NOTES

PAIRWISE INDEPENDENCE OF JOINTLY DEPENDENT VARIABLES

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1. Introduction. It is well known in statistical theory that pairwise independence is necessary but not sufficient for a set of $p \geqq 3$ variables to be mutually independent. The example of a set of three discrete variates that are pairwise independent but not mutually independent that is usually quoted in the statistical literature is due to S. Bernstein (see Cramér [2, page 162] or for a variation, Parzen [4, page 90]). In fact Feller [3, page 117], commenting on this example, states that "Practical examples of pairwise independent events that are not mutually independent apparently do not exist."

Two additional discrete examples of such pairwise independence can readily be presented. For the first, consider the case where p straight line segments are distributed independently and at random on a plane (more appropriately, to avoid marginal effects, these should be great circle segments distributed on the surface of a sphere). The $\frac{1}{2}p(p - 1)$ random variables, X_{ij}, exhibiting pairwise independence equal 1 or 0 according to whether or not segments i and j intersect. In the second example, p balls are distributed independently with equal probability into each of 2 or more urns. The X_{ij}'s in this case equal 1 or 0 according to whether or not balls i and j have been placed in the same urn.

In this note we give a continuous example which may be of both practical interest and pedagogical value.

2. An example. Consider a random sample of n observations from a trivariate non-singular normal distribution with a diagonal variance-covariance matrix. The joint density of the three sample correlation coefficients, say r_{12}, r_{13} and r_{23}, is

$$f(r_{12}, r_{13}, r_{23}) = C(n)(1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23})^{\frac{1}{2}(n-5)} \quad (1)$$

for

$$1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23} > 0 \quad (2)$$

and zero elsewhere.

It is obvious that the three random variables are mutually dependent from positive definiteness and continuity considerations. The pairwise independence of these variables can be shown directly by integrating out in the joint density

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any particular one of three variables over its appropriate range which is obtained from (2). This fact may also be deduced as a consequence of a result of T. W. Anderson [1, problem 9, page 177].

This can be extended to the general case of the $\frac{1}{2}p(p-1)$ jointly distributed correlation coefficients arising when n observations are drawn from a p-variate non-singular normal distribution having a diagonal variance-covariance matrix. This case may be deduced as a consequence of T. W. Anderson's previously mentioned result or by use of the fact that any two correlation coefficients that are based on any three different variates are pairwise independent by the previous argument and any two that are based on four different independent variates are clearly independent. Hence the sample correlation coefficients based on p independent non-singular normal variates exhibit pairwise independence but not mutual independence.

3. Applications. The present result simplifies computation of the variance, in the null case, of linear combinations of correlation coefficients or transforms of correlation coefficients,

$$\text{Var} \sum_{i<j}\sum a_{ij} f(r_{ij}) = \sum_{i<j}\sum a_{ij}^2 \, \text{Var}\, f(r_{ij}). \tag{3}$$

The authors wish to thank Dr. Donald F. Morrison for bringing their attention to this application. The particular linear combination with which Dr. Morrison was concerned was that of the number of the $\frac{1}{2}p(p-1)$ correlation coefficients significant at probability level α. For this instance all $a_{ij} = 1$, $f(r_{ij}) = 1$ or 0 according to whether r_{ij} departs or fails to depart significantly from 0. The variance of each individual $f(r_{ij})$ is then $\alpha(1-\alpha)$ making the variance of the total number of significant correlation coefficients $\frac{1}{2}[p(p-1)\alpha(1-\alpha)]$. While this last result does not depend on the sample size employed, power considerations would make reasonably large sample sizes advisable. Other rapid tests of the null hypothesis, alternative to a likelihood ratio test may also be feasible. Thus one might test the sum of the correlation coefficients where all associations, under the alternative hypothesis, are anticipated to be unidirectional (almost of necessity, positive). The sum of the absolute r's or squares of r's might be more reasonably tested where both large positive and large negative associations are anticipated, but additional work would be needed to implement these possibilities.

REFERENCES

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