FIRST PASSAGE TIMES OF A GENERALIZED RANDOM WALK

By John R. Kinney

Lincoln Laboratory,[1] *Massachusetts Institute of Technology*

Introduction. Let $X(t)$, $t = 1, 2, \cdots$, be independent integer-valued random variables such that $\Pr\{X(t) = i\} = p(i)$, with $p(-m) > 0$, $p(i) = 0$ for $i < -m$, and let $P(z) = E\{z^{X(t)}\}$. The solutions of the functional equation,

$$1 = wP(\lambda(w)),$$

have played a fundamental role in the work of several authors.

R. Otter [5] used this solution for the case $m = 1$, in his study of multiplicative processes. T. E. Harris [4] used it in the examination of first passage times in random walk problems. L. Takács [7] and B. W. Conolly [2] have used the solutions to describe the distribution of the number of persons served during the busy period of a queue.

In the first section of this paper we introduce notation and state some preliminary lemmas. The second section deals with the sums

$$S(t) = S(0) + \sum_{i=1}^{t} X(i),$$

where $S(0)$ is a random variable taking on nonnegative integer values and has $E\{z^{S(0)}\} = K(z) = \sum_{j \geqq 0} k(j)z^j$. The third section deals with the sequence $S^*(t)$ defined inductively by $S^*(0) = S(0)$, $S^*(t) = \max[S^*(t-1), 0] + X(t)$, and the sequence $Z(t) = \max[S^*(t), 0]$. The generating functions of the distributions $\{S(t), \min_{0<i<t} S(i) \geqq 0\}$, $S^*(t)$, and $Z(t)$ are expressed in terms of the solutions of $1 = wP(\lambda(w))$. The distribution of $\{S(t), \min_{0<j<t} S(j) \geqq 0\}$ corresponds to the distribution of a discrete time queue during busy time, and that of $Z(t)$ to the distribution of the transient queue.

The formulae we obtain could be deduced from those of F. Spitzer [6], but we give here a different approach.

1. Notation and Preliminary Lemmas. The following notation will be used. For $i \geqq 0$, $a > 0$, and $n \geqq 0$, let

$$f(n, i, j) = \Pr\{S(j) = i, \min_{0<k<j} S(k) \geqq 0 | S(0) = n\},$$

$$F(n, z, j) = \sum_{i \geqq 0} f(n, i, j)z^i, \qquad \mathfrak{F}(n, z, w) = \sum_{j \geqq 0} F(n, z, j)w^j,$$

$$\mathfrak{F}(z, w) = \sum_{i \geqq 0, j \geqq 0} \Pr\{S(j) = i, \min_{0<k<j} S(k) \geqq 0\} z^i w^j,$$

$$g(n, -a, w) = \Pr\{S(j) = -a, \min_{0<k<j} S(k) \geqq 0 | S(0) = n\},$$

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$$G(n, -a, w) = \sum_{j>0} g(n, -a, j)w^j, \qquad \mathcal{G}(n, z, w) = \sum_{a=1}^{m} G(n, -a, w)z^{-a},$$

$$\tau(n, w) = \sum_{j>0} \Pr\{S(j) < 0, \min_{0<k<j} S(k) \geqq 0 | S(0) = n\}w^j,$$

$$\tau(w) = \sum_{j>0} \Pr\{S(j) < 0, \min_{0<k<j} S(k) \geqq 0\}w^j,$$

$$\mathfrak{F}^*(n, z, w) = \sum_{i \geqq 0, j \geqq 0} \Pr\{S^*(j) = i | S^*(0) = n\}z^i w^j,$$

$$\mathfrak{F}^*(z, w) = \sum_{i \geqq 0, j \geqq 0} \Pr\{S^*(j) = i\}z^i w^j,$$

$$T^*(n, w) = \sum_{j \geqq 0} \Pr\{S^*(j) < 0 | S^*(0) = n\}w^j,$$

$$T^*(w) = \sum_{j \geqq 0} \Pr\{S^*(j) < 0\}w^j,$$

$$\mathcal{H}(z, w) = \sum_{i \geqq 0, j \geqq 0} \Pr\{Z(j) = i\}z^i w^j,$$

and

$$H^*(z) = \lim_{t \to \infty} E\{z^{Z(t)}\}$$

when this limit exists.

In the computations in the subsequent sections we will need

LEMMA 1.

$$\Pr\{S(t + i) = k - a, \min_{t<u<t+i} S(u) \geqq k | S(t) = n + k\}$$
$$= \Pr\{S(i) = -a, \min_{0<u<i} S(u) \geqq 0 | S(0) = n\} = g(n, -a, i)$$

$$\Pr\{S(t + i) = k + j, \min_{t<u<t+i} S(u) \geqq 0 | S(t) = n\}$$
$$= \Pr\{S(i) = j, \min_{0<u<i} S(u) \geqq 0 | S(0) = n\} = f(n, j, i).$$

The same expressions hold when we replace $S(t)$ by $S^*(t)$.

PROOF. Since the $X(t)$ are all independent and have the same distributions, the set of random variables $X(t + 1), \cdots, X(t + i)$ has the same joint probability distribution as $X(1), \cdots, X(i)$. The equations are simple consequences of this. The second statement is a consequence of the fact that

$$\{S^*(i + t) = m, \min_{0<u<i} S^*(u + t) \geqq 0, S^*(t) = n\} \quad \text{and}$$
$$\{S(i + t) = m, \min_{0<u<i} S(u + t) \geqq 0, S(t) = n\}$$

impose the same restrictions on $X(t + 1), \cdots, X(t + i)$, for either positive or negative m.

LEMMA 2. *For $|w| < 1$, the functional equation $1 = wP(\lambda(w))$ has m solutions, $\lambda_1(w), \cdots, \lambda_m(w)$, within the unit circle.*

PROOF. For $|\lambda| = 1$, $|\lambda^m| = 1$ and $|w\lambda^m P(\lambda)| \leqq |w \sum_{i \geqq -m} p(i) = |w|$. Hence we may use Rouché's theorem [1] to see that $\lambda^m - w\lambda^m P(\lambda)$ has m zeros within the unit circle for $0 < |w| \leqq 1$. It may be seen by inspection that $\lambda = 0$ is not one of these, so the same is true of $1 - wP(\lambda)$.

LEMMA 3. *For small non-zero w, the functional equation $1 = \mathcal{G}(0, \lambda(w), w)$ has m distinct solutions, $\lambda_1^*(w), \cdots, \lambda_m^*(w)$, all different from zero.*

PROOF. *In* $g(\lambda, w) = \lambda^m - \lambda^m G(0, \lambda, w)$ we let $w = s^m$, $\lambda = s\zeta$. We obtain $g(\lambda, w) = s^m h(\zeta, s)$. Since the $G(0, -a, w)$ have no constant terms in their power series expansions, it is easy to see that

$$\lim_{s \to 0} h(\zeta, s) = h(\zeta, 0) = \zeta^m - g(0, -m, 1) = \zeta^m - p(-m),$$

and $\lim_{s \to 0} h'(\zeta, s) = h'(\zeta, 0)$, uniformly in $|\zeta| \leqq 1$. The zeros of $h(\zeta, 0)$ are $r_j = [p(-m)]^{1/m} e^{2\pi i j/m}$, $j = 1, \cdots, m$. Let c_j be the circle $|r_j - \zeta| = \epsilon$,

$$\epsilon < \min_{j \neq k} [|r_j|, |r_j - r_k|/2, 1 - |r_j|].$$

Since the limits are uniform in $|\zeta| \leqq 1$,

$$\lim_{s \to 0} \int_{c_j} \frac{h'(\zeta, s)}{h(\zeta, s)}\, d\zeta = \int_{c_j} \frac{h'(\zeta, 0)}{h(\zeta, 0)}\, d\zeta = 2\pi i \qquad j = 1, \cdots, m.$$

Hence, for s sufficiently small, $h(\zeta, s)$ has one of its zeros in each of the c_j, which were chosen so as not to overlap, to avoid zero, and to remain with $|\zeta| < 1$. Since $h(\zeta, s)$ is a polynomial of degree m, this proves the lemma.

2. The Sequence $S(t)$. In this section the functions $G(n, -a, w)$ are expressed in terms of the solutions of $1 = \mathcal{G}(0, \lambda(w), w)$. These solutions are then shown to satisfy $1 = wP(\lambda(w))$. Finally $\mathcal{F}(n, z, w)$ is expressed in terms of the $P(z)$ and $G(n, -a, w)$.

Define the matrix $L = \|L(a, n)\| = \|\lambda_a^{*-n}(w)\|$ $1 \leqq a, n \leqq m$. This matrix has an inverse, since it has a Vandermonde determinant and the $\lambda_a^*(w)$ are distinct and different from zero. Let $A = \|A(a, n)\| = L^{-1}$.

THEOREM 1. *The functions $G(n, -a, w)$ are given by*

$$G(n, -a, w) = \sum_{j=1}^{m} A(a, j) \lambda_j^{*n}(w). \tag{2.1}$$

PROOF. If $S(i) = -a$, $\min_{0 < u < t} S(u) \geqq 0$, $S(0) = n$, there must be a least $k \leqq i$ for which $\min_{0 < u < k} S(u) < n$. The following decompositions can be made. For $n > m - a$,

$$\{S(i) = -a, \min_{0<u<i} S(u) \geqq 0, S(0) = n\}$$

$$= \bigcup_{k=1}^{i} \bigcup_{s=1}^{\min[n,m]} \{S(k) = n - s, \min_{0 \leqq u < k} S(u) \geqq n, S(0) = n\}$$

$$\cap \{S(i) = -a, \min_{k \leqq u < i} S(u) \geqq 0, S(k) = n - s\};$$

for $n \leqq m - a$,

$$\{S(i) = -a, \min_{0<u<i} S(u) \geqq 0, S(0) = n\}$$
$$= \{S(i) = -a, \min_{0\leqq u<i} S(u) \geqq n, S(0) = n\}$$
$$\cup \bigcup_{k=1}^{i} \bigcup_{s=1}^{n} \{S(k) = n - s, \min_{0\leqq u<k} S(u) \geqq n, S(0) = n\}$$
$$\cap \{S(i) = -a, \min_{k\leqq u<i} S(u) \geqq 0, S(k) = n - s\}.$$

Take conditional probabilities and apply Lemma 1 to obtain

$$g(n, -a, i) = \sum_{s=1}^{\min[n,m]} \sum_{k=1}^{i} g(0, -s, k)g(n - s, -a, i - k)$$
$$+ g(0, -n - a, i)\delta(n, [1, m - a])$$

where $\delta(n, [1, m - a]) = 1$ if $1 \leqq n \leqq m - a$, 0 otherwise. For the functions $G(n, -a, w)$ this implies

$$(2.2) \quad G(n, -a, w) = \sum_{s=1}^{\min[n,m]} G(0, -s, w)G(n - s, -a, w)$$
$$+ G(0, -n - a, w)\delta(n, [1, m - a]).$$

For $n \geqq m$, (2.2) is a set of difference equations, and for $n < m$, a set of boundary conditions. Since $\lambda_1^*(w), \cdots, \lambda_m^*(w)$ are the distinct solutions of $1 = \mathcal{G}(0, \lambda, w)$, the solutions of (2.2) can be expressed in the form

$$(2.3) \qquad G(n, -a, w) = \sum_{j=1}^{m} B(a, j)\lambda_j^{*^n}(w),$$

where the $B(a, j)$ are chosen to make the $G(n, -a, w)$ consistent with the first m equations of (2.2).

Define the following matrices:

$$B = \|B(a, n)\|, \qquad M = \|M(a, n)\| = \|\lambda_a^{*^{n-1}}(w)\|,$$
$$G = \|G(a, n)\| = \|G(n - 1, -a, w)\| \qquad 1 \leqq a, \quad n \leqq m$$
$$G^* = L^{-1}M = \|G^*(a, n)\| = \|G^*(n - 1, -a, w\|, \qquad 1 \leqq a, \quad n \leqq m,$$
$$H = \|H(i, k)\|, \qquad H(i, k) = G(0, -(i - k), w),$$
$$0 < i - k < m, \quad 0 \text{ otherwise},$$
$$K = \|K(i, k)\|, \qquad K(i, k) = G(0, -i - k, w),$$
$$1 \leqq i + k < m, \quad 0 \text{ otherwise}.$$

The first m equations of (2.2) may be written $G = GH + K$. The first m equations of (2.3) may be written $G = BM$.

To finish the proof, it will be sufficient to show $B = A = L^{-1}$. That

$$1 = \sum_{s=1}^{n} \lambda_j^{*^{-s}}(w) G^*(0, -s, w) \qquad 1 \leqq j \leqq m$$

may be seen by observing the first row of the product $LG^* = M$. Hence the polynomial $\lambda^m - \sum_{s=1}^{m} \lambda^{m-s} G^*(0, -s, w)$ has the same zeros as

$$\lambda^m - \lambda^m \mathcal{G}(0, \lambda, w).$$

Therefore $G^*(0, -s, w) \equiv G(0, -s, w)$, $1 \leqq s \leqq m$. Multiplying

$$1 = \mathcal{G}(0, \lambda_\alpha^*(w), w)$$

by $\lambda_\alpha^{*^n}(w)$ yields

$$\begin{aligned}
\lambda_\alpha^{*^n}(w) &= \sum_{s=1}^{m} \lambda_\alpha^*(w)^{n-s} G(0, -s, w) \\
&= \sum_{s=1}^{n} \lambda_\alpha^*(w)^{n-s} G(0, -s, w) + \sum_{s=n+1}^{m} \lambda_\alpha^{*^{-(s-n)}}(w) G(0, -s, w) \\
&= \sum_{b=0}^{n-1} \lambda_\alpha^{*^b}(w) G(0, -(n-b), w) + \sum_{b=1}^{m-n} \lambda_\alpha^{*^{-b}}(w) G(0, -n-b, w)
\end{aligned}$$

for $0 \leqq n < m$. In matrix notation this is $M = MH + LK$. Since L has an inverse, $L^{-1}M = L^{-1}M + K$, so $G^* = G^*M + K$. Hence $G^*(n, -a, w)$ satisfies the first m equations of (2.2). However, these equations are a recurrence relations which define the $G(n, -a, w)$ uniquely once the $G(0, -a, w)$ are known. Hence $BM = G = G^* = L^{-1}M$. The matrix M has a Vandermonde determinant and the $\lambda_j^*(w)$ are distinct and not equal to zero, so M has an inverse. Therefore, $B = L^{-1} = A$.

THEOREM 2. *The solutions of* $1 = \mathcal{G}(0, \lambda(w), w)$ *satisfy* $1 = wP(\lambda(w))$.

PROOF. For $i > 0$,

$$\begin{aligned}
&\{S(i) = -a, \min_{0<u<i} S(u) \geqq 0, S(0) = 0\} \\
&\qquad = \bigcup_{k \geqq 0} \{X(1) = k\} \cap \{S(i) = -a, \min_{1<u<i} S(u) \geqq 0, S(1) = k\}.
\end{aligned}$$

Apply Lemma 1 after taking conditional probabilities to obtain

$$g(0, -a, i) = \sum_{k \geqq 0} p(k) g(k, -a, i-1).$$

For $i = 1$, $g(0, -a, 1) = p(-a)$. For the $G(n, -a, w)$, then,

$$G(0, -a, w) = w[p(-a) + \sum_{k \geqq 0} p(k) G(k, -a, w)].$$

Multiply by $\lambda_j^{*^{-a}}(w)$, sum for $1 \leqq a \leqq m$, recall that $1 = \mathcal{G}(0, \lambda_j^*(w), w)$, and apply (2.1) to $G(k, -a, w)$ to obtain

$$1 = w\left[\sum_{a=1}^{m} p(-a) \lambda_j^{*^{-a}}(w) + \sum_{k=0}^{\infty} \sum_{a=1}^{m} \sum_{\alpha=1}^{m} p(k) \lambda_j^{*^{-a}}(w) A(a, \alpha) \lambda_\alpha^{*^k}(w)\right].$$

Since $A = L^{-1}$, this reduces to

$$1 = w\left[\sum_{a=1}^{m} p(-a)\lambda_j^{*^{-a}}(w) + \sum_{k=0}^{\infty} p(k)\lambda_j^{*^{k}}(w)\right] = wP(\lambda_j^*(w)).$$

Since j was arbitrarily chosen, the theorem is proved. From the above theorems, we may deduce

COROLLARY 1. *The set of solutions of* $1 = \mathcal{G}(0, \lambda(w), w)$ *and the set of solutions of* $1 = wP(\lambda(w))$ *within the unit circle are identical.*

COROLLARY 2.

$$G(n, -a, w) = \sum_{\alpha=1}^{m} A(a, \alpha)\lambda_\alpha^n(w)$$

$$\tau(n, w) = \sum_{a=1}^{m} G(n, -a, w) = \sum_{a=1}^{m}\sum_{\alpha=1}^{m} A(a, \alpha)\lambda_\alpha^n(w).$$

Setting $\lambda = 1$ in $\lambda^n - \lambda^n\mathcal{G}(0, \lambda, w) \equiv \prod_{\alpha=1}^{m}(\lambda - \lambda_\alpha(w))$, and recalling that $\tau(0, w) = \sum_{a=1}^{m} G(0, -a, w) = \mathcal{G}(0, 1, w)$, we see

COROLLARY 3.

$$\tau(0, w) = 1 - \prod_{\alpha=1}^{n}(1 - \lambda_\alpha(w)).$$

THEOREM 3.

$$\mathfrak{F}(n, z, w) = \{z^n - \mathcal{G}(n, z, w)\}/\{1 - wP(z)\}$$

$$\mathfrak{F}(z, w) = \left\{K(z) - \sum_{a=1}^{m}\sum_{\alpha=1}^{m} z^{-a}A(a, \alpha)K(\lambda_\alpha(w))\right\} \Big/ [1 - wP(z)].$$

PROOF. Note that

$$\{S(i) = j, \min_{0<u<i} S(u) \geqq 0, S(0) = n\}$$

$$= \bigcup_{k=0} \{S(i-1) = k, \min_{0<u<i-1} S(u) \geqq 0, S(0) = n\} \cap \{X(i) = j - k\}$$

and

$$\{S(i) = -a, \min_{0<u<i} S(u) \geqq 0, S(0) = n\}$$

$$= \bigcup_{k=0} \{S(i-1) = k, \min_{1<u<i-1} S(u) \geqq 0, S(0) = n\} \cap \{X(i) = -a - k\}.$$

Apply Lemma 1 after taking conditional probabilities to obtain

$$f(n, j, i) = \sum_{k\geqq 0} f(n, k, i-1)p(j-k), j > 0;$$

$$g(n, -a, i) = \sum_{k\geqq 0} f(n, k, i-1)p(-u-k), a > 0.$$

This implies

$$F(n, z, i) + \sum_{a=1}^{m} g(n, -a, i)z^{-a} = \sum_{j \geqq 0} \sum_{k \geqq -m} f(n, j, i-1)p(k-j)z^k$$
$$= \sum_{j \geqq 0} f(n, j, i-1)z^j \sum_{k \geqq -m} p(k-j)z^{k-j}.$$

Since $p(-i) = 0$ for $i > m$, this last sum is $P(z)$, so

$$F(n, z, i) + \sum_{a=1}^{m} g(n, -a, i)z^{-a} = F(n, z, i-1)P(z).$$

It follows easily that

$$\mathfrak{F}(n, z, w) - z^n + \mathfrak{G}(n, z, w) = wP(z)\mathfrak{F}(n, z, w).$$

This implies the first statement of the theorem. The elimination of the condition $S(0) = n$ yields $\mathfrak{F}(z, w) = \{K(z) - \sum_{n=0}^{\infty} k(n)\mathfrak{G}(n, z, w)\}/\{1 - wP(z)\}$. It suffices to use (2.1) and rearrange the sum to obtain the second equation of the theorem.

3. The sequences $S^*(t)$ and $Z(t)$. First $T^*(n, w)$ and $T^*(w)$ are found in terms of $\tau(n, w)$ and $\tau(w)$. Then $\mathfrak{F}^*(n, z, w)$, $\mathfrak{F}^*(z, w)$, and $\mathfrak{H}(z, w)$ are expressed in terms of $T^*(n, w)$, $T^*(w)$, $\mathfrak{F}(n, z, w)$, and $\mathfrak{F}(z, w)$. Finally $H^*(z)$ is expressed in terms of $\mathfrak{G}(0, z, 1)$ and $P(z)$.

THEOREM 4.

$$T^*(n, w) = \tau(n, w)/(1 - \tau(0, w)), \quad T^*(w) = \tau(w)/(1 - \tau(0, w)).$$

PROOF. Following methods introduced by Feller [3] in his discussion of recurrent events, we observe that

$$\{S^*(t) < 0, S^*(0) = n\} = \bigcup_{0<i\leqq t} \{S^*(i) < 0, S^*(0) = n\}$$
$$\cap \{S^*(i) < 0, \min_{i<j<t} S^*(j) \geqq 0, S^*(t) < 0\}.$$

It may be seen from the definitions of $S(t)$, $S^*(t)$ that

$$\Pr\{S^*(t) < 0, \min_{i<j<t} S^*(j) \geqq 0 | S^*(i) < 0\}$$
$$= \Pr\{S(t-i) < 0, \min_{0<j<t-i} S(j) \geqq 0 | S(0) = 0\}.$$

Hence, if we take conditional probabilities and introduce generating functions we find

$$T^*(n, w) = T^*(n, w)\tau(0, w) + \tau(n, w).$$

The first equation of the theorem follows from this, and the second follows by eliminating the condition $S(0) = n$.

THEOREM 5.

$$\mathfrak{F}^*(n, z, w) = \mathfrak{F}(n, z, w) + T^*(n, w)[\mathfrak{F}(0, z, w) - 1]$$
$$\mathfrak{F}^*(z, w) = \mathfrak{F}(z, w) + T^*(w)[\mathfrak{F}(0, z, w) - 1]$$
$$\mathfrak{H}(z, w) = \mathfrak{F}(z, w) + T^*(w)\mathfrak{F}(0, z, w).$$

PROOF. Note that

$$\{S^*(t) = i, S^*(0) = n\} = \{S^*(t) = i, \min_{0<j<t} S^*(j) \geqq 0, S^*(0) = n\}$$

$$\cup \bigcup_{0<k<t} \{S^*(t) = i, \min_{k<j<t} S^*(j) \geqq 0, S^*(k) < 0\} \cap \{S^*(k) < 0, S^*(0) = n\}.$$

Since

$$\Pr\{S^*(t) = i, \min_{k<j<t} S^*(j) \geqq 0 | S^*(k) \leqq 0\}$$
$$= \Pr\{S(t) = i, \min_{k<j<t} S(j) \geqq 0 | S(k) = 0\} = f(0, i, t-k),$$

taking conditional probabilities yields

$$\Pr\{S^*(t) = i | S^*(0) = n\}$$
$$= f(n, i, t) + \sum_{k=0}^{t-1} f(0, i, t-k) \Pr\{S^*(k) < 0 | S^*(0) = n\}.$$

For the generating functions this implies

$$\mathfrak{F}^*(n, z, w) = \mathfrak{F}(n, z, w) + T^*(n, w)\{\mathfrak{F}(0, z, w) - 1\}.$$

Elimination of the condition $S^*(0) = n$ yields the second equation of the theorem.

Since $Z(t) = \max[S^*(t), 0]$, $\{S^*(t) = i\} = \{Z(T) = i\}$ for $i > 0$ and

$$\{Z(t) = 0\} = \{S^*(t) = 0\} \cup \{S^*(t) < 0\}.$$

For the generating functions, this implies $\mathfrak{H}(z, w) = \mathfrak{F}^*(z, w) + T^*(w)$. If the expression for $\mathfrak{F}^*(z, w)$ is substituted here, the third statement of the theorem is obtained.

THEOREM 6. *If* $P'(1) < 0$, $\tau'(0, 1) < \infty$, *and* $\lim_{t\to\infty} E\{z^{Z(t)}\} = H^*(z)$, *then for real* z *and* w, $0 < z, w < 1$,

$$H^*(z) = \lim_{w\uparrow 1} (1 - w)\mathfrak{H}(z, w) = \frac{1}{\tau'(0, 1)} \frac{1 - \mathfrak{G}(0, z, 1)}{1 - P(z)}.$$

PROOF. If $P'(1) < 0$, an application of the law of large numbers shows

$$\lim_{t\to\infty} \Pr\{S(t) \geqq 0\} = 0 \quad \text{and so} \quad \lim_{w\to 1} \tau(w) = 1.$$

Since

$$\Pr\{S(t) \geqq 0, \min_{0<j<t} S(j) \geqq 0\} + \sum_{k=1}^{t} \Pr\{S(k) < 0, \min_{0<j<k} S(j) \geqq 0\} = 1,$$

an elementary computation with generating functions shows that

$$(1 - w)\mathfrak{F}(1, w) + \tau(w) = 1.$$

Hence, for z and w real, $0 < z, w < 1$,

$$\lim_{w\uparrow 1} (1 - w)\mathfrak{F}(z, w) \leqq \lim_{w\uparrow 1} (1 - w)\mathfrak{F}(1, w) = \lim_{w\uparrow 1} 1 - \tau(w) = 0.$$

However, $w = 1$ is a simple pole of $T^*(w) = \tau(w)/(1 - \tau(0, w))$. Hence, using the third statement of Theorem 6 together with Theorem 3, we have

$$\lim_{w \uparrow 1} (1 - w)\mathcal{H}(z, w) = \lim_{w \uparrow 1} (1 - w) \frac{\tau(w)}{1 - \tau(0, w)} \frac{1 - \mathcal{G}(0, z, w)}{1 - wP(z)}$$

$$= \frac{1}{\tau'(0, 1)} \frac{1 - \mathcal{G}(0, z, 1)}{1 - P(z)} .$$

For an arbitrary $\epsilon > 0$, take $N(\epsilon)$ so large that for $t > N(\epsilon)$,

$$|E\{z^{Z(t)}\} - H^*(z)| < \epsilon.$$

Then for z and w real, and $z < 1$

$$\lim_{w \uparrow 1} |(1 - w)\mathcal{H}(z, w) - H^*(z)| = \lim_{w \uparrow 1} |(1 - w) \sum_{t=1}^{\infty} [E\{z^{Z(t)}\} - H^*(z)]w^t| \leqq$$

$$\lim_{w \uparrow 1} |(1 - w) \sum_{t=0}^{N(\epsilon)} |E\{z^{Z(t)}\} + H^*(z)| + \lim_{w \uparrow 1} (1 - w) \cdot \epsilon \cdot \sum_{t=N(\epsilon)}^{\infty} w^t = \epsilon,$$

since $E\{z^{Z(t)}\}$ and $H^*(z)$ are bounded for $|z| < 1$. Since ϵ was arbitrary, the theorem is proved.

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