EFFICIENT UTILIZATION OF NON-NUMERICAL INFORMATION IN QUANTITATIVE ANALYSIS: GENERAL THEORY AND THE CASE OF SIMPLE ORDER

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0. Summary. Suppose a single contrast $y = \sum c_j y_j$, where $\sum c_j = 0$, is to be tested as a basis for detecting differences among unknown parameters μ_j, where $y_j = \mu_j + \epsilon_j$, and the ϵ_j are independent and normally distributed with mean zero and variance σ^2. Write $\mu_j = \alpha + \beta x_j$. Then the problem is to detect $\beta \neq 0$. If $\sum x_j = 0$, and $\sum x_j^2 = 1$, the noncentrality of y, referred to its standard deviation, is (β/σ) times the formal correlation coefficient r between the c_j and the x_j.

If the x_j are known, the c_j can be chosen to make the correlation unity. If the x_j are wholly unknown, no single contrast can guarantee power in detecting $\beta \neq 0$. Intermediate situations, where we know something but not everything about the x_j, occur frequently. If our knowledge can be placed in the form of linear inequalities restricting the μ_j (equivalently the x_j) the problem of choosing a *contrast* $\{c_j\}$ which will give relatively good power against the unknown (*latent*) configuration $\{x_j\}$ is a relatively manageable one.

The problem is to obtain a large value of r^2 between $\{c_j\}$ which is at our choice, and $\{x_j\}$, which is only partially known. A conservative approach is to try to select the $\{c_j\}$ so that the minimum value of r^2 compatible with the restrictions on $\{x_j\}$ is maximized, or nearly so.

The maximization of minimum r^2 when response patterns are constrained by linear homogeneous inequalities leads to the mathematical problem of finding the geometric direction whose maximum angle with a given set of directions is least. The solution to this problem is characterized and proven unique (Sections 8, 17–20). No useful algorithm which is absolutely certain to reach the solution in a few steps appears to exist. However, procedures are discussed (Sections 10 and 11) which reach a solution relatively rapidly in the instances we have considered. The procedures are illustrated on selected examples (Sections 15–16).

The general theory is applied (Sections 13–14) to the latent configuration defined by $x_1 \leqq x_2 \leqq x_3 \leqq \cdots \leqq x_n$, which we call *simple rank order*. A formula is found for the *maximin contrast* which maximizes minimum r^2, and its coefficients are given for $n \leqq 20$.

The "linear-2-4" contrast, constructed from the usual linear contrast by

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quadrupling c_1 and c_n, and doubling c_2 and c_{n-1}, is a reasonable approximation to the maximin contrast for small or medium n, and its minimum r^2 remains above 90% of the maximum possible for $n \leqq 50$ (Table 2).

Knowing *only* simple rank order for the μ_j, good practice seems to indicate the use of "maximin" or "linear-2-4" contrasts in careful work. If more information or insight about the x_j is available, some other contrast may be preferable.

I. INTRODUCTION

1. The problem area. Consider a set of y's, $y_1, y_2, \cdots, y_n$, for which

$$y_j = \mu_j + \epsilon_j, \tag{1}$$

where the μ_j are unknown constants and the ϵ_j are a combination of fluctuations and errors. We shall find it convenient to write $\mu_j = \alpha + \beta x_j$, where α, β and the x_j are also unknown constants. Any sequence of c's satisfying $c_1 + c_2 + \cdots + c_n = 0$ defines a *contrast*

$$y = \sum c_j y_j \tag{2}$$

which is a candidate for use in detecting that the μ's are not all equal.

If the joint distribution of the ϵ_j has sufficient symmetry (we need only require that each ϵ_j has mean zero and variance $\delta^2 + \sigma^2$, while the covariance between every pair of ϵ_j is δ^2), then the average value, variance, and noncentrality of y are

$$\text{ave } y = \sum c_j(\alpha + \beta x_j) = \beta \cdot \sum c_j \cdot (x_j - \bar{x}) \tag{3}$$

$$\text{var } y = \sigma^2 \cdot \sum c_j^2 \tag{4}$$

$$\frac{(\text{ave } y)^2}{\text{var } y} = \frac{\beta^2}{\sigma^2} \sum (x_j - \bar{x})^2 \cdot \frac{[\sum c_j \cdot (x_j - \bar{x})]^2}{[\sum c_j^2] \cdot [\sum (x_j - \bar{x})^2]} \tag{5}$$

where the last factor in the noncentrality can be recognized as the square of the formal correlation coefficient between the c_j and the x_j (or the $x_j - \bar{x}$).

If we have equally good ways of estimating the variability of various contrasts, the effectiveness of a particular contrast in detecting that the μ's are not alike (that $\beta \neq 0$) will mainly depend upon this noncentrality (this will be exactly so if, for instance, the ϵ_j are jointly normally distributed), which in its turn depends upon

(a1) the underlying variability σ^2,

(a2) the spread of the μ's, as measured by $\sum (\mu_j - \bar{\mu})^2 = \beta^2 \cdot \sum (x_j - \bar{x})^2$, and

(a3) the value of r^2.

If we knew the x_j, we would choose the c_j to make r^2 equal, or very close, to its maximum value of $+1$. If we know nothing about the x_j and intend to use a single contrast, we might as well choose the c_j at random. Such a random contrast provides a much less sensitive test than one relying upon $\sum (y_j - \bar{y})^2$, or on the range of the y_j, as an indication of a difference among the μ.

The case where we have *incomplete information* about the x_j is of very great practical importance. There are at least four levels of specificity at which this problem can be attacked:

(b1) we may postulate an over-all probability structure, for example, an a priori distribution for the vector $\{x_j\}$, and then try to determine the $\{c_j\}$ to optimize some given criterion,

(b2) we may choose some class of basic statistics, such as contrasts, and some intermediate criterion, such as the least value of r^2 compatible with specified limitations on the $\{x_j\}$, and try to determine the corresponding optimum,

(b3) we may turn to some plausible general procedure of "testing", and trace its consequences for incomplete information of certain sorts,

(b4) we may select, possibly both quite arbitrarily and reasonably wisely, a specific criterion, and develop tables for its use.

We have been unable to see how approach (b1) can be made of practical utility, and we know of no attempts to do so. Our approach in this paper follows (b2), choosing contrasts as the class and maximization of least r^2 over a convex set of admissible $\{x_j\}$ as criterion. Approach (b3), choosing the likelihood-ratio principle, and applying it to the convex set defined by $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ has been adopted by Bartholomew [4], [5], [6]. Approach (b4) has been taken by Jonckheere [8] who uses a statistic based on ranks which is intuitively sensitive when $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, in the special case where each y_j can be replaced by the sample of n_j observations of which it is a mean.

Some comparisons of the power of the special case of our procedure corresponding to the convex set defined by $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ (simple order) and his own procedure have been made by Bartholomew [4]. Further discussion of these issues is, we believe, definitely in order but inappropriate to the present paper.

Wise formulation of problems of the general class discussed here is obviously a matter of great importance, deserving of great care and appropriate space. We note, for one thing, that any detailed discussion of the pros and cons of the choice of the maximum-least-r^2 criterion would be of limited value without an equally careful discussion of the pros and cons of the use of a single contrast.

2. The practical problem. In many cases where we have incomplete information about what the x_j will be like if $\beta \neq 0$, this information can be reasonably approximated by a system of linear equalities among the x_j. Since our information is most unlikely to have "sharp edges", it is important to emphasize the presence and meaning of the words "reasonably approximated." Any $\{c_j\}$ with $c_1 + c_2 + \cdots + c_n = 0$ defines a "valid" test of $\mu_1 = \mu_2 = \cdots = \mu_n$ whatever be the x_j. Precision of significance level will depend only on the adequacy of distribution theory based on assumptions of normality. Power of test for a specific situation will be a monotonic function of r^2. We are guaranteeing a minimum value for r^2 for any $\{x_j\}$ in a specific set. The value of r^2 for $\{x_j\}$ "near" this set will not fall far below this guarantee. Accordingly it will often be appropriate to choose the set to omit, though coming "near" to, some of the $\{x_j\}$ which are possible according to our incomplete information.

If, for instance, we are sure that the x_j will be *very nearly* nondecreasing, but we cannot exclude the possibility that there are small failures of monotonicity, we shall probably be well advised to choose the contrast maximizing the least r^2 for monotone x_j.

3. Remarks. Our basic interest is in the general problem of definite numerical approximation to latent configurations known only qualitatively via a set of constraining *linear* inequalities. We have particularized our aim to be the testing of "μ_j all equal" for clarity, and we invite the reader to particularize it further, say to the case where the y_j are means of repeated observations in a one-way analysis of variance, whose common variance is estimated on $\nu = n(k - 1)$ degrees of freedom by

$$s^2 = \text{error mean square/number, } k\text{, of observations per mean} \tag{6}$$

if this helps his insight.

Applications to problems in the field of psychological scaling [2] and attitude test scoring [1] have been briefly broached elsewhere.

It should be emphasized that, although the simple (rank) order case $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ is of great interest, it is by no means the only case we wish to consider. A general procedure will be developed for finding the "best" sequence $\{c_j\}$, given any particular set of linear homogeneous inequalities on the x's. In the present paper, after the general solution is developed, only simple order and certain illustrative examples will be considered. Subsequent papers will consider cases where more than simple order is known, mainly in terms of inequalities on the *differences* between the x's, as well as cases where less than simple order is known.

One comparison of some importance is the comparison of the t-tests (to which any single-degree-of-freedom approach must lead us) considered here with the F-test based on all $n - 1$ degrees of freedom in $\sum(y_j - y)^2$ and on s^2. This is considered, for the special case of simple order, in a related paper [10], where it is found that the use of a single well-selected contrast is almost certainly preferred to the omnibus F-test, which would be appropriate in the absence of information about order.

II. MAXIMIN r^2

In this part we discuss the geometrical problem of determining the contrast $\{c_j\}$ which makes $r^2_{\min}$, where the minimum is for all admissible $\{x_j - \bar{x}\}$, greatest. In geometrical terms, which we shall use extensively, we seek that direction which makes least the maximum angle with all admissible directions. Our general program is to first set up the geometrical problem, then to characterize its solution (leaving detailed proofs to Sections 18–20), and then to discuss specific steps required to locate solutions in the cases which concern us.

4. Basic geometry. We are concerned with c-vectors $\{c_j\}$ defining contrasts, and hence satisfying

$$c_1 + c_2 + \cdots + c_n = 0 \tag{7}$$

and with x-vectors defining the true (and unknown) configuration of mean values which satisfy

$$(8) \qquad (x_1 - \bar{x}) + (x_2 - \bar{x}) + \cdots + (x_n - \bar{x}) = 0.$$

All our vectors will, consequently, be restricted to an $(n - 1)$-dimensional subspace of n-dimensional Euclidean space. The squared length of the x-vectors (really $(x - \bar{x})$-vectors) will be denoted by $SSD = \sum(x_i - \bar{x})^2$.

The admissible x-vectors will be those satisfying a family of k linear inequalities, such as

$$(9) \qquad x_1 \leqq x_2 \quad \text{or} \quad x_2 - x_1 \leqq x_3 - x_2$$

and will form a convex set (a convex cone with apex at the origin), since nonnegative linear combinations of admissible x-vectors will be admissible. This convex subset of our $(n - 1)$-dimensional subspace will, with trivial exceptions, be a convex region of the subspace.

Any admissible vector satisfies each of the k defining inequalities in one of two ways, either by being $<$ or by being $=$. Any admissible vector thus corresponds to a pattern of $<$'s and $=$'s. In the present paper, *we shall consistently and tacitly assume that the origin,* $(0, 0, \cdots, 0)$, *is the only vector making all* k *inequalities into equalities.*

Each pattern with the maximum number (less than k) of $=$'s corresponds to a *corner* of our convex region, and to a set of vectors which are all positive multiples of one another—to a "direction". A corner will usually be represented by a conveniently normalized x-vector.

A corner is *simple* if only one inequality is a $<$, while all the others are $=$. If all corners are simple, which corresponds to linear independence of inequalities, we have *simple behavior.* We can always identify a corner by specifying the inequalities which are $<$ at it. If we have simple behavior, each corner corresponds to one inequality and vice versa. Many problems show simple behavior and those that do not are usually solved in terms of a problem or several problems which do show simple behavior.

Returning to the patterns of $<$'s and $=$'s with more than the minimum number of $<$'s, we note (i) that each pattern with but one $=$ corresponds to a face of the convex region (and to the hyperplane of which this face is a piece), (ii) that the pattern with all $<$'s corresponds to the *interior* of the convex region and (iii) that patterns with intermediate numbers of $<$'s correspond to *edges* of the convex region of various dimensionalities.

Corners, edges and faces alike, all run to the origin, since $(ex_1, ex_2, \cdots, ex_n)$ for any $e > 0$, no matter how small, satisfies the same (homogeneous linear) inequalities and equalities as $(x_1, x_2, \cdots, x_n)$.

In an inequality of the form $A \geqq B$, $A - B$ is the *slack.* If we have simple behavior, so that each corner has only one nonvanishing slack, we define the standard corner vectors as those whose nonvanishing slacks are $+1$.

In any case the corners generate the convex region, in the sense that, if we have fixed a particular representative vector for each corner, every admissible vector is a linear combination of these vectors with non-negative coefficients. If we have simple behavior, this representation is unique. If we use standard corner vectors, these coefficients are, of course, just the slacks of the admissible vector at the corresponding inequalities.

5. An example of standard corner vectors. Let us consider the restrictions

$$x_1 \leqq x_2 \leqq x_3 \leqq x_4 . \tag{10}$$

Here we have simple behavior, and can match inequalities, corner patterns and standard corner vectors as follows, where we give the standard corner vectors in two forms, one corresponding to values of $x_j - \bar{x}$ (to be used when forming linear combinations of corners) and the other consisting of convenient integer values of x_j (to be used in the maximin solution of Section 13).

Inequality	Corner Pattern	Standard Corner Vectors
$x_1 \leqq x_2$	$< \; = \; =$	$(-0.75, +0.25, +0.25, +0.25)$ or $(0, 1, 1, 1)$
$x_2 \leqq x_3$	$= \; < \; =$	$(-0.50, -0.50, +0.50, +0.50)$ or $(0, 0, 1, 1)$
$x_3 \leqq x_4$	$= \; = \; <$	$(-0.25, -0.25, -0.25, +0.75)$ or $(0, 0, 0, 1)$

If an (admissible) vector with $h_1 \leqq h_2 \leqq h_3 \leqq h_4$ is given, its slacks are $h_2 - h_1$, $h_3 - h_2$, $h_4 - h_3$. If we combine the standard corner vectors with these coefficients, we find that $(h_2 - h_1)(0.75, +0.25, +0.25, +0.25) + (h_3 - h_2)(-0.50, -0.50, +0.50, +0.50) + (h_4 - h_3)(-0.25, -0.25, -0.25, +0.75)$ reduces to (h_1, h_2, h_3, h_4), as it should, whenever $h_1 + h_2 + h_3 + h_4 = 0$.

If we use the other representation,

$$\begin{aligned}(h_1, h_2, h_3, h_4) = (h_2 - h_1)(0, 1, 1, 1) + (h_3 - h_2)(0, 0, 1, 1) \\ + (h_4 - h_3)(0, 0, 0, 1) + h_1(1, 1, 1, 1).\end{aligned} \tag{11}$$

(The appearance of a multiple of a origin shift vector all of whose entries are $+1$ is typical.)

The patterns $(=, <, <)$ $(<, =, <)$ and $(<, <, =)$, corresponding to the conditions $x_1 = x_2$, $x_2 = x_3$ and $x_3 = x_4$, respectively, define the three faces of the region. (In this case, since n is only 4, there are no edges of dimension intermediate between corners and faces.)

We shall see (in Section 13) that a contrast making r^2_{min} as large as possible has coefficients $(-0.866, -0.134, +0.134, +0.866)$ and that this c-vector provides $r^2_{\text{min}} = r^2_{\text{maximin}} = 0.651$.

6. Where is r^2 a minimum? We now consider any one fixed vector $(c_1, c_2, \cdots, c_n)$, the vectors $(x_1 - \bar{x}, x_2 - \bar{x}, \cdots, x_n - \bar{x})$ of some convex region, and the angles between the c-vector and the x-vectors. This angle is a continuous function of the x-vector, and attains an *unrestricted* maximum only when equal to 180°. Consequently, its maximum, when the x-vector is restricted to a region, either (i) is equal to 180°, or (ii) is located on a boundary of the region.

If it has to be on the boundary of a *convex* region, this maximum will be on a face, on an edge, or at a corner. Suppose it to be on a face. Project the c-vector onto the hyperplane of which the face is a part (which surely contains the origin). Maximizing the angle with the original c-vector is equivalent to maximizing the angle with the projected c-vector. The original argument now shows that the maximum is on the boundary of the face. Repeating this argument as often as necessary, we see that, if the maximum is not 180°, it must occur at a corner.

In terms of $r = \cos \theta$, which takes the value -1 when $\theta = 180°$, the result is as follows: The minimum value of r either (i) is -1, or (ii) occurs at a corner.

Suppose now that r is non-negative for each corner. This is equivalent to

$$\sum c_j \cdot (x_j - \bar{x}) \geqq 0 \tag{12}$$

for each corner vector. The same inequality must hold for all non-negative linear combinations of corner vectors. Thus $r > 0$ throughout the convex region so that $r = -1$ is impossible.

We have shown that, if r is non-negative for all corners, its minimum value for admissible x-vectors must occur at a corner. Conversely, if r takes on both signs at corners, there will be an admissible vector with $r = 0$. Thus in working with minimum r^2 we can forget all admissible vectors except a convenient set of corner vectors.

7. Heuristics. We now face a simple problem: Given a finite number of directions, find the direction whose *maximum* angle with any of the given directions is *least*. To understand what results to expect, it is useful to consider a simpler analog: Given a finite set of points, find the point whose *maximum* distance from any of these points is *least*.

If we are given three points in a plane, it is natural to fill in the triangle they determine and to think in terms of this triangle. If the triangle has no angles exceeding a right angle, the point sought will be equidistant from all three vertices. But if the triangle is too flat, such an equidistant point will be outside the triangle and wasteful of maximum distance; we shall do better by using the midpoint of the longest side, which is equidistant from two of the three points and closer to the third. (Clearly no point can have its maximum distance less than half of the longest side, so that, if no distance from this midpoint exceeds this value, this midpoint must be the point we seek.)

The generalization of this result to as many points as we wish may be given in the following form:

If a set of k points is specified, and we seek the point whose maximum distance from any of these points is least, the solution is provided by any point which can be represented as such a linear combination of the specified points that (i) no specified point appears with negative weight, (ii) each specified point appearing with positive weight is at the same distance, and (iii) no other one of the given points is further away than this distance. There is one and only one point satisfying (i), (ii), and (iii).

8. The key result. We show in Sections 17 to 20 that the answer to the maximin angle problem may be put in a similar form:

THEOREM. *If we are given a set of k directions and seek the direction whose maximum angle with any of these directions is least, the solution is provided by a direction representable as a linear combination of the given directions in such a way that* (i) *no given direction appears with negative weight,* (ii) *each given direction appearing with positive weight makes the same angle with the solution, and* (iii) *no other given direction makes a larger angle. There is one and only one direction making angles less than a right angle with the given directions and satisfying* (i), (ii), *and* (iii).

According to (i) the maximin c-vector is a nonnegative linear combination of x-vectors, and hence is an admissible vector. The first step in checking a possible maximin c-vector is to see if it satisfies all the inequalities required of admissible x-vectors. If it does not, it is not maximin.

If the candidate vector is admissible, and the problem has simple behavior, then directions appearing with zero weight correspond to $=$'s in inequalities, while $>$'s correspond to positive weights. Under simple behavior, the second step in checking a candidate vector is to confirm that:

(ii*) Its r^2 with the corners corresponding to inequalities that are $>$ are all equal.

(iii*) Its r^2 with the corners corresponding to inequalities that are $=$ are all at least as large as the common value of the r^2's of (ii*).

Specifically, a *completely equiangular* contrast, one equiangular with all corners, which *also* satisfies the defining inequalities must be the maximin contrast.

There is an important corollary. If we start with a subset of the directions with which we are concerned, and find the one of their linear combinations which satisfies (i), (ii), and (iii) for the subset (but may violate (iii) for some direction not in the subset), then we can calculate both lower and upper bounds for the least maximum angle for the whole set. For the least maximum angle for the whole set is surely *no less* than the least maximum angle (with all directions) of the subset solution (and *no more* than the maximum angle for this candidate). If we are satisfied with a reasonable approximation to the least maximum angle (to r^2_{maximin}) we can often use such bounds to show that we have a practically useful answer without going on to obtain the precise maximin contrast.

9. Equiangular contrasts. We shall clearly wish to calculate the coefficients of contrasts which make equal angles to—are equally correlated with—certain sets of corners. Such a *completely or partially equiangular* contrast will have a common value for the ratio of $[\sum c_j(x_j - \bar{x})]^2$ to $\sum(x_j - \bar{x})^2$ for the corresponding set of corner vectors. It is very convenient to choose this ratio equal to unity. For if this is done,

$$r^2 = \frac{[\sum c_j(x_j - \bar{x})]^2}{(\sum c_j^2)\sum(x_j - \bar{x})^2} = \frac{1}{\sum c_j^2} \tag{13}$$

is easily calculated, while the equations

$$(14) \qquad \sum c_j(x_j - \bar{x}) = \sum c_j x_j = [\sum (x_j - \bar{x})^2]^{\frac{1}{2}} = (SSD)^{\frac{1}{2}}$$

often involve simple coefficients.

All this has been for situations with simple corners; what of situations with complex corners? In practice, so far as we can see, situations with complex corners are going to be handled either (i) by selecting out a subset of corners which behave like simple corners, working with them, and finally testing the "solution" for such a subset against the remaining corners or (ii) by special devices appropriate to the problem. Thus simple behavior is the important case.

10. Finding the solution. We have characterized the solution, but we have still to find it. While it is true that we could find it by

(a) selecting j (linearly independent) directions with $2 \leqq j \leqq k$,

(b) finding the corresponding (partially) equiangular vector,

(c) checking to see if all coefficients are positive by checking to see if all inequalities are satisfied,

(d) checking to see if all the $k - j$ angles with the other given directions are no greater than the j equal angles with the j selected directions,

(e) repeating this process until the checks of (c) and (d) are all satisfied,

we might have to try all $2^k - k - 1$ sets of $j \geqq 2$ directions. For $k = 10$ this would be 1013 cases, and for $k = 20$ would be 1,048,555 cases. Neither number is pleasant, while the last is surely impractical.

Some less exhausting method is desirable. It appears that such a method will be tentative and exploratory, since there seems to be no method guaranteed to find the solution in a small number of steps. We shall describe a method which starts from the "top" and moves down, and one which starts from the bottom and moves up. Detailed examples are given in Sections 15 and 16.

11. Iterative approaches "from the top down" and "from the bottom up". The basic steps of an approach "from the top down" are as follows:

(t0) *Choose a relatively large set of trial corners which has simple behavior* (i.e., are linearly independent). Each trial corner will then have an associated inequality, although these inequalities need not be the same as for the original problem. (See Section 16 for an example.)

(t1) *Calculate, as a trial contrast,* using any symmetry present to simplify Equations (14), *an equiangular contrast for the corners considered.*

(t2) *Check for negative coefficients in the corresponding linear combination* by checking whether the trial contrast satisfies all the associated inequalities of (t0), and pass on to (t3) or (t4) accordingly.

(t3) *If the inequalities are satisfied, find the minimum r^2 for the trial contrast with all the corners including those not included in the subset.* If this is the same as the common value of the angles with the trial set, the exact solution has been reached. If it is an adequate approximation to that value, then an approximate solution has been found. Otherwise another start must be tried.

(t4) *If one or more of the associated inequalities fails, each of the corresponding corner equations is replaced by the inequality converted into an equality, and a new trial contrast is generated.* This contrast will make equal angles with all the trial corners whose equations were not converted, and is suitable for step (t2). (One must then be careful, in a later application of (t3), to include among the "other corners" with which angles are to be checked corners whose equations have thus been converted.)

While many problems yield most easily "from the top down," so that such an approach will be most usually tried first, there are sometimes certain advantages to proceeding "from the bottom up".

Here the steps run as follows:

(b0) *Choose a relatively small set of corners.*

(b1) *Find as a trial contrast*, often by a combination of symmetry arguments and Equations (14), *the contrast which gives* $r^2_{\min}$ *for the* j *initial corners.* (If this process is too difficult, try another starting set.)

(b2) *Calculate* r^2 *for this trial contrast with all other corners and find its minimum value.* If this minimum is the same as $r^2_{\min}$ for the initial set of corners, the solutions has been reached. If the minimum is close enough to the $r^2_{\min}$ for the initial set of corners, a practical solution has been found.

(b3) *If neither a precise or adequately approximate solution is found in* (b2), *the starting set is enlarged by adding that corner* (or those corners) *whose* r^2 *with the trial contrast is least.*

12. Quadratic programming. The problems we face are all instances of quadratic programming [3], [9], [11]; (see [12] for other references) and can thus be solved, if necessary, by the iterative techniques which have been developed for use in solving general quadratic programs. These techniques usually are only feasible when a stored-program computer of substantial size is available. Most of the problems we have considered yield with satisfactory ease to much less powerful tools.

III. SIMPLE ORDER

13. The finite case. We now treat the case where admissibility is defined by

$$x_1 \leqq x_2 \leqq \cdots \leqq x_n . \tag{15}$$

Consider the restriction $x_1 \leqq x_2 \leqq x_3 \leqq x_4$. The three inequalities, and the corresponding corners with their sums of squares of deviations, are given by

$$\begin{aligned} &(a) \quad x_1 \leqq x_2, \quad (0, 1, 1, 1) \text{ with } SSD = 3/4 \\ &(b) \quad x_2 \leqq x_3, \quad (0, 0, 1, 1) \text{ with } SSD = 1 \\ &(c) \quad x_3 \leqq x_4, \quad (0, 0, 0, 1) \text{ with } SSD = 3/4 \end{aligned}$$

If we seek an equiangular contrast with the sum of cross-products equal to the square root of the sum of squared deviations (Equation (14) in Section 9), we require

$$
\begin{aligned}
&(s) \quad c_1 + c_2 + c_3 + c_4 = 0,\\
&(a) \quad c_2 + c_3 + c_4 = (3/4)^{\frac{1}{2}},\\
&(b) \quad c_3 + c_4 = 1^{\frac{1}{2}},\\
&(c) \quad c_4 = (3/4)^{\frac{1}{2}},
\end{aligned}
$$

where (s) is the contrast-ensuring equation.

Proceeding by successive substitution, the solution is (−866, −.134, .134, .866), whose sum of squares is 1.536, so that (Equation (13) of Section 9) its r^2 with each of the three corners is $1/1.536 = .651$. This equiangular contrast clearly satisfies each of the three inequalities, and, hence, must be the desired maximin contrast.

The same construction can be repeated for the general case $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, and leads to

$$c_j = \{(j-1)[1-((j-1)/n)]\}^{\frac{1}{2}} - \{j(1-j/n)\}^{\frac{1}{2}} \tag{16}$$

with the results shown in Table 1 for $n \leqq 20$. Asymptotic behavior for large n will be treated in the next section. In addition to the maximin coefficients and the corresponding values of r^2, Table 1 also provides minimum r^2 values for linearly varying coefficients, and for linearly varying coefficients (i) with the end values doubled, called "linear-2", and (ii) with the end values quadrupled and the next to the end values doubled, called "linear-2-4". For $n = 10$, for example, these coefficients would be proportional to

−9, −7, −5, −3, −1, +1, +3, +5, +7, +9, (linear)
−18, −7, −5, −3, −1, +1, +3, +5, +7, +18 (linear-2)
−36, −14, −5, −3, −1, +1, +3, +5, +14, +36 (linear-2-4)

respectively.

The linear-2-4 contrast yields especially high ratios of minimum r^2 to maximin r^2 (i.e., "maximin efficiency"), as Table 1 shows. There are at least two reasons why, even though we know the formula for the coefficients of the maximin contrast, we may wish to use such a contrast as linear-2-4:

(1) Integer coefficients are often computationally more manageable.

(2) The form of the linear-2-4 coefficients is easier to remember, making the contrast more usable when tables of maximin coefficients are not at hand ($n \leqq 20$) or have not been calculated ($20 \leqq n \leqq 50$, say).

14. Asymptotic behavior. Here we derive, for the case of simple rank order, an approximation to the value of maximin r^2 as a function of n. Equation (16) specifies that the maximin contrast coefficients are formed as (backwards) first differences of:

$$z(t) = \{t(1-t/n)\}^{\frac{1}{2}} \qquad 0 \leqq t \leqq n, \tag{17}$$

TABLE 1

Performance of various contrasts against simple rank order, together with the coefficients of the maximin contrasts, for n ≦ 20.

	Minimum r^2								
	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
maximin	1.000	.750	.651	.596	.557	.530	.510	.492	.478
min* for linear	1.000	.750	.600	.500	.429	.375	.333	.300	.273
min for linear-2	1.000	.750	.649	.588	.546	.512	.485	.462	.441
min for linear-2-4	1.000	.750	.649	.588	.549	.522	.493	.473	.456
	Minimum r^2 as % of maximum r^2 ("maximin efficiency")								
	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
linear	100	100	92	84	77	71	65	61	57
linear-2	100	100	100	99	98	97	95	94	92
linear-2-4	100	100	100	99	99	98	97	96	95

j	Values of c_j for maximin contrast								
	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
1	−.707	−.816	−.866	−.894	−.913	−.926	−.935	−.943	−.949
2	.707	.000	−.134	−.201	−.242	−.269	−.289	−.305	−.316
3		.816	.134	.000	−.070	−.114	−.144	−.167	−.184
4			.866	.201	.070	.000	−.045	−.076	−.100
5				.894	.242	.114	.045	.000	−.032
6					.913	.269	.144	.076	.032
7						.926	.289	.167	.100
8							.935	.305	.184
9								.943	.316
10									.949
	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
SSD	1.000	1.333	1.536	1.679	1.794	1.886	1.961	2.032	2.091

	Minimum r^2									
	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$	$n = 17$	$n = 18$	$n = 19$	$n = 20$
maximin	.467	.457	.447	.439	.433	.427	.420	.415	.410	.406
min* for linear	.250	.231	.214	.200	.188	.176	.167	.158	.150	.143
min for linear-2	.423	.407	.392	.378	.366	.354	.343	.333	.324	.315
min for linear-2-4	.443	.433	.423	.418	.412	.407	.402	.397	.393	.389
	Minimum r^2 as % of maximin r^2 ("maximin efficiency")									
	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$	$n = 17$	$n = 18$	$n = 19$	$n = 20$
linear	54	51	48	46	43	41	40	38	37	35
linear-2	91	89	88	86	85	83	82	80	79	78
linear-2-4	95	95	95	95	95	95	96	96	96	96

TABLE 1—*Continued*

j	Values of c_j for maximin contrast†									
	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$	$n = 17$	$n = 18$	$n = 19$	$n = 20$
1	−.953	−.957	−.961	−.964	−.966	−.968	−.970	−.972	−.973	−.975
2	−.326	−.333	−.340	−.346	−.351	−.354	−.358	−.362	−.364	−.366
3	−.198	−.209	−.218	−.226	−.233	−.238	−.243	−.248	−.252	−.255
4	−.118	−.133	−.145	−.155	−.163	−.170	−.178	−.183	−.188	−.192
5	−.056	−.075	−.090	−.102	−.113	−.122	−.130	−.136	−.142	−.147
6	.000	−.024	−.043	−.059	−.071	−.082	−.092	−.100	−.107	−.113
7	.056	.024	.000	−.019	−.035	−.047	−.059	−.068	−.077	−.084
8	.118	.075	.043	.019	.000	−.015	−.029	−.040	−.050	−.058
9	.198	.133	.090	.059	.035	.015	.000	−.014	−.024	−.034
10	.326	.209	.145	.102	.071	.047	.029	.014	.000	−.011
	$n = 11$	$n = 12$	$n = 13$	$n = 14$	$n = 15$	$n = 16$	$n = 17$	$n = 18$	$n = 19$	$n = 20$
SSD	2.141	2.189	2.235	2.277	2.312	2.344	2.379	2.411	2.437	2.465

* The minimum occurs for the linear and linear-2 sequences at the corner labelled by $(n - 1)$ zeros and one 1. This is also true for linear-2-4 for $n \leqq 7$; but the linear-2-4 minimum occurs at the corner with 4 ones for $n = 8$, 9, or large; at the corner with 5 ones for $n = 10$, 11, 17, 18, 19, 20; at the corner with six ones for $n = 12$, 13, 15, 16; and at the corner with 7 ones for $n = 14$.

† For $n \geqq 11$ use symmetry to complete the table.

namely

$$(18) \qquad c_j = z(j-1) - z(j) \quad \text{where} \quad j = 1, 2, \cdots, n.$$

A natural approximation to these differences is given by

$$(19) \qquad c\left(t + \frac{1}{2}\right) = -\frac{dz}{dt} = \frac{2t/n - 1}{2\{t(1 - t/n)\}^{1/2}}.$$

This approximation is poor at the extremes but rather good elsewhere. To find an approximation to $\sum c_j^2$, therefore, one may use the exact values at the two extreme values of j, and an integral approximation elsewhere. The net result is

$$(20) \qquad \sum_{j=1}^{n} c_j^2 \doteq c_1^2 + c_n^2 + \int_1^{n-1} \frac{(2t/n - 1)^2}{4t(1 - t/n)}\, dt \doteq 1 + \tfrac{1}{2}\cdot\log(n - 1)$$

so that

$$(21) \qquad r^2 \doteq 2/[2 + \log(n - 1)].$$

At $n = 20$, this approximation yields .4044, compared with the true value .4056; at $n = 50$, the approximation is .3395, the true value .3399. The values given in Table 2 for scattered $n > 50$ have been computed from this approximate formula. Linear and linear-2-4 results are included for comparison.

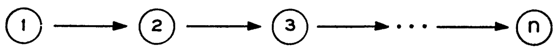
TABLE 2

Performance of various contrasts against simple order Minimum r^2 for selected n's with $2 \leqq n \leqq 1000$

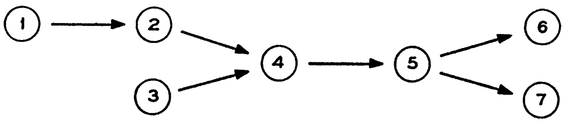
	$n = 2$	$n = 5$	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 200$	$n = 500$	$n = 1000$
Maximin	1.000	.596	.478	.406	.340	.303	.274	.244	.225
Linear	1.000	.500	.273	.143	.059	.030	.015	.006	.003
Linear-2-4	1.000	.533	.456	.389	.306	.231	.155	.079	.043

IV. EXAMPLES OF ITERATIVE SOLUTIONS

15. Other tree diagrams. If the inequalities among the x's are indicated by arrows, the case so far treated may be diagrammed thus



This case led to linear equations which could be solved one value at a time by successive substitutions. Such easy solution occurs for any "tree" diagram because of the one-to-one correspondence between inequalities and corners. For example, consider the diagram



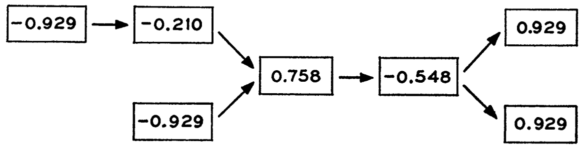
for which inequalities and corners are as follows:

$$
\begin{array}{llll}
(a) & x_1 \leqq x_2, & (0, 1, 1, 1, 1, 1, 1) \text{ with } SSD = 6/7 \\
(b) & x_2 \leqq x_4, & (0, 0, 1, 1, 1, 1, 1) \text{ with } SSD = 10/7 \\
(c) & x_3 \leqq x_4, & (1, 1, 0, 1, 1, 1, 1) \text{ with } SSD = 6/7 \\
(d) & x_4 \leqq x_5, & (0, 0, 0, 0, 1, 1, 1) \text{ with } SSD = 12/7 \\
(e) & x_5 \leqq x_6, & (0, 0, 0, 0, 0, 1, 0) \text{ with } SSD = 6/7 \\
(f) & x_5 \leqq x_7, & (0, 0, 0, 0, 0, 0, 1) \text{ with } SSD = 6/7
\end{array}
$$

Equations (14) for a completely equiangular contrast become

$$
\begin{array}{ll}
(s) & c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = 0 \\
(a) & c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = (6/7)^{\frac{1}{2}} \\
(b) & c_3 + c_4 + c_5 + c_6 + c_7 = (10/7)^{\frac{1}{2}} \\
(c) & c_1 + c_2 + c_4 + c_5 + c_6 + c_7 = (6/7)^{\frac{1}{2}} \\
(d) & c_5 + c_6 + c_7 = (12/7)^{\frac{1}{2}} \\
(e) & c_6 = (6/7)^{\frac{1}{2}} \\
(f) & c_7 = (6/7)^{\frac{1}{2}}
\end{array}
$$

whence we may easily find the *four* end coefficients $c_1 = c_3 = (6/7)^{\frac{1}{2}}$, $c_6 = c_7 = (6/7)^{\frac{1}{2}}$ and continue inward. The resulting contrast, converted to decimal form and arranged in the original pattern becomes



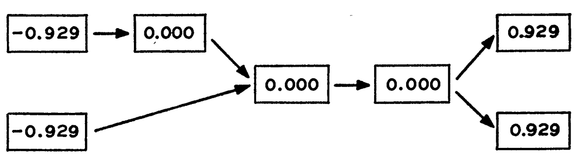
We now clearly see that one inequality, (d) has been violated, so that this completely equiangular contrast, for which $\sum c_j^2 = 4.359$ and $r_{\min}^2 = .229$, is not the maximin contrast.

The discussion of Section 11 suggests that we try either (i) proceeding "down" by dropping a corner or (ii) starting "up" from two or more widely-spread corners.

We detail the latter procedure here. Extremity of angles makes it reasonable to begin with the *four* corners (a), (c), (e) and (f). We impose the *equalities* associated with corners (b) and (d), which we are for the moment not considering, thus forcing the solution to give zero weight to these corners. We then have the modified equations, whose solution will still be only a trial contrast.

$$
\begin{array}{ll}
(s) & c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = 0 \\
(a) & \phantom{c_1 + {}} c_2 + c_3 + c_4 + c_5 + c_6 + c_7 = (6/7)^{\frac{1}{2}} \\
(b) & \phantom{c_1 + {}} c_2 = c_4 \\
(c) & c_1 + c_2 + c_4 + c_5 + c_6 + c_7 = (6/7)^{\frac{1}{2}} \\
(d) & c_4 = c_5 \\
(e) & c_6 = (6/7)^{\frac{1}{2}} \\
(f) & c_7 = (6/7)^{\frac{1}{2}}
\end{array}
$$

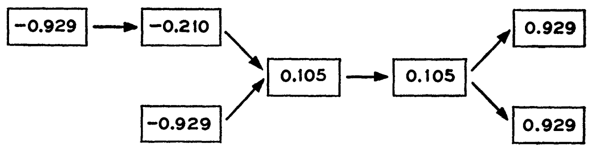
Their solution is $[-(6/7)^{\frac{1}{2}}, 0, -(6/7)^{\frac{1}{2}}, 0, 0, (6/7)^{\frac{1}{2}}, (6/7)^{\frac{1}{2}}]$ or,



with $\sum c_j^2 = 24/7 = 3.430$ and hence r^2 with corners (a), (c) (e) and (f) of $1/3.430 = .292$. Turning to corner (b), whose $(SSD)^{\frac{1}{2}}$ is $(10/7)^{\frac{1}{2}}$, we find $\sum c_i x_i$ equal to $(6/7)^{\frac{1}{2}}$. The value of r^2 with this corner is thus smaller than the common value .292, namely $(6/10)(.292) = .175$. This violates Condition (iii) (of the Theorem of Section 8), so we cannot have found the solution. The first try from the bottom has thus only provided us with upper and lower bounds, 0.292 and 0.175 for the maximin r^2. The final solution may actually be reached in one more step by adding (b) to the subset (a), (c), (e), (f) already considered; eliminating c_5, we have

$$\begin{array}{ll}
(s) & c_1 + c_2 + c_3 + 2c_4 + c_6 + c_7 = 0 \\
(a) & c_2 + c_3 + 2c_4 + c_6 + c_7 = (6/7)^{\frac{1}{2}} \\
(b) & c_3 + 2c_4 + c_6 + c_7 = (10/7)^{\frac{1}{2}} \\
(c) & c_1 + c_2 \qquad 2c_4 + c_6 + c_7 = (6/7)^{\frac{1}{2}} \\
(e) & c_6 = (6/7)^{\frac{1}{2}} \\
(f) & c_7 = (6/7)^{\frac{1}{2}}
\end{array}$$

of which the solution is

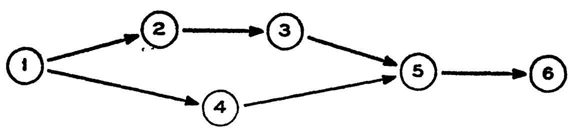


All inequalities are now clearly satisfied. The sum of squares of coefficients is now 3.495, so that $r^2_{\text{min}} = 0.286$ (as compared with $r^2_{\text{min}} = .229$ for the completely equiangular contrast).

Note how close the upper bound obtained "from below" of 0.292 is to the actual 0.286. The values of r^2 are 0.286 for each corner except corner (d) and 0.643 for this corner. By allowing the correlation with corner (d) to increase beyond the others, we have been able to increase all r^2 to at least 5/4ths their previous values, and have reached the maximin.

16. An example lacking simple behavior. In the previous section, we illustrated an iterative solution "from the bottom up." Now we illustrate a solution "from the top down", and introduce the complication of nonsimple behavior.

Suppose that we have a diagram with one or more loops but no aid from symmetry. For example, suppose that the available inequalities are $x_1 \leqq x_2 \leqq x_3 \leqq x_5 \leqq x_6$ and $x_1 \leqq x_4 \leqq x_5$, for which the diagram is



In this example, we have the inequalities

$$\begin{array}{ll}
(a) & x_1 \leqq x_2 \\
(b) & x_2 \leqq x_3 \\
(c) & x_3 \leqq x_5 \\
(d) & x_1 \leqq x_4 \\
(e) & x_4 \leqq x_5 \\
(f) & x_5 \leqq x_6
\end{array}$$

Each corner corresponds to a minimal set of arrows whose deletion disconnects the diagram. Enumerating these sets, we find seven corners, one of them simple, the other six "compound." Each of the six compound corners makes two of the inequalities $<$ (with excess of unity) and the other four $=$. If the corners are identified by the $<$ inequalities, we have

$$\begin{array}{ll} (ad) & (0, 1, 1, 1, 1, 1) \text{ with } SSD = 5/6 \\ (ae) & (0, 1, 1, 0, 1, 1) \text{ with } SSD = 8/6 \\ (bd) & (0, 0, 1, 1, 1, 1) \text{ with } SSD = 8/6 \\ (be) & (0, 0, 1, 0, 1, 1) \text{ with } SSD = 9/6 \\ (cd) & (0, 0, 0, 1, 1, 1) \text{ with } SSD = 9/6 \\ (ce) & (0, 0, 0, 0, 1, 1) \text{ with } SSD = 8/6 \\ (f) & (0, 0, 0, 0, 0, 1) \text{ with } SSD = 5/6 \end{array}$$

We know that the maximin contrast will also be a maximin contrast for some linearly independent set of five corners. There are $\binom{7}{5} = 21$ sets of five corners, but not all are linearly independent. Thus we can reduce the number of explorations by learning which sets of five involve linear dependence. We could do this directly, examining the standard corner sequence and noting, for example, that $(ad) + (be) = (ae) + (bd)$, or we could proceed in another way, one which provides us with conveniently simplified equations.

The contrast-ensuring condition (s) is always available, and by its use we may reduce the corner equations to the forms,

$$\begin{array}{lrl} (ad) & c_1 = -(5/6)^{\frac{1}{2}} & \\ (ae) & c_1 + c_4 = -(8/6)^{\frac{1}{2}} & \\ (bd) & c_1 + c_2 = -(8/6)^{\frac{1}{2}} & \\ (be) & c_1 + c_2 + c_4 = -(9/6)^{\frac{1}{2}} & \text{and} \quad c_3 + c_5 + c_6 = (9/6)^{\frac{1}{2}} \\ (cd) & c_1 + c_2 + c_3 = -(9/6)^{\frac{1}{2}} & \text{and} \quad c_4 + c_5 + c_6 = (9/6)^{\frac{1}{2}} \\ (ce) & c_5 + c_6 = (8/6)^{\frac{1}{2}} & \\ (f) & c_6 = (5/6)^{\frac{1}{2}} & \end{array}$$

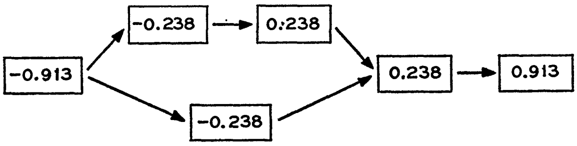
A value for c_4 is determinable from three different pairs of corner equations as follows:

$$\begin{array}{ll} (ad \text{ and } ae) & c_4 = -(8/6)^{\frac{1}{2}} + (5/6)^{\frac{1}{2}} \\ (bd \text{ and } be) & c_4 = -(9/6)^{\frac{1}{2}} + (8/6)^{\frac{1}{2}} \\ (cd \text{ and } ce) & c_4 = (9/6)^{\frac{1}{2}} - (8/6)^{\frac{1}{2}} \end{array}$$

No two of these values are consistent, so that no two of these pairs can appear among the selected five corners. This reduces us to 12 pentads.

These are tried successively until one is found which successfully satisfies the conditions stated in Section 8. The uniqueness result guarantees that only one such success is possible.

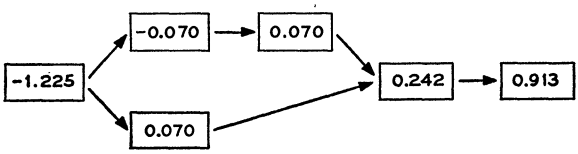
Trial of the pentad (*ad*, *ae*, *bd*, *ce*, *f*) leads to the partly equiangular contrast



The sums of products with unused corners (*be*) and (*cd*) are respectively 1.389 > $(9/6)^{\frac{1}{2}}$ and .913 < $(9/6)^{\frac{1}{2}}$. The last inequality shows that this contrast has a *lower* correlation with corner (*cd*) than with the other corners, thereby violating Condition (iii) of the Theorem. Expressing this violation in a standard way, we say that this pentad "fails on corner (*cd*)." When such failure is encountered, we set aside the pentad, hoping to find the solution from another starting point.

Eight more of the pentads fail on one or both unused corners. Three pentads, however, do not fail on the unused corners, and it will be instructive to consider these in detail.

Pentad (*ae*, *be*, *cd*, *ce*, *f*) yields



The sums of products with (*ad*) and (*bd*) are 1.225 > $(5/6)^{\frac{1}{2}}$ and 1.295 > $(8/6)^{\frac{1}{2}}$. All of the original inequalities (*a*)-(*f*) are satisfied by this contrast, and it would seem that this is the solution, with $r_{\min}^2 = .415$. However, a subtlety arises in cases with compound corners. The above contrast is not a convex combination of the corners, as required by our Theorem. This may readily be seen by solving for the linear combination of corners that produces the contrast, $\{c_j\}$.

$$\{c_j\} = 1.155(ae) + .140(be) + 1.295(cd) - 1.123(ce) + .671(f) - 1.225(s)$$

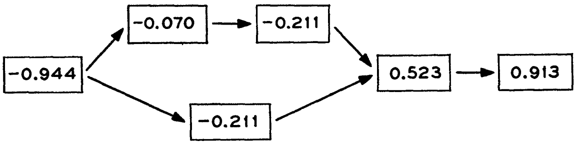
Corner (*ce*) enters with negative weight, signifying that further manipulation is needed.

At this point it is necessary to construct a single inequality which has an excess of unity at corner (*ce*) and zero at (*ae*), (*be*), (*cd*), and (*f*). (We are in effect looking now at a hypothetical five-inequality problem with precisely the present pentad as its standard *simple* corners.) This single inequality turns out to be

$$(g) \quad x_5 - x_4 - x_3 + x_1 \geqq 0.$$

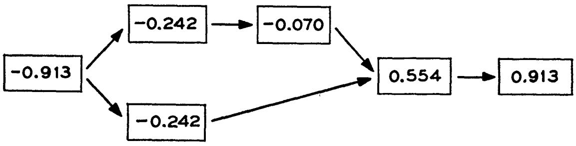
Trial proceeds according to (t4) of Section 11 by forcing equality to hold for

(g) and finding the equiangular contrast among (ae), (be), (cd), and (f), leading to



with $r^2 = .478$ vs. (ae), (be), (cd), and (f). The r^2 is improved, as one expects. However, this contrast now fails on (bd) and is set aside in favor of a new trial pentad.

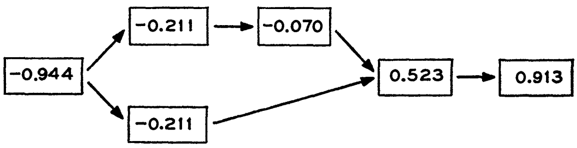
The pentad (ad, ae, bd, cd, f) yields



which does not fail on (be) or (ce) and has $r^2_{\min} = .477$. However, this is not a convex combination, (ad) entering with negative weight. The hypothetical inequality for which (ad) is the standard corner in this pentad is

$$(h) \quad x_1 - x_2 - x_4 + x_5 \geqq 0.$$

Forcing equality (h) and writing corner equations for (ae), (bd), (cd), and (f) gives the contrast



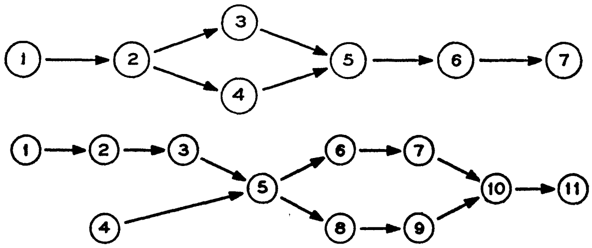
with $r^2_{\min} = .478$. This contrast is indeed a convex combination (except for the coefficient of the scale shift vector (s)).

$$\{c_j\} = .733(ae) + .141(bd) + .592(cd) + .390(f) - .944(s).$$

The contrast succeeds on corners (be), (ce) and (ad), and thus is the required solution.

This solution may also be reached via the remaining pentad (ae, bd, be, cd, f). After the first step, corner equation (be) must be replaced by an equality which turns out to be (h).

Similar exhaustion techniques appear to be necessary whenever the diagram has loops which cannot be dealt with by symmetry arguments or by special devices. It may be worthwhile to notice that the examples below are essentially tree-like and may be straightforwardly solved, since symmetry allows us to put $c_3 = c_4$ in the first and $c_6 = c_8$ and $c_7 = c_9$ in the second, so that we need consider only symmetrical corners, whose possible compoundness we need never notice.



V. PROOFS

In this part we develop and establish the geometric facts which we have already used. Certain simple facts about unit vectors in n-dimensional space come first, and are then shown to imply the characterization and uniqueness of the solution.

17. Three results of elementary geometry. We begin by establishing the following:

LEMMA. *Given $k + 1$ nonzero vectors, u_0, u_1, $\cdots$, u_k in a linear space*, either

(1) *There are non-negative constants a_j such that $u_0 = \sum a_j u_j$*, or

(2) *There is a hyperplane through the origin with u_0 definitely on one side and all the u_j either in it or on the other side.*

If (1) fails, then the set K of all vectors of the form $\sum a_j u_j$, all $a_j \geqq 0$, is a closed convex set including the origin which does not contain u_0. Consequently there is a neighborhood U of u_0 which does not intersect K. This neighborhood can of course be taken to be convex. If L consists of all vectors of the form au, with $a > 0$ and u in such a convex U, then L is an open convex set which is disjoint from K. By a classical result [7], there is a hyperplane which separates K from L in the weak sense that L is on one side of the hyperplane while the points of K are either in the hyperplane or on its other side. Since the origin is in K, and is a limit point of L, this hyperplane must contain the origin, establishing (2).

In our Euclidean n-space, where each hyperplane through the origin is perpendicular to a unit vector, this result can be restated as follows:

LEMMA A. *Given $k + 1$ unit vectors u_0, u_1, $\cdots$, u_k in Euclidean n-space*, then either

(1) $u_0 = \sum a_j u_j$ with all $a_j \geqq 0$, or

(2) *there is a unit vector w such that $u_0 \cdot w < 0$ but $u_j \cdot w \geqq 0$ for $j = 1$* to k.

Next we have:

LEMMA B. *If the u_j, u, and v are distinct unit vectors with* (i) $u \neq v$, (ii) $u \cdot u_j \geqq \varphi$ *for all j*, (iii) $v \cdot u_j \geqq \varphi$ *for all j, where $\varphi > 0$, then there are infinitely many distinct unit vectors w with $w \cdot u_j > \varphi$ for all j. These w come arbitrarily close to u.*

If $au + bv$, where $a, b > 0$, is a unit vector w,

$$1 = (au + bv) \cdot (au + bv) = a^2 + b^2 + 2ab(u \cdot v) < (a + b)^2$$

so that $a + b > 1$ and

$$(w \cdot u_j) = (au + bv) \cdot u_j = a(u \cdot u_j) + b(v \cdot u_j) \geqq (a + b)\varphi > \varphi.$$

Thus any such $au + bv$ serves as a w, which will be arbitrarily close to u when b is arbitrarily small.

Finally:

LEMMA C. *If* $u = \sum a_j u_j$, *and* $u \cdot u_j = \varphi$ *for all* j, *then the only other unit vector* $v = \sum b_j u_j$ *making equal angles with all* u_j *is* $v = -u$.

Suppose first $\varphi = 0$. Then $u \cdot u = u \cdot \sum a_j u_j = \sum a_j(u \cdot u_j) = \sum a_j \cdot 0 = 0$ which is impossible. If $\varphi \neq 0$, then set $t = (1/\varphi)u$, so that $u_j \cdot t = 1$ for all u_j in P. If v makes all $v \cdot u_j = \psi$, then, by the argument just made $\psi \neq 0$, and we may set $s = (1/\psi)v$. We then have, for all u_j in P,

$$u_j \cdot s = 1 \quad \text{and} \quad u_j(t - s) = 0.$$

Since $t - s$ is a linear combination of the u_j, it follows that $(t - s) \cdot (t - s) = 0$, that $t = s$, that $\psi = \pm\varphi$ and that $v = \pm u$. The result is proved.

18. Necessity of the equiangular-or-closer condition. Consider now a set A of unit vectors u_j and any unit vector u. The maximum angle θ between u and any u_j is determined by the minimum value of $u_j \cdot u$, the one increasing as the other decreases. If u cannot be represented in the form $u = \sum a_j u_j$ with all $a_j \geqq 0$, then by Lemma A there is a unit vector w such that

$$u_j = v_j \cos \eta_j + w \sin \eta_j, \qquad u = v \cos \eta - w \sin \eta$$

where v and the v_j are perpendicular to w, $0 \leqq \eta_j \leqq \pi/2$ and $0 < \eta \leqq \pi/2$. (Note that $\eta = 0$ is excluded.) Consequently

$$u \cdot u_j = (v \cdot v_j) \cos \eta_j \cos \eta - \sin \eta_j \sin \eta < (v \cdot v_j) \cos \eta_j = v \cdot u_j$$

so that u cannot maximize the least $u \cdot u_j$, and consequently cannot minimize the greatest angle between u and any u_j.

Consider now some u which does maximize the least $u \cdot u_j$ (and minimize the greatest angle). It is certainly of the form $u = \sum a_j u_j$ with all $a_j \geqq 0$. Suppose further that $u \cdot u_j > 0$ for all j. Let M consist of those u_k for which $u \cdot u_k = \min_j[u \cdot u_j]$ that is for which the angle between u and u_k is equal to the greatest angle between u and any u_j.

For any v, define θ_M as the maximum, for u_k in M, of the angle between v and u_k, and define θ_A as the corresponding maximum over all u_j in A. Consider θ_j, the angle between v and u_j, u_j not in M. For $v = u$, $\theta_j < \theta_M$, and, since both θ_j and θ_M are continuous, this inequality holds in a neighborhood of u. Consequently there is a neighborhood of u in which $\theta_j < \theta_M$ for all u_j not in M.

We assert now that u minimizes the greatest angle with the u_j in M. If this were not so, then there would be a unit vector v with

$$v \cdot u_k \geqq \min_{(M)} v \cdot u_k > \min_{(M)} u \cdot u_k = \min_{(A)} u \cdot u_j$$

for all u_k in M. And by Lemma B there would be w arbitrarily near u for which the same result would hold. But in a certain neighborhood of u, $\theta_M = \theta_A$, and some of these w would thus have to make θ_A smaller than for u, contrary to hypothesis.

Moreover, since u does minimize the greatest angle with the u_j in M, it must be representable as $\sum a_j u_j$ with all $a_j \geqq 0$ and all u_j in M. This completes the proof of the

NECESSITY LEMMA. *If u minimizes the greatest angle with the u_j of any set A, and this minimum is $< 180°$, then there is a subset P of A such that*

(1) *u is a linear combination with positive coefficients of the u_j of P*

(2) *u makes equal angles with each u_j in P*

(3) *The angles between u and any u_j not in P are no larger than these equal angles.*

Given any representation $u = \sum a_j u_j$ with all $a_j \geqq 0$ and all u_j in M, we need only omit those u_j with $a_j = 0$ to obtain the subset P.

19. Sufficiency of the equiangular-or-closer condition. We may now easily show that: If $u = \sum a_j u_j$ with $a_j \geqq 0$ and u_j in a set P, and if $u \cdot u_j = \varphi > 0$ *for all u_j in P, then φ is the maximum value of the least $w \cdot u_j$ for u_j in P.*

Suppose the contrary, then for some unit vector w, we have $w \cdot u_j > \varphi = u \cdot u_j$ for all u_j in P.

Multiplying by a_j and summing gives $w \cdot u = w \cdot (\sum a_j u_j) > u \cdot (\sum a_j u_j) = u \cdot u = 1$ which is impossible.

This shows that the necessary conditions of the last section are also sufficient.

20. Uniqueness of Solution. All that remains is to show that, given any set of unit vectors u_j, with the maximum of the least $w \cdot u_j$ equal to $\varphi > 0$, only one u makes equal angles with some subset of the u_j, and is, at the same time, a convex combination of the same u_j. There are a finite number of subsets, and by Lemma C at most two u's have this relation to any subset. Hence at most a finite number of u's make $u_j \cdot u \geqq \varphi$ for all j. Suppose two of these, say v and w, were distinct, then by Lemma B there would have to be infinitely many more, which is impossible. Hence the u that makes $u_j \cdot u \geqq \varphi > 0$ for all j is unique, and the proof of theorem of Section 8 is complete.

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