THE DISTRIBUTION OF THE DETERMINANT OF A COMPLEX WISHART DISTRIBUTED MATRIX

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Summary. Let $\boldsymbol{\xi}' = (\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_p)$ denote a p-variate zero mean complex Gaussian random variable with nonsingular Hermitian covariance matrix $\Sigma_\xi = E\boldsymbol{\xi}\bar{\boldsymbol{\xi}}' = \|\sigma_{jk}\|$. The *generalized variance* of $\boldsymbol{\xi}$ is $\sigma_\xi^2 \equiv \det(\Sigma_\xi)$. The real and imaginary parts of the complex random variables $\mathbf{Z}_j$ $j = 1, 2, \cdots p$ are taken to have the special covariance structure described in Goodman [1] and [2] so that the Hermitian covariance matrix Σ_ξ then determines the probability structure of the random variable $\boldsymbol{\xi}$. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_s, \cdots, \boldsymbol{\xi}_n$ denote n independent and identically distributed p-variate zero mean complex Gaussian random variables with Hermitian covariance matrix Σ_ξ. The sample Hermitian covariance matrix $\hat{\boldsymbol{\Sigma}}_\xi \equiv (1/n)\sum_{s=1}^n \boldsymbol{\xi}_s\bar{\boldsymbol{\xi}}_s' \equiv \|\hat{\boldsymbol{\sigma}}_{jk}\|$ is then complex Wishart distributed. The *sample generalized variance* of $\boldsymbol{\xi}$ is $\hat{\boldsymbol{\sigma}}_\xi^2 \equiv \det(\hat{\boldsymbol{\Sigma}}_\xi)$. The random variable $(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2$ is distributed as is the product of p independent χ^2 random variables with $2n$, $2(n-1), \cdots, 2(n-p+1)$ degrees of freedom respectively.

Definition 1.1. Let $\boldsymbol{\xi}' = (\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_p)$ denote a p-variate zero mean complex Gaussian random variable with nonsingular Hermitian covariance matrix $\Sigma_\xi = E\boldsymbol{\xi}\bar{\boldsymbol{\xi}}' = \|\sigma_{jk}\|$. The *generalized variance* of $\boldsymbol{\xi}$ is $\sigma_\xi^2 \equiv \det(\Sigma_\xi)$.

Comment 1.1. Throughout the paper the real and imaginary parts of the complex random variables $\mathbf{Z}_j$, $j = 1, 2, \cdots p$ are taken to have the special covariance structure described in Goodman [1] and [2] so that the Hermitian covariance matrix Σ_ξ then determines the probability structure of the random variable $\boldsymbol{\xi}$.

Definition 1.2. Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_s, \cdots, \boldsymbol{\xi}_n$ denote n independent and identically distributed p-variate zero mean complex Gaussian random variables with Hermitian covariance matrix Σ_ξ. The sample Hermitian covariance matrix $\hat{\boldsymbol{\Sigma}}_\xi \equiv (1/n)\sum_{s=1}^n \boldsymbol{\xi}_s\bar{\boldsymbol{\xi}}_s' \equiv \|\hat{\boldsymbol{\sigma}}_{jk}\|$. The *sample generalized variance* of $\boldsymbol{\xi}$ is $\hat{\boldsymbol{\sigma}}_\xi^2 \equiv \det(\hat{\boldsymbol{\Sigma}}_\xi)$.

Theorem 1.1. *The random variable $(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2$ is distributed as is the product of p independent χ^2 random variables with $2n$, $2(n-1), \cdots, 2(n-p+1)$ degrees of freedom respectively.*

Proof. The method of proof is as follows: The characteristic function of the random variable $\ln[(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2]$ is computed. The characteristic function of a random variable which is the sum of p independent $\ln \chi^2$ random variables with $2n$, $2(n-1), \cdots, 2(n-p+1)$ degrees of freedom respectively is computed. The two characteristic functions are compared and seen to be equal. The characteristic function of the random variable $\mathbf{V} = \ln[(2n)^p\hat{\boldsymbol{\sigma}}_\xi^2/\sigma_\xi^2]$ is

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$$\psi_{\mathbf{V}}(t) = E\exp[it\mathbf{V}] = E\exp\,[it\ln\,[(2n)^p\,\hat{\sigma}_\xi^2/\sigma_\xi^2]] \tag{1.1}$$
$$= E\,[(2n)^p\hat{\sigma}_\xi^2/\sigma_\xi^2]^{it} = E\,[(2n)^p\,|\,\hat{\mathbf{\Sigma}}_\xi\,|/|\,\Sigma_\xi\,|]^{it} = E\,[2^p\,|\,\mathbf{A}\,|/|\,\Sigma_\xi\,|]^{it}$$

where

$$\mathbf{A} \equiv n\hat{\mathbf{\Sigma}}_\xi\,.$$

The probability density of the Hermitian matrix **A** is (See Goodman [2])

$$p_W(A) = \frac{|A|^{n-p}}{I(\Sigma_\xi)}\exp\,[-\,\mathrm{tr}(\Sigma_\xi^{-1}A)] \tag{1.2}$$

where

$$I(\Sigma_\xi) = \pi^{\frac{1}{2}p(p-1)}\Gamma(n)\,\cdots\,\Gamma(n-p+1)|\Sigma_\xi|^n.$$

The density is defined over the domain D_A where A is Hermitian positive semi-definite. From (1.2) one also has formally

$$\int_{D_A}|A|^{n+it-p}\exp\,[-\,\mathrm{tr}\,(\Sigma_\xi^{-1}A)] = \pi^{\frac{1}{2}p(p-1)}\Gamma(n+it) \cdots\Gamma(n+it-p+1)\,|\,\Sigma_\xi\,|^{n+it}. \tag{1.3}$$

Now, from (1.1), (1.2), and (1.3)

$$\begin{aligned}\psi_{\mathbf{V}}(t) &= \int_{D_A}\left[2^p\frac{|A|}{|\Sigma_\xi|}\right]^{it}p_W(A)\\ &= \frac{2^{pit}}{|\,\Sigma_\xi\,|^{it}I(\Sigma_\xi)}\int_{D_A}|\,A\,|^{n+it-p}\exp\,[-\,\mathrm{tr}\,(\Sigma_\xi^{-1}A)]\\ &= \frac{2^{pit}\pi^{\frac{1}{2}p(p-1)}\Gamma(n+it)\,\cdots\,\Gamma(n+it-p+1)\,|\,\Sigma_\xi\,|^{n+it}}{|\,\Sigma_\xi\,|^{it}\,\pi^{\frac{1}{2}p(p-1)}\Gamma(n)\cdots\Gamma(n-p+1)|\,\Sigma_\xi\,|^n}\\ &= 2^{pit}\,\frac{\Gamma(n+it)\,\cdots\,\Gamma(n+it-p+1)}{\Gamma(n)\,\cdots\,\Gamma\,(n-p+1)}\,.\end{aligned} \tag{1.4}$$

The probability density function of a χ_{2k}^2 random variable is

$$p(v) = (1/2^k\Gamma(k))v^{k-1}e^{-\frac{1}{2}v}. \tag{1.5}$$

From (1.5) one has formally

$$\int_0^\infty v^{k+it-1}e^{-\frac{1}{2}v}\,dv = 2^{k+it}\,\Gamma(k+it). \tag{1.6}$$

The characteristic function of a $\ln\chi_{2k}^2$ random variable is

$$\begin{aligned}\psi_{\ln\chi_{2k}^2}(t) &= E\,e^{it\ln\chi_{2k}^2} = E(\chi_{2k}^2)^{it} = \int_0^\infty v^{it}\,p(v)\,dv\\ &= \int_0^\infty\frac{v^{k+it-1}}{2^k\Gamma(k)}\,e^{-\frac{1}{2}v}dv = \frac{2^{k+it}\Gamma(k+it)}{2^k\Gamma(k)} = 2^{it}\,\frac{\Gamma(k+it)}{\Gamma(k)}\,.\end{aligned} \tag{1.7}$$

From (1.7) one has that the characteristic function of the sum of independent $\ln \chi^2_{2n}$, $\ln \chi^2_{2(n-1)}$, $\cdots$, $\ln \chi^2_{2(n-p+1)}$ random variables is

$$(1.8) \qquad \psi_{\ln \chi^2_{2n} + \cdots + \ln \chi^2_{2\,(n-p+1)}}(t) = 2^{pit} \frac{\Gamma(n + it) \cdots \Gamma(n - p + 1 + it)}{\Gamma(n) \cdots \Gamma(n - p + 1)} .$$

From (1.4) and (1.8) one observes that

$$(1.9) \qquad \psi_{\mathbf{V}}(t) = \psi_{\ln \chi^2_{2n} + \cdots + \ln \chi^2_{2(n-p+1)}}(t) .$$

Comment. The distribution of the determinant of a Wishart distributed matrix is given here for comparison. Let $\xi_R' = (\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_p)$ denote a p-variate zero mean real Gaussian random variable with nonsingular symmetric covariance matrix $\Sigma_{R\xi} = E\xi_R\xi_R' = \|\sigma_{Rjk}\|$. The *generalized variance* of ξ_R is $\sigma^2_{R\xi} \equiv \det(\Sigma_{R\xi})$. Let $\xi_{1R}, \xi_{2R}, \cdots, \xi_{sR}, \cdots, \xi_{nR}$ denote n independent and identically distributed p-variate zero mean real Gaussian random variables with nonsingular symmetric covariance matrix $\Sigma_{R\xi}$. The sample symmetric covariance matrix

$$(1.10) \qquad \hat{\mathbf{\Sigma}}_{R\xi} = (1/n) \sum_{s=1}^{n} \xi_{sR}\xi_{sR}' = \|\hat{\mathbf{\sigma}}_{Rjk}\|$$

is then Wishart distributed. The *sample generalized variance* of ξ_R is

$$\hat{\mathbf{\sigma}}^2_{R\xi} \equiv \det(\hat{\mathbf{\Sigma}}_{R\xi}).$$

The random variable $n^p(\hat{\mathbf{\sigma}}^2_{R\xi}/\sigma^2_{R\xi})$ is distributed as is the product of p independent χ^2 random variables with $n, n - 1, \cdots, n - p + 1$ degrees of freedom respectively. (See Wilks [3].)

REFERENCES

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