A COMBINATORIAL THEOREM FOR PARTIAL SUMS

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1. Introduction. Let $(x_1, \cdots, x_n)$ be a sequence of real numbers, $s_k = \sum_{j=1}^{k} x_j$ and $M_k = \max(s_1, \cdots, s_k)$. As usual let the superscript $^+$ mean maximize with zero. In a recent paper of M. Dwass [2], a theorem equivalent to the following is proved (generalizing a result of Kac [3] and Spitzer [4]):

Theorem. $\sum_\sigma (M_n^+ - M_{n-1}^+) = s_n^+$ *where* σ *ranges over all cyclic permutations of* $(x_1, \cdots, x_n)$.

In this note we give a generalization of this theorem. It will be seen that several recent results of L. Takács [5], [6] may be derived from this extension.

2. The basic theorems. We begin with a preliminary lemma. Let $X = (x_1, \cdots, x_n)$ be a sequence of real numbers. Let $m(X)$ denote the rth *largest* term of X (or zero if $r > n$) and let $m(X, y)$ abbreviate $m((x_1, \cdots, x_n, y))$.

Lemma. *If* $y \geqq 0$ *then*

$$m(X, y)^+ - m(X)^+ = m(X, y) - m(X, 0). \tag{1}$$

Proof. There are three cases:

(i) Suppose $m(X, y) \leqq 0$. Since $y \geqq 0$ then $m(X, 0) = m(X, y)$ and $m(X)^+ = m(X, y)^+ = 0$ and (1) follows.

(ii) Suppose $m(X, y) > 0$ and $m(X) > 0$. Then $m(X, y)^+ = m(X, y)$ and $m(X)^+ = m(X) = m(X, 0)$ and (1) follows.

(iii) Suppose $m(X, y) > 0$ and $m(X) \leqq 0$. Since $y \geqq 0$ then a moment's reflection shows that $m(X)^+ = 0 = m(X, 0)$. But we have $m(X, y)^+ = m(X, y)$ and so (1) follows.

This completes the proof.

Now denote the partial sum $\sum_{j=1}^{k} x_j$ by s_k. Suppose $1 \leqq r \leqq n$ and let $m_k = m((s_1, \cdots, s_k))$. Then we have

Theorem 1. $\sum_\sigma (m_n^+ - m_{n-1}^+) = s_n^+$ *where* σ *ranges over all cyclic permutations of* $(x_1, \cdots, x_n)$.

Proof. If $s_n < 0$ then the theorem is immediate since in this case $s_n^+ = 0 = m_n^+ - m_{n-1}^+$ for all permutations of the x_i. Assume $s_n \geqq 0$ and note that

$$\begin{aligned} m_n &= m((x_1, x_1 + x_2, \cdots, x_1 + \cdots + x_n)) \\ &= x_1 + m((0, x_2, x_2 + x_3, \cdots, x_2 + \cdots + x_n)). \end{aligned}$$

Therefore by the lemma

$$\begin{aligned} \sum_\sigma (m_n^+ - m_{n-1}^+) &= \sum_\sigma (m_n - m((x_1, x_1 + x_2, \cdots, x_1 + \cdots + x_{n-1}, 0))) \\ &= \sum_\sigma (x_1 + m((0, x_2, x_2 + x_3, \cdots, x_2 + \cdots + x_n)) \\ &\qquad - m((0, x_1, x_1 + x_2, \cdots, x_1 + \cdots + x_{n-1}))) \\ &= x_1 + \cdots + x_n = s_n \end{aligned}$$

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since the sum is taken over all cyclic permutations of $(x_1, \cdots, x_n)$. This proves the theorem.

Theorem 1 is a special case of a more general result. Let $(x_1, \cdots, x_{t+u})$ be a sequence of real numbers and let $m_j(k)$ denote $m((x_{k+1}, x_{k+1} + x_{k+2}, \cdots, x_{k+1} + \cdots + x_{k+j}))$ for $0 \leqq k \leqq t$ and $1 \leqq j \leqq u$. Then exactly as before we have

THEOREM 2. *If* $1 \leqq r \leqq u$ *and* $\sum_{j=1}^{u} x_{k+j} \geqq 0$ for $1 \leqq k \leqq t$ *then*

$$\sum_{k=1}^{t} (m_n(k)^+ - m_{n-1}(k)^+) = m_u(t) - m_u(0) + \sum_{k=1}^{t} x_k .$$

If we let $t = u = n$ and $x_{n+j} = x_j$ for $1 \leqq j \leqq n$ in Theorem 2 then $m_n(n) = m_n(0)$ and we obtain Theorem 1. A similar substitution in Theorem 2 yields

THEOREM 3. *Let* $(x_1, \cdots, x_n)$ *be a sequence of real numbers and let* $1 \leqq m \leqq n$. *Suppose the sum of any* m *consecutive* x_j *is nonnegative where the* x_j *are considered cyclically* i.e., x_1 *follows* x_n, *etc. Then for* $m - 1 \leqq q \leqq p \leqq n$ *we have* (*using the notation of Theorem* 1)

$$\sum_{\sigma} (m_p^+ - m_q^+) = (p - q)s_n^+$$

where σ *ranges over all cyclic permutations of* $(x_1, \cdots, x_n)$.

3. Concluding remarks. It may be noted that several results of L. Takács follow directly from Theorem 1. For example, if we assume that the x_j are integers and $x_1 + \cdots + x_n = 1$, then Theorem 1 asserts that for any integer r satisfying $1 \leqq r \leqq n$ we have

$$\sum_{\sigma} (m_n^+ - m_{n-1}^+) = (x_1 + \cdots + x_n) = 1$$

where σ ranges over all cyclic permutations of $(x_1, \cdots, x_n)$. Since each summand $m_n^+ - m_{n-1}^+$ is a nonnegative integer then there must be *exactly one* cyclic permutation of $(x_1, \cdots, x_n)$ such that $m_n^+ - m_{n-1}^+ > 0$. This inequality holds, however, if and only if there are exactly r of the partial sums s_j which are $\geqq s_n = 1$, i.e., if and only if there are exactly r positive partial sums. This is just Theorem 1 of Takács [5]. Similarly, by replacing x_k by $1 - x_{n+1-k}$ and taking $r = 1$ in Theorem 1, we obtain the following interesting result of Takács ([6], Theorem 1) (which may also be derived from an elegant result of Dwass [2]):

THEOREM. *If* $x_1, \cdots, x_n$ *are nonnegative integers such that* $x_1 + \cdots + x_n = k \leqq n$ *then there are exactly* $n - k$ *cyclic permutations of* $x_1, \cdots, x_n$ *such that the* jth *partial sum is less than* j *for* $j = 1, 2, \cdots, n$.

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