of integration being from $-\infty$ to ∞ in each variable. Let A be a matrix belonging to G_1 for which $A(S + N\bar{X}\bar{X}')A' = I$. Then $A'A = (S + N\bar{X}\bar{X}')^{-1} = S^{-1} - NS^{-1}\bar{X}\bar{X}'S^{-1}/(1 + N\bar{X}'S^{-1}\bar{X})$, so that $N\bar{X}'A'A\bar{X} = N\bar{X}'S^{-1}\bar{X}/(1 + N\bar{X}'S^{-1}\bar{X}) = R_1 + R_2$. Since $A_{11}(S_{11} + N\bar{X}_{[1]}\bar{X}'_{[1]})A'_{11} = I$, we obtain, similarly, $N\bar{X}'_{[1]}A'_{11}A_{11}\bar{X}_{[1]} = N\bar{X}'_{[1]}S_{11}^{-1}\bar{X}_{[1]}/(1 + N\bar{X}_{(1)}S_{11}^{-1}\bar{X}_{[1]}) = R_1$, so that we can define $N^{\frac{1}{2}}A\bar{X}$ as a vector Y such that $Y'_{[1]}Y_{[1]} = R_1$, $Y'_{[2]}Y_{[2]} = R_2$, where $Y_{[1]} = (Y_1 \cdots Y_q)'$ and $Y_{[2]} = (Y_{q+1} \cdots Y_p)'$. Writing $qA^{-1} = h$, we have $\partial g/\partial h = |\det A_{11}|^p|\det A_{22}|^{p-q}$. From (2.6), we get

$$\begin{aligned} I_1 &= C \exp[-\tfrac{1}{2}(\delta_1 + \delta_2)]\, (\det S)^{\frac{1}{2}N}\, |\det A_{11}|^{N+p-q}\, |\det A_{22}|^{N-q} \\ &\quad \times \int (\det(h_{11}h'_{11}))^{\frac{1}{2}(N-q)}\, (\det(h_{22}h'_{22}))^{\frac{1}{2}(N-p)} \\ &\quad \cdot \exp\left[-\frac{1}{2}\operatorname{tr}\left\{\sum_{j\leq i=1}^{2} h_{ij}h'_{ij} - 2\sum_{j\leq i=1}^{2} \rho_{[i]}\, Y_{[j]}\, h'_{ij}\right\}\right] dh, \end{aligned} \tag{2.7}$$

the integration again being from $-\infty$ to ∞ in each variable. For $i > j$, the integration with respect to h_{ij} yields a factor $(2\pi)^{\frac{1}{2}q(p-q)} \exp[\frac{1}{2}\delta_2 R_1]$. For $j = i = 1$, we obtain a factor (apart from the constant term)

$$\exp[\tfrac{1}{2}R_1\delta_1]E(\chi_q^2(R_1\delta_1))^{\frac{1}{2}(N-q)}\prod_{j=1}^{q-1} E(\chi_j^2)^{\frac{1}{2}(N-q)}. \tag{2.8}$$

This follows from the fact that $\det(h_{11} + \rho_{[1]}Y'_{[1]})(h_{11} + \rho_{[1]}Y'_{[1]})'$ is distributed as the product $\chi_q^2(R_1\delta_1)\prod_{i=1}^{q-1}\chi_i^2$ where χ_i^2 is a central chi-square with i degrees of freedom and $\chi_q^2(\beta)$ is a noncentral chi-square random variable with q degrees of freedom and noncentrality parameter $\beta(= E(\chi_q^2(\beta) - q)$. Similarly, for $j = i = 2$, we obtain apart from the constant term, a factor

$$\exp\left[\tfrac{1}{2}R_2\delta_2\right]E(\chi_{p-q}^2(R_2\delta_2))^{\frac{1}{2}(N-p)}\prod_{j=1}^{p-q-1}E(\chi_i^2)^{\frac{1}{2}(N-p)}. \tag{2.9}$$

Thus finally, we have, with C' a constant

$$\begin{aligned} I_1 = {} & C'\,(\det S)^{\frac{1}{2}N}\,|\det A_{11}|^{N+p-q}\,|\det A_{22}|^{N-q} \\ & \cdot\exp\left[-\sum_{i=1}^{2}\delta_i + \sum_{j=1}^{2}R_j\sum_{i\geq j}\delta_i/2\right] \\ & \cdot E[\chi_q^2(R_1\delta_1)]^{\frac{1}{2}(N-q)}E[\chi_{p-q}^2(R_2\delta_2)^{\frac{1}{2}(N-p)}\prod_{i=1}^{q-1}E(\chi_i^2)\prod_{j=1}^{p-q-1}E(\chi_j^2). \end{aligned} \tag{2.10}$$

Of course, I_0 is just the value of I_1 when $\delta_2 = 0$. Hence,

$$\begin{aligned} \frac{dp_1^*(R_1, R_2)}{dp_0^*(R_1, R_2)} = {} & \exp\left[-\frac{1}{2}\delta_2(1 - R_1)\right] \\ & \cdot\sum_{r=0}^{\infty}\frac{(R_2\delta_2)^r}{r!}\,\frac{\Gamma(\frac{1}{2}(N - q) + r)\Gamma\frac{1}{2}(p - q)}{\Gamma[\frac{1}{2}(p - q) + r]\Gamma\frac{1}{2}(N - q)}. \end{aligned} \tag{2.11}$$

From Cochran and Bliss (1948) or using (2.6) with $H_1{:}\delta_2 = 0,\ \delta_1 > 0$ and $H_0{:}\delta_2 = 0,\ \delta_1 = 0$, together with the fact that the probability density of R when $\delta_1 = \delta_2 = 0$ is $f(r_1, r_2) = \Gamma(N/2)/[\Gamma[\frac{1}{2}(N - p)]\Gamma[\frac{1}{2}(p - q)]\Gamma(\frac{1}{2}q)]\cdot \gamma_1^{\frac{1}{2}q-1}\gamma_2^{\frac{1}{2}(p-q)-1}(1 - r_1 - r_2)^{\frac{1}{2}(N-p)-1}$, we obtain

$$\begin{aligned} dP_0^*(r_1, r_2) = {} & \exp\left[-\tfrac{1}{2}\delta_1\right] \\ & \sum_{r=0}^{\infty}\frac{(\frac{1}{2}\delta_1 r_1)^r}{r!}\,\frac{r_1^{\frac{1}{2}q-1}r_2^{\frac{1}{2}(p-q)-1}(1 - r_1 - r_2)^{\frac{1}{2}(N-p)-1}}{B(\frac{1}{2}(N - q), \frac{1}{2}q + r)B(\frac{1}{2}(N - p), \frac{1}{2}(p - q))}. \end{aligned} \tag{2.12}$$

Hence, from (2.11) and (2.12), we have

$$\begin{aligned} dP_1^*(r_1, r_2) = {} & \exp\left[-\tfrac{1}{2}(\delta_1 + \delta_2) - \tfrac{1}{2}\delta_2 r_1\right] \\ & \sum_{r=0}^{\infty}\frac{(r_1\frac{1}{2}\delta_1)^r}{r!}\,\frac{\Gamma(\frac{1}{2}N + r)\Gamma(\frac{1}{2}q)}{\Gamma(\frac{1}{2}q + r)\Gamma(\frac{1}{2}N)} \\ & \sum_{r=0}\frac{(r_2\frac{1}{2}\delta_2)^r}{r!}\,\frac{\Gamma[\frac{1}{2}(N - q) + r]\Gamma[\frac{1}{2}(p - q)]}{\Gamma[\frac{1}{2}(p - q) + r]\Gamma[\frac{1}{2}(N - q)]} \\ & \frac{\Gamma(\frac{1}{2}N)r_1^{\frac{1}{2}q-1}r_2^{\frac{1}{2}(p-q)-1}(1 - r_1 - r_2)^{\frac{1}{2}(N-p)-1}}{\Gamma(\frac{1}{2}q)\Gamma[\frac{1}{2}(p - q)]\Gamma[\frac{1}{2}(N - p)]}. \end{aligned} \tag{2.13}$$

REMARK. From (2.12), the probability density function of $Z = (1 - R_1 - R_2)/(1 - R_1)$ and R_1 under H_0 is

$$\exp\left[-\frac{1}{2}\delta_1\right]\sum_{r=0}^{\infty}\frac{(\frac{1}{2}\delta_1 r_1)^2 r_1^{\frac{1}{2}q-1}(1 - r_1)^{\frac{1}{2}(N-q)-1}Z^{\frac{1}{2}(N-p)-1}(1 - Z)^{\frac{1}{2}(p-q)-1}}{r!\,B(\frac{1}{2}(N - q), \frac{1}{2}q + r)B(\frac{1}{2}(N - p), \frac{1}{2}(p - q))}. \tag{2.14}$$

Hence under H_0, Z is beta-distributed with parameters $\frac{1}{2}(N-p)$, $\frac{1}{2}(p-q)$ and is independent of R_1.

From (2.12), it is easy to see that R_1 is sufficient for δ_1. Let $\phi(r_1, r_2)$ be any invariant level α test of H_0 against H_1. In order to find the uniformly most powerful similar one, it is necessary now (see for example pages 130–131 Lehmann (1959)) to check whether the family of distributions $\{P_{\delta_1}(R_1), \delta_1 \geqq 0\}$ is boundedly complete or not, where:

DEFINITION. A family of distributions $\{P_{\delta_1}(R_1), \delta_1 \varepsilon \Omega\}$ is boundedly complete, if

$$E_{\delta_1}(h(R_1)) = \int h(r_1)\, dP_{\delta_1}(r_1) = 0 \tag{2.15}$$

for all $\delta_1 \varepsilon \Omega$ and for any real valued measurable function $h(r_1)$ implies that, $h(r_1) = 0$ almost everywhere with respect to each of the measure $P_{\delta_1}(R_1)$.

LEMMA 2.1. *The family of distributions* $\{P_{\delta_1}(R_1), \delta_1 \geqq 0\}$ *is boundedly complete.*

PROOF. Let $\phi(R_1)$ be any real valued bounded function of R_1. Then

$$E_{\delta_1}(\phi(R_1)) = \exp\left[-\frac{1}{2}\delta_1\right] \sum_{r=0}^{\infty} \left(\frac{1}{2}\delta_1\right)^r a_r \int_0^1 \phi(r_1)(r_1)^{\frac{1}{2}q+r-1}$$

$$(1-r_1)^{\frac{1}{2}(N-q)-1}\, dr_1 = \exp\left[-\frac{1}{2}\delta_1\right] \sum_{r=0}^{\infty} \left(\frac{1}{2}\delta_1\right)^r a_r \int_0^1 \phi^*(r_1) r_1^r\, dr_1$$

where $a_r = B[\frac{1}{2}(p-q), \frac{1}{2}(N-p)+1]/\{B[\frac{1}{2}(N-q), \frac{1}{2}q+r]\, B[\frac{1}{2}(N-p), \frac{1}{2}(p-q)]\}$ and $\phi^*(r_1) = \phi(r_1) r_1^{\frac{1}{2}q-1}(1-r_1)^{\frac{1}{2}(N-q)-1}$ Hence, $E_{\delta_1}\phi(R_1) = 0$ implies that

$$\sum_{r=0}^{\infty} \left(\frac{1}{2}\delta_1\right)^r a_r \int_0^1 \phi^*(r_1) r_1^r\, dr_1 = 0.$$

Since the right hand side of this equation is a polynomial in δ_1, all its coefficients must be zero. In other words, $\int_0^1 \phi^*(r_1) r_1^r\, dr_1 = 0$ for $r = 0, 1, 2, \cdots$. Let $\phi^*(R_1) = \phi^{*+}(R_1) - \phi^{*-}(R_1)$, where ϕ^{*+} and ϕ^{*-} denote the positive and negative parts of ϕ^* respectively. Hence, we have $\phi^{*+}(r_1) r_1^r\, dr_1 = \phi^{*-}(r_1) r_1^r\, dr_1$ for $r = 0, 1, 2, \cdots$ which imply that $\phi^{*+}(r_1) = \phi^{*-}(r_1)$ for all r_1, except possibly on a set of measure zero. Hence, $\phi^*(r_1) = 0$ a.e. $\{P_{\delta_1}(R_1), \delta_1 \geqq 0\}$, i.e., $\phi(R_1) = 0$ a.e. $\{P_{\delta_1}(R_1), \delta_1 \geqq 0$.

Since R_1 is complete, it is well-known that ϕ has Neyman structure with respect to R_1 (see for example Lehmann (1959) p. 134) i.e.

$$E_{H_0}[\phi(R_1, R_2) \mid R_1] = \alpha. \tag{2.16}$$

Now to find the uniformly most powerful test among all similar invariant tests, we need the probability ratio

$$\begin{aligned} \frac{dP_1^*(R_2/R_1)}{dP_0^*(R_2/R_1)} &= \frac{dP_1^*(R_2, R_1)\cdot dP_0^*(R_1)}{dP_0^*(R_2, R_1)\cdot dP_1^*(R_1)} \\ &= \exp\left[-\frac{1}{2}\delta_2(1-R_1)\right] \sum_{r=0}^{\infty} \frac{(R_2 \frac{1}{2}\delta_2)^r}{r!} \frac{\Gamma[\frac{1}{2}(N-q)+r]\Gamma[\frac{1}{2}(p-q)]}{\Gamma[\frac{1}{2}(p-q)+r]\Gamma[\frac{1}{2}(N-q)]} \end{aligned} \tag{2.17}$$

(by 2.11). Since the distribution of R_2 on each surface $R_1 = r_1$ is independent of δ_1, the condition (2.16) reduces the problem to that of testing a simple hypothesis: $\delta_2 = 0$ against the alternative: $\delta_2 > 0$ for each value of R_1. In this conditional situation, by Neyman and Pearson's fundamental lemma, the most powerful level α invariant test $\phi(R_2/R_1 = r_1)$ for testing $\delta_2 = 0$ against the simple alternative $\delta_2 = \delta_2$ is (from 2.17)) given by

$$\phi(R_2/R_1 = r_1) = 1,$$

$$\text{(2.18)} \qquad \text{if } \sum_{r=0}^{\infty} \frac{(R_2 \frac{1}{2}\delta_2)^r}{r!} \frac{\Gamma[\frac{1}{2}(N-q)+r]\Gamma[\frac{1}{2}(p-q)]}{\Gamma[\frac{1}{2}(N-q)]\Gamma[\frac{1}{2}(p-q)+r]} \geqq C(r_1),$$

$$\phi(R_2/R_1 = r_1) \equiv 0, \quad \text{otherwise},$$

where $C(r_1)$ is chosen in such a way that $E_{\delta_2=0}\phi(R_2 \mid R_1 = r_1) = \alpha$. Since $R_2 = (1 - R_1)(1 - Z)$, (2.18) reduces to $\phi(Z \mid R_1 = r_1) = 1$, if $Z \leqq C'$; and $= 0$ otherwise, where C' is given by $E_{\delta_2=0}\phi(Z \mid R_1 = r_1) = \alpha$. Since Z is independent of r_1, C' is independent of r_1. Furthermore, $\phi(Z \mid R_1 = r_1)$ is independent of δ_2. Hence, we have the following theorem:

THEOREM 2.1. *Given the observations* $X' \cdots X^N$ $(N > p)$, *the likelihood ratio test of* $H_0: \Gamma_{[2]} = 0$ *against the alternative* $\Gamma_{[2]} \neq 0$, *when* ξ, Σ *are both unknown, is uniformly most powerful invariant similar.*

3. Conditional power. From (2.17) and (2.12), it follows that the conditional distribution of Z given R_1 is a non-central beta $B[\frac{1}{2}(N - p), \frac{1}{2}(p - q)]$ with non-centrality parameter $\delta_2(1 - R_1)$. Hence, in this conditional situation the likelihood ratio test is uniformly most powerful invariant for testing $\delta_2 = 0$ against $\delta_2 > 0$.

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