ON A CLASS OF SIMPLE RANDOM WALKS[1]

By J. R. Blum, D. L. Hanson and S. V. Parter[2]

University of New Mexico and Sandia Corporation

Let $\{X_n, n \geqq 1\}$ be a sequence of independent and identically distributed random variables such that each X_n assumes only non-negative integral values with $P\{X_n = i\} = p_i$ for $i = 0, 1, 2, \cdots$. Let $S_n = \sum_{i=1}^n X_i$ and let $m > 0$ and $k \geqq 0$ be rational. In this paper we give a procedure for evaluating

$$q(m, k) = P\{S_n < nm + k, n = 1, 2, \cdots\} \tag{1}$$

(i.e. for evaluating the probability of the collection of random walks in the plane which start at the origin and at the nth step move one unit to the right and X_n units up and remain strictly below the line $y = mx + k$).

With $m = 1$, $k = 0$, and the assumption that $EX_n = \mu < 1$, the solution was obtained by Meyer Dwass (unpublished) who showed using mainly a combinatorial argument that $P\{S_n < n, n = 1, 2, \cdots\} = 1 - \mu$. Note that if we set $Y_i = X_i - m$ and $T_n = \sum_{i=1}^n Y_i$ then $q(m, k) = \lim_{j\to\infty} P\{\max_{1\leqq i\leqq j} T_i < k\}$. A number of papers have dealt with the problem of evaluating the probabilities $P\{\max_{1\leqq i\leqq j} T_i < k\}$. We mention only a few here. Baxter and Donsker [2] and Pyke [4] treat instead of a discrete time parameter, a continuous time parameter where $\{T_t\}$ is a separable process with stationary independent increments (for Pyke $\{T_t\}$ is Poisson). Kinney [3] considers probabilities *related to* those of interest here. Spitzer [5] gives the explicit formula $q(m, 0) = \exp[-\sum_{k=1}^\infty a_k/k]$ where $a_k = P\{T_k > 0\}$, without any restrictions other than that the random variables $Y_1, Y_2 \cdots$ are independent and identically distributed. However it is difficult to see how his formula reduces to the simple one we shall give for the class of random variables we consider.

Throughout we shall make certain assumptions which do not affect the generality of our result as is easily verified:

(2)
- (i) m and k are integers.
- (ii) $p_0 > 0$.
- (iii) If $r \geqq 2$ is an integer with $m = 0(\text{mod } r)$ then there exists an integer $i \not\equiv 0(\text{mod } r)$ such that $p_i > 0$.
- (iv) $\mu < m$ (and consequently it follows from the strong law of large numbers that $q_\infty(m) = \lim_{k\to\infty} q(m, k) = 1$).

With regard to (iv) we shall see below that $q(m, k) = 0$ if $\mu \geqq m$.

We take as the starting point of our investigation the infinite system of equations satisfied by the numbers $q(m, k)$, i.e.

$$q(m, k) = \sum_{i=0}^{m+k-1} p_i q(m, k + m - i), \qquad k = 0, 1, \cdots. \tag{3}$$

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[2] On leave from Cornell University, Summer 1961.

Note that the equations (3) *define the numbers* $q(m, k)$ *completely once the values* $q(m, 0), \cdots, q(m, m-1)$ *are given*, and our problem reduces to computing these numbers. In the remainder of the paper we shall hold m fixed and use the notation $q_k = q(m, k)$, $q_\infty = q_\infty(m)$. Let $P(t)$ and $Q(t)$ be the generating functions corresponding to the sequences $\{p_i\}$ and $\{q_i\}$ respectively, i.e.

$$P(t) = \sum_{i=0}^{\infty} p_i t^i \quad \text{and} \quad Q(t) = \sum_{i=0}^{\infty} q_i t^i.$$

As functions of the complex variable t, P and Q are analytic for $|t| < 1$ and P is uniformly continuous for $|t| \leqq 1$. Let $S(t) = P(t)Q(t)$ and let s_i be the coefficient of t^i in the expansion of $S(t)$. Then the equations (3) take the form

$$(4) \qquad q_k = s_{k+m} - q_0 p_{k+m}, \qquad k = 0, 1, \cdots.$$

Let $\sigma_i = s_i - q_0 p_i$ for all i. After manipulating (4) appropriately we see that $P(t)$ and $Q(t) \not\equiv 0$ must satisfy the functional equation

$$(5) \qquad q_0 P(t) + \sum_{i=0}^{m-1} \sigma_i t^i = Q(t)[P(t) - t^m], \qquad |t| < 1.$$

Now let t approach one from below through real values. Then it is easily verified that $\lim_{t\uparrow 1} Q(t)(1-t) = q_\infty$, $\lim_{t\uparrow 1}[P(t) - t^m]/(1-t) = m - \mu$. Since $P(1) = 1$ we have

$$(6) \qquad q_0 + \sum_{i=0}^{m-1} \sigma_i = q_\infty(m - \mu).$$

Dwass' result follows at once from (6) since $\sigma_0 = 0$ by definition, and if $m = 1$ and $\mu < 1$ then $q_\infty = 1$ giving $q_0 = 1 - \mu$. Another consequence of (6) is the remark made above that we may assume that $\mu < m$. Note that $\sigma_i \geqq 0$ for all i and if $\mu \geqq m$ then $q_0 = \sigma_0 = \sigma_1 = \cdots = \sigma_{m-1} = 0$ and an inspection of (3) shows that in that case $q_k = 0$ for all k.

THEOREM. *Suppose* (2) *holds. Then* $P(t) - t^m$ *has precisely* $m - 1$ *zeros* $t_1, \cdots, t_{m-1}$ *in* $|t| < 1$. *Setting* $t_0 = 0$ *we have*: (a) $q_0 = (m - \mu)/\prod_{i=0}^{m-1}(1 - t_i)$. (b) $q_1, \cdots, q_{m-1}$ *may be solved for successively from the equations* $\sigma_i = \sum_{j=0}^{i-1} p_j q_{i-j}$ $i = 1, \cdots, m-1$; *where* $\sigma_1, \cdots, \sigma_{m-1}$ *are obtained by equating coefficients in the equation*

$$q_0 t^{m-1} + \sum_{i=1}^{m-1} \sigma_i t^i = q_0 \prod_{i=1}^{m-1}(t - t_i).$$

PROOF. The fact that $P(t) - t^m$ has $m - 1$ zeros in $|t| < 1$ involves fairly standard analytic techniques and is similar to the argument given by Bailey [1]. Let $h(t) = q_0 t^{m-1} + \sum_{i=1}^{m-1} \sigma_i t^{i-1}$. Then (5) reduces to

$$(7) \qquad q_0 f(t) + t h(t) = Q(t) f(t).$$

Unless $Q(t) \equiv 0$ (which is impossible) $h(t)$ must have the same zeros as $f(t)$ in $|t| < 1$ and since the degree of h is $m - 1$, we can write

$$h(t) = q_0 \prod_{i=1}^{m-1} (t - t_i). \tag{8}$$

Taking the limit as $t \to 1$ in (7) gives (a). (b) follows from (8), the definition of h, and the definition of σ_i.

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