SOME ASYMPTOTIC RESULTS FOR A COVERAGE PROBLEM

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1. Introduction. A quantity whose distribution is of considerable interest in calculation of microscopic behavior of heterogeneous materials is the intercept fraction of the phases of the mixture (i.e., the fraction of a linear path intercepted by a particular phase). For example, in nuclear reactor theory, one will be interested in the fraction of a neutron path through a given phase. In this paper, we study the statistical behavior of the intercept fraction, for a path of fixed length, under the following idealization (more precisely defined in Section 2):

1. Linear sections of a phase are selected at random and placed on a very long line at random, without overlap.
2. The given path length is placed at random on the long line.

Some related experimental work has been done [1], with photomicrographs of sections on solid Boron Carbide-Zirconium mixtures. In this work a number of linear paths of fixed length and parallel to one axis of the photograph were taken at positions along the other axis of the photograph and the length of Boron Carbide covered by each path was measured. It is interesting to note that the frequency of zero fraction intercept predicted by the idealization was in good agreement with experimental results. The small differences found were in the direction suggested by the fact that the sampled line had to be of finite thickness. That is, the predicted frequency of zero intercept tended to be slightly higher than observed.

2. Assumptions and Summary of Results. We assume we have a sample of line segments $\Delta_1, \Delta_2, \cdots, \Delta_n$ which are independent random drawings from a universe of segments with probability density $p(\Delta)$, $0 \leqq \Delta \leqq \Delta_M$. Now we suppose that the segments $\Delta_1, \Delta_2, \cdots, \Delta_n$ are placed on the interval $(0, L)$ in such a way that all admissible configurations of the segments are equally likely and that $L \geqq n\Delta_M$. We call a configuration admissible if

(a) There is no overlapping among segments.

(b) No segments overlap zero or L.

Now we consider a line of length $\lambda \ll L$ and place it at random on $(0, L)$ with the restriction that no overlap with zero or L occurs.

Finally we define the intercept (or coverage) of λ, λF, as that part of λ covered by the segments $\Delta_1, \Delta_2, \cdots, \Delta_n$, and ask for the distribution of λF. In particular, we consider the limiting distribution of λF as $n \to \infty$, for $n\mu/L = V$, $0 < V < 1$, where μ is average segment size.

We find that

$$\text{I.} \lim_{n\to\infty} \Pr\{\lambda F = 0\} = (1 - V) \exp - \alpha\lambda, \qquad \text{where } \alpha = V/(1 - V)\mu.$$

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$$\text{II. } \lim_{n\to\infty} \Pr\{\lambda F = \lambda\} = \frac{V}{\mu}\int_{\lambda}^{\Delta_M} (x - \lambda)p(x)\,dx, \qquad \text{if } \lambda < \Delta_M,$$
$$= 0, \qquad \text{if } \lambda \geqq \Delta_M.$$

III. For $0 < \lambda F < \lambda$, there are a number of distinct continuous contributions to the cumulative probability which unfortunately are extremely dependent on the specific nature of $p(\Delta)$. Because of the large numbers and complexity of these contributions, we defer a more or less detailed listing of this result and only note that for the simplest of these contributions, the probability integral is given by

$$(1\text{-}V)\sum_{1}^{\infty}\int_{0}^{\min(\lambda F, s\Delta_M)} \frac{[\alpha(\lambda - x)]^s}{s!} \exp - \alpha(\lambda - x)p_s(x)\,dx,$$

where $p_s(x)$ is the s-fold convolution of $p(\Delta)$.

$$\text{IV. } E\lambda F = \lambda V - \frac{\lambda V}{\mu}\left\{\int_{\min(\lambda,\Delta_M)}^{\Delta_M} xp(x)\,dx - \int_{\lambda}^{\max(\lambda,\Delta_M)} (x - \lambda)p(x)\,dx\right\}.$$

V. It was not feasible to obtain a variance for λF for the asymptotic distribution. If, however, one further assumes that $\lambda \to \infty$, it is found that, for large λ, Var $\lambda F = \mu V(1 - V)^2 [1 + (\sigma^2/\mu^2)]\lambda$, where σ^2 is the variance of the distribution $p(\Delta)$, and is assumed finite.

VI. As a matter of further interest, the probability distribution of "admissible configurations" mentioned above is apparently a novel generalization of the joint distribution of n independent uniformly distributed random variables. In addition to its use in the present problem, one can derive from this distribution, a solution to a one dimension nearest neighbor problem for non-infinitesimal one dimensional "particles"; this result is at least suggestive of a correction for the three dimensional problem when the three dimensional particles are not infinitesimal. Details are given later in the discussion. The nearest neighbor problem is of interest in both physics (e.g., Chandresekhar [2]) and biology (e.g., Clarke and Evans [3]).

3. Joint Distribution of Admissible Configurations. Consider some one of the admissible configurations of $\Delta_1, \Delta_2, \cdots, \Delta_n$. Denote the segment closest to zero by δ_1, next closest by δ_2 and so on so that $\delta_1, \delta_2, \cdots, \delta_n$ is some permutation of $\Delta_1, \Delta_2, \cdots, \Delta_n$, numbered according to segment order on $(0, L)$. Let $x_j (j = 1, 2, \cdots, n)$ denote the position of the midpoint of δ_j on $(0, L)$. Then our stipulation of equi-probability of admissible configurations requires that the joint probability density of $x_1, x_2, \cdots, x_n$, say $h(\mathbf{x})$, using a vectorial notation, be given by

$$h(\mathbf{x}) = \left(\int_{E_n} d\mathbf{x}\right)^{-1}, \tag{3.1}$$

where E_n is the domain of possible values of $\mathbf{x}$ for all possible vectors $\boldsymbol{\delta}$ (of which there are $n!$)

For given $\boldsymbol{\delta}$, it is almost obvious that one must have

$$\begin{aligned} \sum_1^{n-1} \delta_h + \tfrac{1}{2}\delta_n &\leqq x_n \leqq L - \tfrac{1}{2}\delta_n , \\ \sum_1^{n-2} \delta_h + \tfrac{1}{2}\delta_{n-1} &\leqq x_{n-1} \leqq x_n - \tfrac{1}{2}(\delta_{n-1} + \delta_n), \\ &\vdots \\ \sum_1^{j-1} \delta_h + \tfrac{1}{2}\delta_j &\leqq x_j \leqq x_{j+1} - \tfrac{1}{2}(\delta_j + \delta_{j+1}), \\ &\vdots \\ \tfrac{1}{2}\delta_1 &\leqq x_1 \leqq x_2 - \tfrac{1}{2}(\delta_1 + \delta_2). \end{aligned} \tag{3.2}$$

One readily finds that, for given $\boldsymbol{\delta}$, integration over the region just defined gives a volume $(L - \sum_1^n \delta_h)^n/n!$ and, since there are $n!$ possible vectors $\boldsymbol{\delta}$, one has

$$p(x) = 1/(L - \textstyle\sum_1^n \delta_h)^n \tag{3.3}$$

for each admissible configuration. We note that this is a straightforward generalization of the joint distribution of n independent variates each uniform on $(0, L)$. In addition to being basic to the discussion of this paper, it is of interest to note that one can, for this case, obtain the exact distribution of the nearest neighbor distance defined by

$$d_3 = \min_j \{|z_i - z_j| - \tfrac{1}{2}(\Delta_i + \Delta_j)\}, \tag{3.4}$$

where z_i and z_j are not ordered but simply denote the position of the midpoints of Δ_i and Δ_j. Because this distribution is not directly relevant to the present problem, we defer a somewhat more detailed discussion to an appendix.

From (3.3) and (3.2) we can deduce various marginal distributions.[1] In particular, we can show that the distribution density for x_j, the jth order statistic, given that δ_j is a particular one of the Δ's and that $\delta_1, \cdots \delta_{j-1}, \delta_{j+1} \cdots, \delta_n$ correspond to a particular set of Δ's, is

$$\begin{aligned} &h(x_j) \mid \boldsymbol{\delta} \\ &= \frac{n!}{(j-1)!(n-j)!} \frac{\left(x_j - \sum_1^{j-1} \delta_h - \tfrac{1}{2}\delta_j\right)^{j-1} \left(L - \sum_{j+1}^{n} \delta_h - \tfrac{1}{2}\delta_j - x_j\right)^{n-j}}{(L - T_n)^n}, \end{aligned} \tag{3.5}$$

where $T_n = \sum_1^n \delta_h$ and

$$\sum_1^{j-1} \delta_h + \tfrac{1}{2}\delta_j \leqq x_j \leqq L - \sum_{j+1}^{n} \delta_h - \tfrac{1}{2}\delta_j .$$

[1] We shall use the notation $h(x_{i_1}, x_{i_2}, \cdots, x_{i_r} \mid \boldsymbol{\delta})$, $r \leqq n$, to mean the joint distribution of the i_1th, $\cdots$, i_rth order statistics given $\delta_{i_1}, \cdots, \delta_{i_r}$ and also the sets of δ's to the left and right of the corresponding order statistics. For brevity we shall also verbally describe this distribution as the density of $x_{i_1}, \cdots, x_{i_r}$ given $\boldsymbol{\delta}$.

We can also show that the density of x_j and x_{j+s-1} ($j = 1, 2, \cdots, n - s + 1$; $s = 2, 3, \cdots, n$) given $\boldsymbol{\delta}$ is given by

$$(3.6)\quad p(x_j, x_{j+s-1} \mid \boldsymbol{\delta}) = \theta_{njs}\Big(x_j - \sum_1^{j-1} \delta_h - \tfrac{1}{2}\delta_j\Big)^{j-1}$$

$$\Big[x_{j+s-1} - x_j - \sum_{j+1}^{j+s-2} \delta_h - \tfrac{1}{2}(\delta_j + \delta_{j+s-1})\Big]^{s-2}\Big(L - \sum_{j+s}^{n} \delta_h - \tfrac{1}{2}\delta_{j+s-1} - x_{j+s-1}\Big)^{n-j+s-1},$$

where $\sum_1^{j-1} \delta_h + \frac{1}{2}\delta_j \leqq x_j \leqq x_{j+s-1} - \sum_{j+1}^{j+s-2} \delta_h - \frac{1}{2}(\delta_j + \delta_{j+s-1})$,

$$\sum_1^{j+s-2} \delta_h + \tfrac{1}{2}\delta_{j+s-1} \leqq x_{j+s-1} \leqq L - \sum_{j+s}^{n} \delta_h - \tfrac{1}{2}\delta_{j+s-1},$$

and $\theta_{njs} = n!/[(j - 1)!\,(s - 2)!\,(n - j - s + 1)!]$. From these examples it is clear how one can, by analogy with the order statistic distribution of n independent rectangular variates on $(0, L)$, immediately set down any of the multivariate marginal distributions stemming from (3.3).

4. Outline of Derivation of Distribution of Coverage. Consider first $\Pr\{\lambda F = 0\}$. Denoting by y the position on $(0, L)$ of the midpoint of the line of length λ, we observe that, for given $\boldsymbol{\delta}$, in order to have zero coverage we must have either some one of the following events occurring:

$$(4.1\text{a})\qquad \{x_j + \tfrac{1}{2}\delta_j \leqq y - \tfrac{1}{2}\lambda \quad\text{and}\quad x_{j+1} - \tfrac{1}{2}\delta_{j+1} \geqq y + \tfrac{1}{2}\lambda\}, \qquad j = 1, 2, \cdots, n - 1,$$

or

$$(4.1\text{b})\qquad x_1 - \tfrac{1}{2}\delta_1 \geqq y + \tfrac{1}{2}\lambda,$$

or

$$(4.1\text{c})\qquad x_n + \tfrac{1}{2}\delta_n \leqq y - \tfrac{1}{2}\lambda.$$

We consider this case in some detail, both because of its simplicity and because it will serve to exemplify accurately the type if not the amount of tedious solving of simultaneous inequalities which appears to be essential in the solution of the problem.

From (4.1a) and (3.6), for a specific value of j, and from our assumptions on y, we must have

$$\max\{\tfrac{1}{2}\lambda, x_j + \tfrac{1}{2}\delta_j + \tfrac{1}{2}\lambda\} \leqq y \leqq \min\{x_{j+1} - \tfrac{1}{2}\delta_{j+1} - \tfrac{1}{2}\lambda, L - \tfrac{1}{2}\lambda\},$$

$$(4.2)\qquad \sum_1^{j-1} \delta_h + \tfrac{1}{2}\delta_j \leqq x_j \leqq x_{j+1} - \tfrac{1}{2}(\delta_j + \delta_{j+1}),$$

$$\sum_1^{j} \delta_h + \tfrac{1}{2}\delta_{j+1} \leqq x_{j+1} \leqq L - \sum_{j+2}^{n} \delta_h - \tfrac{1}{2}\delta_{j+1}.$$

The bounds on y in (4.2) will evidently always be given by

$$(4.2\text{a})\qquad x_j + \tfrac{1}{2}\delta_j + \tfrac{1}{2}\lambda \leqq y \leqq x_{j+1} - \tfrac{1}{2}\delta_{j+1} - \tfrac{1}{2}\lambda.$$

However, for (4.2a) to give rise to a non-zero probability, we must have

$$x_j \leqq x_{j+1} - \tfrac{1}{2}(\delta_j + \delta_{j+1}) - \lambda. \tag{4.2b}$$

Thus, from (4.2),

$$\sum_1^{j-1} \delta_h + \tfrac{1}{2}\delta_j \leqq x_j \leqq x_{j+1} - \tfrac{1}{2}(\delta_j + \delta_{j+1}) - \lambda, \tag{4.2c}$$

which will only give rise to a non-zero probability if

$$x_{j+1} \geqq \sum_1^{j} \delta_h + \tfrac{1}{2}\delta_{j+1} + \lambda. \tag{4.2d}$$

Hence, one can reduce (4.2) to

$$\begin{gathered} x_j + \tfrac{1}{2}\delta_j + \tfrac{1}{2}\lambda \leqq y \leqq x_{j+1} - \tfrac{1}{2}\delta_{j+1} - \tfrac{1}{2}\lambda, \\ \sum_1^{j-1} \delta_h + \tfrac{1}{2}\delta_j \leqq x_j \leqq x_{j+1} - \tfrac{1}{2}(\delta_j + \delta_{j+1}) - \lambda, \\ \sum_1^{j} \delta_h + \tfrac{1}{2}\delta_{j+1} + \lambda \leqq x_{j+1} \leqq L - \sum_{j+2}^{n} \delta_h - \tfrac{1}{2}\delta_{j+1}. \end{gathered} \tag{4.3}$$

Now we perform the transformations

$$\begin{aligned} u &= x_j - \sum_1^{j-1} \delta_h - \tfrac{1}{2}\delta_j, \\ w &= x_{j+1} - \sum_1^{j} \delta_h - \tfrac{1}{2}\delta_{j+1}, \end{aligned} \tag{4.4}$$

to reduce (4.3) to

$$\begin{gathered} u + \sum_1^{j} \delta_h + \tfrac{1}{2}\lambda \leqq y \leqq w + \sum_1^{j} \delta_h - \tfrac{1}{2}\lambda, \\ 0 \leqq u \leqq w - \lambda, \\ \lambda \leqq w \leqq (L - T_n). \end{gathered} \tag{4.5}$$

(4.4) applied to (3.6) leads to a joint density for y, u, w,

$$\frac{n!}{(j-1)!(n-j-1)!} \frac{u^{j-1}(L - T_n - w)^{n-j-1}}{(L - T_n)^n L}. \tag{4.6}$$

Integrating on y according to (4.4) one gets (4.6) multiplied by $(w - u - \lambda)$, the important point being that the resultant expression is independent of the segment size distribution except for the total length, T_n, of the n segments. Since each possible $\boldsymbol{\delta}$ has probability $1/n!$, it is evident that averaging over the allocation of the Δ's to the various orders does not change the result so far obtained. Now, however, we sum on j from 1 to $n - 1$ to get

$$\frac{n!}{(n-2)!} \frac{(w - u - \lambda)(L - T_n - w + u)^{n-2}}{(L - T_n)^n L}. \tag{4.7}$$

Letting $z = w - u - \lambda$ be a transform on u, and integrating on w and z one gets the exact result for given T_n,

$$\text{(4.8)} \qquad \Pr\{\lambda F = 0 \mid T_n\} = \frac{(L - T_n - \lambda)^{n+1}}{(L - T_n)^n L}.$$

Writing $L = n\mu/V$, $T_n = n\bar{\Delta}$ we have

$$\text{(4.9)} \qquad \Pr\{\lambda F = 0\} = \int_0^{\Delta_M} [1 - V\lambda/[n(\mu - V\bar{\Delta})]]^n[1 - (V\bar{\Delta}/\mu)]g_n(\bar{\Delta})\, d\bar{\Delta},$$

where $g_n(\bar{\Delta})$ is the density of $\bar{\Delta}$. It then follows by standard arguments that

$$\text{(4.10)} \qquad \lim_{n\to\infty} \Pr\{\lambda F = 0\} = (1 - V) \exp - V\lambda/(1 - V)\mu.$$

Now consider $\Pr\{\lambda F = \lambda\}$. First we note that if $\lambda \geqq \Delta_M$, complete coverage can only be achieved when two or more of the Δ's form a continuous line segment of length $\geqq \lambda$ and cover λ; but from the continuity of the distribution of segment midpoint coordinates, the probability of two or more of the Δ's forming a continuous line segment is zero. Thus we need only consider $\Pr\{\lambda F = \lambda\}$ if $\lambda < \Delta_M$. In such case, one must have for $\lambda F = \lambda$, one of the disjoint events

$$\text{(4.11)} \qquad \{x_j - \tfrac{1}{2}\delta_j \leqq y - \tfrac{1}{2}\lambda;\, x_j + \tfrac{1}{2}\delta_j \geqq y + \tfrac{1}{2}\lambda\}, \qquad j = 1, 2, \cdots, n.$$

Taking into account (3.5) and the requirements on the limits for a non-zero probability, (4.11) becomes, for a particular j,

$$\text{(4.11a)} \qquad \begin{aligned} &x_j - \tfrac{1}{2}\delta_j + \tfrac{1}{2}\lambda \leqq y \leqq x_j + \tfrac{1}{2}\delta_j - \tfrac{1}{2}\lambda, \\ &\sum_1^{j+1} \delta_h + \tfrac{1}{2}\delta_j \leqq x_j \leqq \mathrm{L} - \sum_{j+1}^{n} \delta_h - \tfrac{1}{2}\delta_j, \qquad \lambda < \delta_j \leqq \Delta_M. \end{aligned}$$

Integrating the joint density of y, x_j, δ_j over the indicated limits and using $L = n\mu/V$, one gets for a given j and any permutation of the Δ's

$$\text{(4.11b)} \qquad \Pr\{\lambda F = \lambda \mid j\} = \frac{V}{\mu n} \int_\lambda^{\Delta_M} (x - \lambda)p(x)\, dx,$$

independently of j. Thus the result cited earlier follows immediately.

Now we go on to discuss briefly the work involved in computing other contributions to the cumulative probability function of λF. If we ask for $\Pr\{\lambda F \leqq C\}$ we can distinguish, aside from the cases already considered, three distinct cases:

A. s of the segments lie completely within $(y - \tfrac{1}{2}\lambda, y + \tfrac{1}{2}\lambda)$; $n - s$ of the segments lie completely outside $(y - \tfrac{1}{2}\lambda, y + \tfrac{1}{2}\lambda)$; the segment farthest to the left on (0, L) and still wholly within $(y - \tfrac{1}{2}\lambda, y + \tfrac{1}{2}\lambda)$ may be the jth segment on $(0, L)$ counting from zero; $j = 1, 2, \cdots, n - s + 1$; $s = 1, 2, \cdots, n$. For a given s and j and permutation and position of the Δ's it is evident that $\lambda F = \sum_j^{j+s-1} \delta_h$ and one can verify that, in addition to the restrictions imposed by the distribution of x and y, one must also satisfy, except for $j = 1$ or $s = n$,

$$\begin{aligned} &\max\{x_{j-1} + \tfrac{1}{2}\lambda + \tfrac{1}{2}\delta_{j-1},\, x_{j+s-1} - \tfrac{1}{2}\lambda + \tfrac{1}{2}\delta_{j+s-1}\} \\ &\qquad \leqq y \leqq \min\{x_j + \tfrac{1}{2}\lambda - \tfrac{1}{2}\delta_j,\, x_{j+s} - \tfrac{1}{2}\lambda - \tfrac{1}{2}\delta_{j+s}\}, \qquad 0 \leqq \sum_1^{j+s-1} \delta_h \leqq C. \end{aligned}$$

For the exceptional case where $j = 1$ but $s \neq n$, the inequalities to be satisfied become, for given s,

$$x_s - \tfrac{1}{2}\lambda + \tfrac{1}{2}\delta_s \leqq y \leqq \min\{x_1 + \tfrac{1}{2}\lambda - \tfrac{1}{2}\delta_1,\ x_{s+1} - \tfrac{1}{2}\lambda - \tfrac{1}{2}\delta_{s+1}\},\quad 0 \leqq \sum_1^s \delta_h \leqq C,$$

and for $j = 1$, $s = n$,

$$x_n - \tfrac{1}{2}\lambda + \tfrac{1}{2}\delta_n \leqq y \leqq x_1 + \tfrac{1}{2}\lambda - \tfrac{1}{2}\delta_1, \qquad 0 \leqq \sum_1^n \delta_h \leqq C.$$

By an analysis similar to but somewhat more arduous than the cases already considered, one obtains as the *exact* probability, for this case, that $\lambda F \leqq C$,

$$\frac{1}{L}\sum_{s=1}^{n}\int_0^{\min(C, s\Delta_M)}\int_0^{L-\lambda}\binom{n}{s}\frac{(\lambda - x)^s(L - \lambda - y)^{n-s+1}}{(L - x - y)^n} p_s(x)p_{n-s}(y)\,dy\,dx, \tag{4.12}$$

where $p_s(x)$, $p_{n-s}(y)$ are s- and $(n - s)$-fold convolutions respectively of the segment size distribution. Since $x + y = \sum_1^n \Delta_h$ we transform (4.12) to a sum of integrals over the joint distribution of x and standardized $x + y$. An appropriate asymptotic argument gives, as cited in the summary,

$$(1 - \mathrm{V})\sum_{s=1}^{\infty}\int_0^{\min(C, s\Delta_M)}\frac{[\alpha(\lambda - x)]^s}{s!}\exp - \alpha(\lambda - x)p_s(x)\,dx. \tag{4.13}$$

B. $s - 1$ of the segments lie completely within $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$; one segment lies partially within $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$ and is the segment farthest to the left on $(0, L)$ and still having an interval in common with $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$; $n - s$ segments lie completely outside $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$. The partially covered segment may be the jth segment on $(0, L)$ counting from zero; $j = 1, 2, \cdots, n - s + 1$; $s = 1, 2, \cdots, n$. For a given s and j and permutation and position of the Δ's, one can show that, for admissible positions,

$$\lambda F = x_j - y + \tfrac{1}{2}(\lambda + \delta_j) + \sum_{j+1}^{j+s-1}\delta_h$$

and verify that, in addition to the restriction imposed by the distribution of $\mathbf{x}$ and y one must also satisfy, for $\lambda F \leqq C$,

$$\max\left\{\begin{matrix} x_j - C + \frac{1}{2}(\lambda + \delta_j) + \sum_{j+1}^{j+s-1}\delta_h \\ x_j + \frac{1}{2}(\lambda - \delta_j) \\ x_{j+s-1} - \lambda - \frac{1}{2}(\delta_{j+s-1}) \end{matrix}\right\} \leqq y \leqq \min\left\{\begin{matrix} x_j + \frac{1}{2}(\lambda + \delta_j) \\ x_{j+s} - \frac{1}{2}(\lambda + \delta_{j+s}) \end{matrix}\right\}.$$

It is clear that there are again exceptional values of j and s for which the above inequality must be slightly modified. We omit a detailed consideration. Further, by symmetry, it is clear that there will be an identical contribution to

$\Pr\{\lambda F \leqq C\}$ when the single segment partially covered is the segment farthest to the right on $(0, L)$ and still having an interval in common with $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$. By increasingly tedious analysis of the relevant inequalities one can show that except for terms of $O(n^{-1})$ the contributions to $\Pr\{\lambda F \leqq C\}$ for case B are given by

$$(4.14\text{a})^2 \quad \frac{2n}{L}\sum_{s=1}^{n}\binom{n-1}{s-1}\int_0^C\int_0^{C-x}\int_0^{L-\lambda}\int_0^x \frac{(L-\lambda-x-w)}{(L-x-y-w)^n} \cdot(\lambda-z-y)^{s-1}(L-\lambda-x-w-z)^{n-s}p(w,x,y)\,dz\,dw\,dy\,dx$$

and

$$(4.14\text{b})^3 \quad \frac{2n}{L}\sum_{s=1}^{n}\binom{n-1}{s-1}\int_0^C\int_{C-x}^C\int_0^{L-\lambda}\int_0^{C-y} \frac{(L-\lambda-x-w)}{(L-x-y-w)^n} \cdot(\lambda-z-y)^{s-1}(L-\lambda-x-w-z)^{n-s}p(w,x,y)\,dz\,dw\,dy\,dx,$$

where $p(w, x, y) = p_{n-s}(w)p_{s-1}(y)p(x)$ and, as before, $p_r(x)$ is the r-fold convolution of $p(x)$. Again noting $x + y + w = \sum_1^n \Delta_h$ we can transform (4.14a) and (4.14b) to a series of integrals over the joint distribution of z, x, y, and standardized $x + y + w$. From an asymptotic argument whose details we omit, one obtains

$$(4.15\text{a})^3 \quad \frac{2V}{\alpha\mu}\sum_{s=1}^{\infty}\int_0^C\int_0^{C-x}\int_0^x \frac{[\alpha(\lambda-y-z)]^{s-1}}{(s-1)!} \cdot\exp-\alpha(\lambda-y-z)p_{s-1}(y)p(x)\,dz\,dy\,dx$$

and

$$(4.15\text{b})^3 \quad \frac{2V}{\alpha\mu}\sum_{s=1}^{\infty}\int_0^C\int_{C-x}^C\int_0^{C-y} \frac{[\alpha(\lambda-y-z)]^{s-1}}{(s-1)!} \cdot\exp-\alpha(\lambda-y-z)p_{s-1}(y)p(x)\,dz\,dy\,dx.$$

C. $(s - 2)$ of the segments lie completely within $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$; two segments lie partially within $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$; $n - s$ segments are completely outside $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$. The partially covered segment on the left of $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$ may be the jth segment on $(0, L)$ counting from zero; $j = 1, 2, \cdots, n - s + 1$; $s = 2, 3, \cdots, n$.

For a given s and j and permutation and admissible position of the Δ's, one can show that

$$\lambda F = x_j - x_{j+s-1} + \lambda + \tfrac{1}{2}(\delta_j + \delta_{j+s-1}) + \sum_{j+1}^{j+s-2}\delta_h$$

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[2] For notational simplicity we have written upper limits to x and y as C and $C - x$, respectively; they should, of course, be $\min(C, \Delta_M)$ and $\min[C - x, (s-1)\Delta_M$ respectively.

[3] Remarks similar to footnote reference two apply here.

and verify that, in addition to the restrictions imposed by the distribution of $\mathbf{x}$ and y, one must also satisfy, for $\lambda F \leqq C$,

$$\max\{x_j + \tfrac{1}{2}(\lambda - \delta_j),\, x_{j+s-1} - \tfrac{1}{2}(\lambda + \delta_{j+s-1})\} \leqq y \leqq \min\{x_j + \tfrac{1}{2}(\lambda + \delta_j),\, x_{j+s-1} - \tfrac{1}{2}(\lambda - \delta_{j+s-1})\}$$

$$x_j \leqq C + x_{j+s-1} - \tfrac{1}{2}(\delta_j + \delta_{j+s-1}) - \sum_{j+1}^{j+s-2} \delta_h - \lambda.$$

There are again exceptional values of j and s for which the above inequalities must be modified. We omit a detailed consideration.

By analyses of the type outlined above we can derive expressions for contributions to $\Pr\{\lambda F \leqq C\}$ of essentially the type already obtained for previous cases. Unfortunately, this last case involves a rather large number of distinct sub-cases; for brevity we give a typical result for this case,

$$\frac{2V}{\mu} \sum_2^\infty \int_0^{\Delta_M} \int_0^{x_1} \int_{C-x_2}^{C} \int_{\lambda-C}^{\lambda-y} \frac{[\alpha(\lambda - y) - \alpha z][\alpha z]^{s-2}}{(s-2)!} \cdot [\exp - \alpha z] p_{s-2}(y) p(x_2) p(x_1)\, dz\, dy\, dx_2\, dx_1,$$

if $\Delta_M \leqq C \leqq \lambda/2$. Other contributions for this case differ only in limits of integration and values of C for which they are applicable.

We close this section by remarking that one could in the results above replace the convolution densities with appropriate Fourier integrals; reasonable specifications of $p(x)$ (e.g., that it be representable as a polynomial in x) would then allow all necessary integration to proceed in a straightforward manner. However, it seems unlikely that such a computation would result in a useful representation and therefore no such computation has been attempted.

One might also mention that the Fourier integral representation allows summation of all of the infinite series obtained above. The resultant integrals, however, do not appear tractable even under specific assumptions about the nature of $p(x)$.

5. Mean and Variance of Coverage. Although the exact limiting distribution of λF, has been given in Section 4, it has not been possible to derive useful forms for the variance of λF using this distribution result, except as described later in this section under additional restrictions on the distribution.

One can, however, compute the asymptotic ($n \to \infty$, $n\mu/L \to V$) expected value of λF using the obvious fact that λF is actually a sum of n (non-independent) random variables each with the same expectation. Thus if we denote the portion of a segment of length Δ_i which is covered by $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$ by C_i, then $E\lambda F = \sum_1^n EC_i$. To compute EC_i, it is necessary to consider two cases,

A. The segment is completely covered.

B. The segment is partially covered.

In case A, the particular segment can be the "j"th segment, $j = 1, 2, \cdots, n$, in order on $(0, L)$ and for any permutation of the Δ's and particular j, the coordinate of the midpoint of the segment must satisfy

$$y - \tfrac{1}{2}(\lambda - \delta_j) \leqq x_j \leqq y + \tfrac{1}{2}(\lambda - \delta_j)$$

in addition to the requirements imposed by the distribution of $\mathbf{x}$ and y. Again a reduction of the relevant inequalities leads to a contribution to $E\lambda F$, neglecting terms of $O(n^{-1})$,

$$\frac{V}{\mu}\int_0^{\lambda} x(\lambda - x)p(x)\,dx.$$

For case B, for particular j and permutation of the Δ's, the additional requirement on x_j is [if the particle covers the left boundary of $(y - \tfrac{1}{2}\lambda, y + \tfrac{1}{2}\lambda)$]

$$y - \tfrac{1}{2}(\lambda + \delta_j) \leqq x_j \leqq y - \tfrac{1}{2}(\lambda - \delta_j).$$

A reduction of the relevant inequalities leads to a contribution for this case (which, from symmetry, we double)

$$\frac{1}{2}\frac{V}{\mu}\int_0^{\lambda} x^2 p(x)\,dx.$$

Thus one has, neglecting terms of $O(n^{-1})$,

$$E\lambda F = \frac{\lambda V}{\mu}\int_0^{\min(\lambda,\Delta_M)} xp(x)\,dx.$$

However if $\lambda < \Delta_M$, it follows from (4.11b) that one has an additional contribution to $E\lambda F$ given by

$$\frac{\lambda V}{\mu}\int_{\lambda}^{\Delta_M} (x - \lambda)p(x)\,dx.$$

Hence one can write

$$E\lambda F = \lambda V - \frac{\lambda V}{\mu}\left\{\int_{\min(\lambda,\Delta_M)}^{\Delta_M} xp(x)\,dx - \int_{\lambda}^{\max(\lambda,\Delta_M)} (x - \lambda)p(x)\,dx\right\}.$$

Unfortunately, it does not appear possible to calculate the variance in the manner suggested by the preceding. Once again, in such a calculation, one encounters integrals of the type derived in Section 4. This suggests that the variance of λF depends on $p(x)$ in a quite complex way. This suggestion gains further credence if one considers, as suggested at the close of Section 4, representing the convolution densities by appropriate Fourier transforms. One can then see quite readily that for quite general $p(x)$ both the density and moments of λF involve not only

$$M(\alpha) = \int_0^{\Delta_M} \exp \alpha x p(x)\,dx,$$

but derivatives thereof as well as related functions.

One can, however, obtain a variance for λ large (or more properly the variance of the asymptotic distribution for λ large) from the following considerations:

It is trivially true that $E(\lambda F)^2 \geqq \lambda^2 V^2$. We also have $E(\lambda F)^2 \leqq \lambda E \lambda F = \lambda^2 V$. Hence, for some θ^2, $0 < \theta^2 < 1$, one has $\text{Var}\, \lambda F = \theta^2 \lambda^2 V(1 - V)$. Note that θ^2 may be a function of λ, V, or the segment size distribution, $p(x)$. Now we consider that part of the density of λF arising from (4.13); one has, for $C \leqq s\Delta_M$,

$$(5.1) \qquad (1 - V) \frac{[\alpha(\lambda - C)]^s}{s!} \exp - \alpha(\lambda - C) p_s(C).$$

In (5.1) we make the following transformations:

$$(5.2a) \qquad t = \left[s - \frac{V(\lambda - C)}{\mu(1 - V)} \right] \Big/ \left[\frac{V(\lambda - C)}{\mu(1 - V)} \right]^{\frac{1}{2}}$$

and

$$(5.2b) \qquad z = (C - \lambda V)/\theta\lambda[V(1 - V)]^{\frac{1}{2}}.$$

(5.2a) suggests itself because of the Poisson factor in (5.1); (5.2b) is of course the natural standardization for C. If one goes through the details of an asymptotic argument for fixed z and t we find that for $\lambda \to \infty$ we can write (5.1) as

$$(5.2) \quad \frac{(1 - V)^2}{2\pi} \frac{\mu}{\sigma} \left(\frac{1 - V}{V} \right)^{\frac{1}{2}} \exp \left[\frac{-t^2}{2} - \frac{1}{2} \frac{\phi^2 \mu^2}{V(1 - V)\sigma^2} \left(z - \frac{(V(1 - V))^{\frac{1}{2}}}{\phi} t \right)^2 \right]$$

providing $\theta = [(\mu/V\lambda)]^{\frac{1}{2}}\phi$, where ϕ is a constant independent of λ. Integrating on t one finds that z will be asymptotically $N(0, 1)$ providing

$$\theta^2 = \frac{\mu(1 - V)[1 + (\sigma^2/\mu^2)]}{\lambda}.$$

Thus one has as the variance of λF, for large λ,

$$(5.3) \qquad \text{Var}\, \lambda F = \mu V(1 - V)^2 [1 + (\sigma^2/\mu^2)].$$

It is easy to show that (5.3) also holds for $\lambda \to \infty$ and $V \to 0$ but $V\lambda/\mu$ approaching a constant; this result follows immediately upon appropriate substitutions in the various integrals and going to the limits on λ and V.

Also note that (5.2) has an "extraneous" factor (after integration on t) of $(1 - V)^2$. This is simply the probability, for large λ, that one has non-zero coverage and that all segments which are covered at all are completely covered. A similar computation for (4.15a) and (4.15b) shows that the asymptotic probability of non-zero coverage and partial coverage for one of the two end segments on $(y - \frac{1}{2}\lambda, y + \frac{1}{2}\lambda)$ is $2V(1 - V)$. By subtraction, since $\lim_{\lambda \to 0} \Pr\{\lambda F = 0\} = 0$ and $\lim_{\lambda \to 0} \Pr\{\lambda F = 1\} = 0$, the probability of non-zero coverage and partial coverage of both of the two end segments is V^2.

One can verify that the argument we have given depends on the fact that $V\lambda/\mu \to \infty$ and $\lambda \to \infty$. For $V\lambda/\mu \to \infty$ and $\mu \to 0$, we can show, assuming (5.3) holds and $\sigma^2 \leqq k\mu^2$ where k is a constant independent of μ, that the coverage converges in probability to λV. Also note that as $\lambda \to 0$, the distribution of coverage approaches a two point distribution,

$$\lim_{\lambda \to 0} \Pr\{\lambda F = 0\} = 1 - V; \qquad \lim_{\lambda \to 0} \Pr\{\lambda F = \lambda\} = V. \tag{5.4}$$

APPENDIX

A Nearest Neighbor Problem

The distribution of Section 3 may be used to formulate and solve a nearest neighbor problem which is at least suggestive of a correction to the same problem in three dimensions when particle size and volume % are not vanishingly small.

In the simplest version of the one-dimensional problem one assumes a set of values $x_1, x_2, \cdots, x_n$ randomly and independently selected on $0 \leqq x \leqq L$, and asks for the distribution of

$$d_1 = \min_i | x_i - L/2 |. \tag{A.1}$$

It is then easy to show that the cumulative distribution function of d_1 is given by

$$F(d_1) = 1 - [1 - (2d_1/L)]^n, \qquad 0 \leqq d_1 \leqq L/2. \tag{A.2}$$

The problem is essentially unchanged if we define the nearest neighbor distance as

$$d_2 = \min_j | x_i - x_j | \text{ for some fixed } i. \tag{A.3}$$

It can be shown that the cumulative of d_2 is

$$\begin{aligned} F(d_2) &= 1 - [1 - (2d_2/L)]^n + 2\{[1 - (d_2/L)]^n - (1 - (2d_2/L)^n\}/n \\ &\qquad \text{if } 0 \leqq d_2 \leqq L/2, \\ &= 1 - (2/n)[1 - (d_2/L)]^n, \ (L/2) \leqq d_2 \leqq L, \end{aligned} \tag{A.4}$$

which reduces to (A.2) for n large.

The generalization of the above, based on Section 3, is to consider a set of segments $\Delta_1, \Delta_2, \cdots, \Delta_n (\sum_1^n \Delta_i < L)$ with midpoint coordinates on (0, L) $z_1, z_2, \cdots, z_n$. We then define the nearest neighbor distance analogous to (A.3) as

$$d_3 = \min_j \{| z_i - z_j | - \tfrac{1}{2}(\Delta_i + \Delta_j)\} \text{ for some fixed } i. \tag{A.5}$$

It can be shown that

$$F(d_3) = 1 - \left(1 - \frac{2d_3}{(1-\alpha)L}\right)^n + \frac{2}{n}\left[\left(1 - \frac{d_3}{(1-\alpha)L}\right)^n - \left(1 - \frac{2d_3}{(1-\alpha)L}\right)^n\right] \tag{A.6}$$

$$\text{if } 0 \leqq d_3 \leqq (1-\alpha)L/2$$

$$= 1 - \frac{2}{n}\left(1 - \frac{d_3}{(1-\alpha)L}\right)^n, \quad \frac{(1-\alpha)L}{2} < d_3 \leqq (1-\alpha)L,$$

where $\alpha = \sum_1^n \Delta_i/L$. From (A.6) one finds the expected nearest neighbor distance is given by

$$[(1-\alpha)L/2(n+1)][1+(2/n)], \tag{A.7}$$

which differs from the analogous result based on (A.2) by the factor $(1-\alpha)[1+(2/n)]$ and from the result based on (A.4) by the factor $(1-\alpha)$. This suggests that as a plausible approximation for the three dimensional expected nearest neighbor distance [usually computed in generalization of (A.2)] one should use

$$\left[\frac{3(1-\alpha)\rho}{4\pi}\right]^{\frac{1}{3}} \frac{\Gamma(1/3)}{3} \tag{A.8}$$

where $\rho = V/n$, V is the volume under consideration, α is the volume fraction taken up by particles, and n is the total number of particles.

The result (A.6) is obtained by noting that z_i may be any one of the n order statistics on $(0, L)$ so that if z_i is the rth order statistic the nearest neighbor segment will correspond either to the $(r-1)$th or $(r+1)$th order statistic. This leads to a tedious analysis of inequalities along the lines indicated in Section 4. Fortunately, it turns out to be unnecessary to pursue the analysis in complete detail since one finds that $F(d_3)$ depends only on $(1-\alpha)L$ and n. Since for $\alpha = 0$ one must obtain (A.4) and since the coupling of $(1-\alpha)L$ prevents any change in form of the distribution for variation in α, it follows that $F(d_3)$ is identical with $F(d_2)$ with $(1-\alpha)L$ replacing L.

We also note that for (A.6) one has as the sth moment, ignoring terms of order higher than $O(n^{-2})$,

$$Ed_3^s = \frac{(1-\alpha)^s L^s \Gamma(s+1)\Gamma(n+1)}{2^s\Gamma(n+s+1)}. \tag{A.9}$$

In particular

$$\operatorname{Var} d_3 = \frac{(1-\alpha)^2 L^2 n}{4(n+1)^2(n+2)} \cong \frac{(1-\alpha)^2 L^2}{4n^2}.$$

If one assumes that the Δ's have a distribution and $n\mu/L = \alpha$

$$Ed_3 \cong ((1-\alpha)\mu)/2\alpha; \qquad \operatorname{Var} d_3 \cong ((1-\alpha)^2\mu^2)/4\alpha^2.$$

There are a large number of further statistical problems that one could define based on the distribution of Section 3. One that seems worth mentioning in closing is the distribution of a randomly selected inter-segment distance (including distance from zero to 1st segment and distance from last segment to L). It is easy to show, denoting inter-segment distance by $\bar{s}$ and probability density of $\bar{s}$ by $h(\bar{s})$ that as $n \to \infty$,

$$h(\bar{s}) = \alpha/(\mu(1 - \alpha)) \exp -\alpha\bar{s}/(\mu(1 - \alpha)) \quad 0 \leqq \bar{s} < \infty. \tag{A.10}$$

Thus $E\bar{s} = (1 - \alpha)\mu/\alpha$, $\mathrm{Var}\,\bar{s} = ((1 - \alpha)^2\mu^2)/\alpha^2$.

Note that (A.10) is completely analogous to the solution of a similar problem for n independently distributed rectangular variates on $(0, L)$. The only difference in (A.10) is the nature of the constant.

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