LIMIT THEOREMS FOR RANDOMLY SELECTED PARTIAL SUMS[1]

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1. Summary. Let $\{S_n\}$ be the partial sums of a sequence (not necessarily independent) of random variables $\{X_n\}$, and let $\{m_u\}$ be a set of integer-valued random variables depending on an index $u \geqq 0$. Suppose that m_u/u converges in probability to a constant as $u \to \infty$ and that S_n obeys the central limit theorem (when it is normed properly, as must also be the other variables below). Anscombe [1] has shown that if the S_n do not fluctuate too much, in a sense made precise below, then the random sum S_{m_u} also obeys the central limit theorem. Anscombe's condition is closely related to one introduced by Prohorov [6] in connection with the Erdös-Kac-Donsker invariance principle. In Section 2 this relationship is investigated; in particular, it is shown that if the sequence $\{X_n\}$ satisfies the invariance principle then S_{m_u} is asymptotically normal. The invariance principle has been proved in [2] for various dependent sequences $\{X_n\}$, to each of which this result is then applicable. In Section 3 an invariance principle is formulated and proved for the random partial sums; this result enables one to find, for example, the limiting distribution of $\max_{k \leqq m_u} S_k$. In Section 4, these theorems are applied to renewal processes.

2. The central limit theorem for random sums. Let $\{X_1, X_2, \cdots\}$ be a sequence (possibly dependent) of random variables on some probability measure space $(\Omega, \mathcal{B}, \mathbf{P})$, and put $S_k = \sum_{i=1}^k X_k (S_0 = 0)$. Then $\{X_n\}$ satisfies the central limit theorem with norming factors[2] $n^{\frac{1}{2}}\sigma$ if

$$\lim_{n\to\infty} \mathbf{P}\{S_n/n^{\frac{1}{2}}\sigma \leqq \alpha\} = \Phi(\alpha), \tag{2.1}$$

where $\Phi(\alpha)$ is the unit normal distribution function.

In some cases, $\{X_n\}$ satisfies a stronger limit theorem. To state this result we need some auxiliary definitions and theorems. Let C be the space of continuous functions on [0, 1], with the uniform topology, and let $\mathcal{C}$ be the σ-field of Borel sets. If ν_n and ν are probability measures on $\mathcal{C}$, then ν_n is said to converge weakly to ν (written $\nu_n \Rightarrow \nu$) if $\int f\, d\nu_n \to \int f\, d\nu$ for any function f on C which is bounded and continuous; see [6] or [2]. Prohorov [6] has proved the following theorem. For $x \varepsilon C$, let $\omega_x(\delta) = \sup\{|x(s) - x(t)|: |s - t| \leqq \delta\}$ be the oscillation function of x.

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[2] Other changes of scale, as well as changes of location are of course possible. We confine ourselves for simplicity to norming by a multiple of $n^{\frac{1}{2}}$.

THEOREM 2.1. *The following pair of conditions is necessary and sufficient for* $\nu_n \Rightarrow \nu$. (i) *For any integer* c *and real numbers* $\alpha_1, \cdots, \alpha_c$, *we have*

$$\lim_{n\to\infty} \nu_n\{x: x(i/c) \leqq \alpha_i, 1 \leqq i \leqq c\} = \nu\{x: x(i/c) \leqq \alpha_i, 1 \leqq i \leqq c\},$$

provided the right-hand member, considered as a function of $(\alpha_1, \cdots, \alpha_c)$, *is continuous there.* (ii) *For any positive* ϵ *and* η, *there exist a positive* δ *and an integer* n_0 *such that if* $n \geqq n_0$ *then*

$$\nu_n\{x: \omega_x(\delta) \geqq \epsilon\} < \eta.$$

Condition (i) says that the finite dimensional distributions, i.e., the distributions of the vector $(x(1/c), \cdots, x(c/c))$ induced by ν_n and by ν, converge in the proper manner. Condition (ii) has the effect of limiting the measures of sets of x's which oscillate violently. We will have occasion below to use Theorem 2.1 with n replaced by a parameter which goes to ∞ through a continuum of values.

An extension of the central limit theorem for $\{X_n\}$ is obtained by constructing measures on $\mathcal{C}$ in the following way. For each $\omega \varepsilon \Omega$, let $p(u) = p(u, \omega)$ be the function on $[0, \infty)$ defined by $p(u) = S_{[u]} + (u - [u])X_{[u]+1}$. For $n = 1, 2, \cdots$, define $p_n(t) = p_n(t, \omega)$ for $0 \leqq t \leqq 1$ by $p_n(t) = p(tn)/n^{\frac{1}{2}}\sigma$. Thus $p_n(\cdot)$ is that element of C which is linear on each interval $[(k-1)/n, k/n]$ and satisfies $p_n(k/n) = S_k/n^{\frac{1}{2}}\sigma$. It is not difficult to show that the mapping $\omega \to p(\cdot, \omega)$ is measurable; hence if $P_n(A) = \mathbf{P}\{p_n \varepsilon A\}$ for $A \varepsilon \mathcal{C}$, then P_n is a probability measure on $\mathcal{C}$. We say that $\{X_n\}$ satisfies the invariance principle with norming factors $n^{\frac{1}{2}}\sigma$ if $P_n \Rightarrow W$, where W denotes Wiener measure.

One reason the invariance principle is of interest is that it can be used to find the limiting distributions of functions of the partial sums. If f is any function on C which is continuous except on a set of ν-measure 0, and if $\nu_n \Rightarrow \nu$ then $\lim_n \nu_n\{x: f(x) \leqq \alpha\} = \nu\{x: f(x) \leqq \alpha\}$ at continuity points α of the limit function; see [6] and [2]. Thus, if $P_n \Rightarrow W$ and f is continuous except on a set of W-measure 0, then $\lim_n \mathbf{P}\{f(p_n) \leqq \alpha\} = W\{x: f(x) \leqq \alpha\}$ at continuity points α of the limit. For example, if $f(x) = \sup_t x(t)$ then

$$\lim_{n\to\infty} \mathbf{P}\{\max_{k\leqq n} S_k/n^{\frac{1}{2}}\sigma \leqq \alpha\} = F(\alpha),$$

where $F(\alpha) = W\{x: \sup_t x(t) \leqq \alpha\}$; $F(\alpha) = 0$ for $\alpha < 0$, while

$$F(\alpha) = (2/\pi)^{\frac{1}{2}} \int_0^\alpha e^{-\frac{1}{2}u^2}\, du \tag{2.2}$$

for $\alpha \geqq 0$. Similarly an arc sine law can be derived by taking $f(x)$ to be the Lebesgue measure of the set of t in $[0, 1]$ such that $x(t) \geqq 0$. To see that the invariance principle actually contains the central limit theorem, take $f(x)$ to be $x(1)$.

By specializing Theorem 2.1 to the case in which $\nu_n = P_n$ and $\nu = W$, and using the definitions of P_n and W, it can be seen that $P_n \Rightarrow W$ if and only if the following two conditions are satisfied.

CONDITION (F). *For any integer c we have*

$$\lim_{n\to\infty} \mathbf{P}\{(S_{[in/c]} - S_{[(i-1)n/c]}/n^{\frac{1}{2}}\sigma \leqq \alpha_i,\ 1 \leqq i \leqq c\} = \prod_{i=1}^{c} \Phi(\alpha_i).$$

CONDITION (P). *For any positive ϵ and η there exist a positive δ and an integer n_0 such that if $n \geqq n_0$ then*

$$\mathbf{P}\{\max_{\substack{|j-k|\leqq n\delta \\ j,k\leqq n}} |S_j - S_k| \geqq \epsilon n^{\frac{1}{2}}\} < \eta.$$

Thus we have the following special case of Prohorov's result.

THEOREM 2.2. *The sequence $\{X_n\}$ satisfies the invariance principle with norming factors $n^{\frac{1}{2}}\sigma$ if and only if Conditions* (F) *and* (P) *are satisfied.*

The sufficiency of the conditions in Theorem 2.2 is not very useful for establishing the invariance principle for particular sequences $\{X_n\}$, especially ones which are dependent. The original Erdös-Kac-Donsker approach, as extended in [2] is more widely applicable (see also [4] and [5]). We will not discuss this approach here; the point is that the invariance principle has been proved for various interesting sequences $\{X_n\}$, for example for certain Markov, m-dependent, and moving-average processes. We will apply Theorems 2.1 and 2.2 to these known results to derive new central limit theorems and invariance principles involving partial sums with random indices.

For $u \geqq 0$, let m_u be a nonnegative-integer-valued random variable on $(\Omega, \mathcal{B}, \mathbf{P})$. It is frequently of interest to know whether S_{m_u} is approximately normal for large u. Anscombe [1] has shown that this is true if, among other things, the partial sums S_n do not fluctuate excessively, in the sense of the following condition.[3]

CONDITION (A). *For any positive ϵ and η, there exist a positive δ and an integer n_0 such that if $n \geqq n_0$ then*

$$\mathbf{P}\{\max_{|k-n|\leqq n\delta} |S_n - S_k| \geqq \epsilon n^{\frac{1}{2}}\} < \eta.$$

Anscombe has proved the following result.

THEOREM 2.3. *Suppose that $\{X_n\}$ satisfies the central limit theorem with norming factors $n^{\frac{1}{2}}\sigma$, i.e., that* (2.1) *holds, and that Condition* (A) *is satisfied. If*

$$\text{(2.3)} \qquad \operatorname{p\,lim}_{u\to\infty} m_u/u = \theta,$$

where θ is a positive constant, then

$$\text{(2.4)} \qquad \lim_{u\to\infty} \mathbf{P}\{S_{m_u}/\theta^{\frac{1}{2}}u^{\frac{1}{2}}\sigma \leqq \alpha\} = \Phi(\alpha).$$

By Slutsky's theorem (Section 20.6 of [3]), it is possible in (2.4) to replace $S_{m_u}/\theta^{\frac{1}{2}}u^{\frac{1}{2}}\sigma$ by $S_{m_u}/m_u^{\frac{1}{2}}\sigma$.

If the X_n are independent and if the variance of $S_n - S_m$ is $O(n - m)$, then

[3] Anscombe's condition and theorem are actually more general than the ones given here; we have specialized his results to the case relevant to this paper. See also Rényi [7] and [8].

it follows easily from Kolmogorov's inequality that Condition (A) holds, so that Theorem 2.3 applies. (Condition (P) is difficult to verify directly even in this case.) If the X_n are dependent, however, Kolmogorov's inequality is in general no longer available, and Condition (A) may be hard to prove. Suppose, however, that the invariance principle (with norming factors $n^{\frac{1}{2}}\sigma$) is known to hold for a particular sequence $\{X_n\}$. It then follows from Theorem 2.2 that Conditions (F) and (P) hold. But (2.1) is the special case of Condition (F) with $c = 1$, and Condition (A) is obviously weaker than Condition (P). Thus the hypotheses of Theorem 2.3 are fulfilled, and we have the following result.

THEOREM 2.4. *If* $\{X_n\}$ *satisfies the invariance principle with norming factors* $n^{\frac{1}{2}}\sigma$, *then* (2.3) *implies* (2.4).

Thus the random sum $S_{m_u}/\theta^{\frac{1}{2}}u^{\frac{1}{2}}\sigma$ is asymptotically normal, provided (2.3) holds, if $\{X_n\}$ is any of those sequences, described above, to which the invariance principle applies.

We will give an alternate proof of this result, which depends on Theorem 2.1, but not on Theorems 2.2 and 2.3. Some of the facts established in the course of the proof will be used in Section 3. Since (ii) of Theorem 2.1 holds, it follows that for all ϵ, η, there exist δ, n_0 such that if $n \geqq n_0$ then, with probability exceeding $1 - \eta$,

$$\sup_{\substack{|v-w| \leqq \delta n \\ v,w \leqq n}} |p(v) - p(w)| < \epsilon\eta^{\frac{1}{2}}. \tag{2.5}$$

We may suppose that $\delta < 1$ and $n_0 > 1$. If $u \geqq n_0$, let n be the next integer larger than $u + \delta u$; then $n_0 \leqq n \leqq u + \delta u + 1 \leqq 3u$. If $v \leqq u$ and $|v - w| \leqq \delta u$ then v, $w \leqq n$, so that, if (2.5) holds,

$$|p(v) - p(w)| < \epsilon\eta^{\frac{1}{2}} \leqq 3\epsilon u^{\frac{1}{2}}.$$

Thus $u \geqq n_0$ implies that

$$\sup_{\substack{|v-w| \leqq \delta u \\ v \leqq u}} |p(v) - p(w)| < 3\epsilon u^{\frac{1}{2}}$$

holds with probability exceeding $1 - \eta$. Absorbing the 3 into the ϵ, we may say that for all ϵ, η, there exist δ, u_0 such that if $u \geqq u_0$ then

$$\mathbf{P}\{\sup_{\substack{|v-w| \leqq \delta u \\ v \leqq u}} |p(v) - p(w)|/u^{\frac{1}{2}} \geqq \epsilon\} < \eta. \tag{2.6}$$

Given ϵ and η, choose δ and u_0 so that if $u \geqq u_0$ then

$$\sup_{|w-\theta u| \leqq \delta u} |p(w) - p(u\theta)| < \epsilon u^{\frac{1}{2}} \tag{2.7}$$

with probability exceeding $1 - \eta$, which is possible by (2.6). At the same time, choose u_0 so large that if $u \geqq u_0$ then

$$|m_u - \theta u| < \delta u \tag{2.8}$$

with probability exceeding $1 - \eta$, which is possible if (2.3) holds. If both (2.7) and (2.8) hold then

(2.9) $$|p(m_u) - p(u\theta)| < \epsilon u^{\frac{1}{2}}.$$

Thus, if $u \geqq u_0$, (2.9) holds with probability exceeding $1 - 2\eta$; therefore

(2.10) $$\text{p lim}_{u\to\infty} |p(m_u) - p(u\theta)|/u^{\frac{1}{2}} = 0.$$

Since, by (i) of Theorem 2.1, the distribution of $p(u\theta)/\theta^{\frac{1}{2}}u^{\frac{1}{2}}\sigma$ converges to $\Phi(\alpha)$, and since $p(m_u) = S_{m_u}$, Theorem 2.4 now follows from Slutsky's theorem.

3. The invariance principle for random sums. If $\{X_n\}$ satisfies the invariance principle, then still stronger results on the limiting behavior of randomly selected partial sums can be proved. We will define a function $q(u)$ which is associated with the random partial sums in a manner analogous to that in which $p(u)$ is associated with the nonrandom ones. Let us assume from now on that $m_u = m_u(\omega)$ is, with probability one, a nondecreasing, right-continuous function of u, which increases by unit jumps. Let $\xi_0(\omega) = 0$, and let $\xi_1(\omega), \xi_1(\omega), \xi_2(\omega), \cdots$ be the successive discontinuities of $m_u(\omega)$ as a function of u, so that $m_u = i$ if $\xi_i \leqq u < \xi_{i+1}$. Let

(3.1) $$m_u' = i + (u - \xi_i)/(\xi_{i+1} - \xi_i) \qquad \text{if } \xi_i \leqq u \leqq \xi_{i+1};$$

thus m_u' is that function of u which is linear on each interval $[\xi_i, \xi_{i+1}]$ and agrees with m_u at its jumps. Finally, take $q(u) = p(m_u')$.[4] Now, for $u \geqq 0$, $0 \leqq t \leqq 1$, put $q_u(t) = q(ut)/\theta^{\frac{1}{2}}u^{\frac{1}{2}}\sigma$. Then $q_u(\cdot)$ is, for each ω, an element of C; define a measure Q_u on $\mathcal{C}$ by $Q_u(A) = \mathbf{P}\{q_u \varepsilon A\}$ for $A \varepsilon \mathcal{C}$.

We inquire after conditions which ensure that $Q_u \Rightarrow W$. The following result stands to Theorem 2.4 as the invariance principle does to the central limit theorem.

THEOREM 3.1. *If $\{X_n\}$ satisfies the invariance principle with norming factors $n^{\frac{1}{2}}\sigma$, and if*

(3.2) $$\text{p lim}_{u\to\infty} \{\sup_{v\leqq u} |m_v' - \theta v|/u\} = 0,$$

where θ is a positive constant, then $Q_u \Rightarrow W$.

PROOF. Since m_u increases by unit jumps, we have $|m_u - m_u'| \leqq 1$; it follows from (3.2) that

(3.3) $$\text{p lim}_{u\to\infty} \{\sup_{v\leqq u} |m_v' - \theta v|/u\} = 0,$$

and in particular that $\text{p lim}_u m_u'/u = \theta$. Now (2.6) holds for the same reasons as before. Hence by the same argument which we used to prove (2.10), we can prove

$$\text{p lim}_{u\to\infty} |p(m'_{ui/c}) - p(\theta ui/c)|/u^{\frac{1}{2}} = 0, \qquad i = 1, \cdots, c.$$

But $p(m'_{ui/c}) = q_u(i/c)$; it follows that if Condition (i) of Theorem 2.1 holds

[4] It would perhaps be more natural to take $q(u) = p(m_u)$, but this would lead outside C to the space D of functions with discontinuities of the first kind. The theory of weak convergence is much more involved for D than for C; see [6].

with $\nu_n = P_n$, $\nu = W$, then it also holds if ν_n is replaced by Q_u and $\nu = W$. Therefore we need only show that Condition (ii) of Theorem 2.1 holds (with ν_n replaced by Q_u and $\nu = W$). By transforming (2.6) slightly, it is easily seen that if ϵ, η are given then there exist δ, u_0 such that if $u \geqq u_0$ then

$$\sup_{\substack{|v-w| \leqq u\delta \\ v \leqq u(1+\delta)}} |p(v\theta) - p(u\theta)| < \epsilon u^{\frac{1}{2}} \tag{3.4}$$

with probability exceeding $1 - \eta$. Further, by (3.3), u_0 can be chosen so that if $u \geqq u_0$ then, with probability exceeding $1 - \eta$, we have

$$\sup_{v \leqq u} |m'_v - \theta v| < \delta u/(2 + \theta). \tag{3.5}$$

If both (3.4) and (3.5) hold and if $v, w \leqq u$, $|v - w| \leqq u\delta$, then $|m'_v - m'_w| < \delta u$ and $m'_v \leqq u(1 + \delta)$, so that $|p(m'_u) - p(m'_v)| < \epsilon u^{\frac{1}{2}}$. Thus

$$\sup_{\substack{|v-w| \leqq u\delta \\ v,w \leqq u}} |p(m'_v) - p(m'_w)| < \epsilon u^{\frac{1}{2}},$$

with probability exceeding $1 - 2\eta$. This implies that for all ϵ, η there exist δ, u_0 such that if $u \geqq u_0$ then $\mathbf{P}\{\omega_{Q_u}(\delta) \geqq \epsilon\} < \eta$. This is Condition (ii) of Theorem 2.1 for Q_u, which completes the proof.

Theorem 3.1 can be used to derive the asymptotic distribution of $\max_{k \leqq m_u} S_k$, for example.

4. Application to renewal processes. Let $\{Y_1, Y_2, \cdots\}$ be a sequence of positive random variables on $(\Omega, \mathcal{B}, \mathbf{P})$. Suppose that μ is a positive constant such that if $X_n = Y_n - \mu$, then $\{X_n\}$ satisfies the invariance principle with norming factors $n^{\frac{1}{2}}\sigma$. Let

$$m_u = \max\left\{k: \sum_{i=1}^{k} Y_i \leqq u\right\}, \tag{4.1}$$

with $m_u = 0$ if $Y_1 > u$. Then, as a function of u, m_u is nondecreasing, right-continuous, and increases by unit jumps. Moreover, by the definitions of m'_u and $q(u)$, we have $q(u) = u - m'_u\mu$.

In order to apply the results of the preceding section, we must show that (3.2) holds for some θ; we will prove it with $\theta = 1/\mu$:

$$\text{p}\lim_{u\to\infty} \{\sup_{v \leqq u} |m_v - v/\mu|/u\} = 0. \tag{4.2}$$

Since $\{X_n\}$ satisfies the invariance principle, the distribution function of

$$\max_{k \leqq n} |S_k|/n^{\frac{1}{2}}\sigma$$

converges, namely to $G(\alpha) = W\{x: \sup_t |x(t)| \leqq \alpha\}$. Therefore

$$\text{p}\lim_{n\to\infty} \{\max_{k \leqq n} |S_k|/n\} = 0. \tag{4.3}$$

We will deduce (4.2) from (4.3). By the definition (4.1), $m_v \geqq k$ if and only if $S_k \leqq v - \mu k$. Now

$$\sup_{v \leqq u} (m_v - v/\mu)/u \geqq \epsilon \tag{4.4}$$

implies that $m_v \geqq [\epsilon u + v/\mu]$ for some $v \leqq u$, or that

$$\frac{S_{[\epsilon u + v/\mu]}}{[\epsilon u + u/\mu]} \leqq \frac{v - \mu[\epsilon u + v/\mu]}{[\epsilon u + u/\mu]} \tag{4.5}$$

for some $v \leqq u$. Now

$$\frac{v - \mu[\epsilon u + v/\mu]}{[\epsilon u + u/\mu]} \leqq -\frac{\mu\epsilon - \mu/u}{\epsilon + 1/\mu - 1/u}$$

for all u, v, and the right-hand member of this inequality is less than

$$-\delta = -\mu\epsilon/2(\epsilon + 1/\mu)$$

for all sufficiently large u. Moreover, if $v \leqq u$ then $[\epsilon u + v/\mu] \leqq [\epsilon u + u/\mu] = n_u$. Thus there is some u_0 such that if $u \geqq u_0$ and if (4.5) holds for some $v \leqq u$, then $S_k/n_u \leqq -\delta$ for some $k \leqq n_u$. Therefore, if $u \geqq u_0$, (4.4) implies

$$\min_{k \leqq n_u} S_k/n_u \leqq -\delta. \tag{4.6}$$

Since $\lim_u n_u = \infty$, it follows from (4.3) that the probability of (4.6) goes to 0; hence the probability of (4.4) goes to 0. A symmetric argument shows that the probability of

$$\inf_{v \leqq u} (m_v - v/\mu) \leqq -\epsilon$$

goes to 0, which completes the proof of (4.2).

Thus Theorem 3.1 applies with $\theta = 1/\mu$. Now

$$q_u(t) = q(ut) = -(m'_{ut} - ut/\mu)/u^{\frac{1}{2}}\sigma\mu^{-\frac{3}{2}},$$

and if $Q_u(A) = \mathbf{P}\{q_u \varepsilon A\}$, then $Q_u \Rightarrow W$. From the symmetry of the Wiener distribution, it is clear that the result remains valid if q_u is replaced by $-q_u$. Therefore we have the following theorem.

THEOREM 4.1. *Let $\{Y_n\}$ be a sequence of positive random variables such that for some $\mu > 0$, $\{Y_n - \mu\}$ satisfies the invariance principle with norming factors $n^{\frac{1}{2}}\sigma$. Define m_u by* (4.1), *m'_u by* (3.1), *and put $r_u(t) = (m'_{ut} - ut/\mu)/u^{\frac{1}{2}}\sigma\mu^{-\frac{3}{2}}$ for $u \geqq 0$, $0 \leqq t \leqq 1$. If $R_u(A) = \mathbf{P}\{r_u \varepsilon A\}$, then $R_u \Rightarrow W$ as $u \rightarrow \infty$.*

The special case of this theorem in which the Y_n are independent, identically distributed, and integer-valued was proved, by very different methods, in [2]. For the central limit theorem for renewal processes, see [9] and [10].

As an example, consider the random variable $\sup_t r_u(t)$, which equals

$$\sup_{v \leqq u} (m'_v - v/\mu)/u^{\frac{1}{2}}\sigma\mu^{-\frac{3}{2}}. \tag{4.7}$$

Under the conditions of Theorem 4.1 the distribution function of this random variable converges to the $F(\alpha)$ defined by (2.2). Since $|m'_v - m_v| \leqq 1$, this result still holds if m'_v is replaced by m_v in (4.7). Similarly, an arc sine law can be derived.

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