ORDER STATISTICS OF PARTIAL SUMS[1]

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1. Introduction. In recent years there have emerged a number of remarkable identities, which connect the distributions of certain quantities arising in the fluctuation theory of sums of independent random variables with the distributions of much simpler quantities depending only on the individual partial sums. Authors contributing to the theory include Baxter [1], [2], Pollaczek [4], Sparre Andersen [5], [6] and Spitzer [7].

The present paper is a further study along these lines; its chief aim is to link up certain results of Spitzer and Pollaczek, to be described in detail in the next section. The method used is an extension of that presented in [8] and may be described as algebraic, in contrast to Spitzer's combinatorial approach and the function-theoretic treatment by Pollaczek. A similar algebraic approach has been developed by Baxter.

An outline of the paper is as follows. In Section 2 we collect definitions and state the main results. These all stem from a fundamental integral equation, whose derivation is the theme of Section 3. The algebraic tools required to treat the integral equation are developed and applied in Section 4. In Section 5 we give the proofs of the results stated in Section 2 and make some additional remarks. In Section 6 certain formulas for continuous-time additive processes are obtained by a passage to the limit.

2. Definitions and chief results. Let $\{X_n\}$ be a sequence of independent random variables with common distribution function $f(x) = \Pr\{X < x\}$ and characteristic function φ. Let $\{S_n\}$ be the sequence of partial sums, with $S_0 \equiv 0$. For a real number x let $N_n(x)$ denote the number of S_k, $0 \leqq k \leqq n$, that exceed or equal x, and write $N_n = N_n(0 + 0)$, the number of positive partial sums among the first n. The order statistics of the $n + 1$ quantities $S_0, S_1, \cdots, S_n$ are designated by $R_{n,0} \geqq R_{n,1} \geqq \cdots \geqq R_{n,n}$, and it is convenient to write $\bar{R}_n = \max_{0 \leqq k \leqq n} S_k = R_{n,0}$ and $\underline{R}_n = \min S_k = R_{n,n}$. For a real number x write $x^+ = \max(x, 0)$, $x^- = \min(x, 0)$ and $e(x) = 1$ or 0 according as x is positive or not. Note that the order statistics of S_k^+, $0 \leqq k \leqq n$, are precisely the numbers $R_{n,k}^+$. If R and S are random variables we write

$$E(\exp i[\rho R + \sigma S]) = \text{cf}\,(R \,\&\, S),$$

suppressing the real arguments ρ and σ on the right, as may be done without

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ambiguity; similarly cf (R) is understood to be the characteristic function of R, with argument ρ. Finally, for a random variable Y and event A we write

$$E(Y; A) = E(YI_A),$$

where I_A is the indicator of A.

We can now state the results. The first theorem concerns relations between generating functions of certain characteristic functions.

THEOREM 2.1. *The following identities hold, if* $|w|$ *and* $|z|$ *are less than one:*

$$\sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} z^k \text{ cf } (R_{n,k} \,\&\, S_n) = \exp\left\{\sum_{n=1}^{\infty} n^{-1}[w^n \text{ cf } (S_n^+ \,\&\, S_n) + (wz)^n \text{ cf } (S_n^- \,\&\, S_n)]\right\} \tag{2.1a}$$

$$\sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} z^k \text{ cf } (R_{n,k}^+ \,\&\, S_n) = \{(1-z)(1-wz\varphi(\sigma))\}^{-1} \cdot \left\{\exp\left[\sum_{n=1}^{\infty} n^{-1} w^n (1-z^n) \text{ cf } (S_n^+ \,\&\, S_n)\right] - z\right\} \tag{2.1b}$$

$$\sum_{n=0}^{\infty} w^n \text{ cf } (N_n \,\&\, S_n) = \exp \sum_{n=1}^{\infty} n^{-1} w^n \text{ cf } (ne(S_n) \,\&\, S_n). \tag{2.1c}$$

The above remain true when the second argument, S_n, *is deleted from all cf symbols, i.e. when* $\sigma = 0$.

Note that the cf's appearing on the right are in principle determined solely by the distributions of the individual S_n; thus, for example,

$$\text{cf } (ne(S_n) \,\&\, S_n) = [\exp(i\rho n) - 1]E(\exp(i\sigma S_n); S_n > 0) + \varphi(\sigma)^n.$$

Formula (2.1b) provides the link between the results of Spitzer and Pollaczek alluded to in Section 1. Spitzer [7] proved the case of (2.1b) in which $z = 0$, namely, the identity

$$\sum_{n=0}^{\infty} w^n \text{ cf } (\bar{R}_n^+ \,\&\, S_n) = \exp \sum_{n=1}^{\infty} n^{-1} w^n \text{ cf } (S_n^+ \,\&\, S_n),$$

and deduced many interesting consequences. Pollaczek [4] obtained a version of (2.1b) in which S_n is omitted (i.e. $\sigma = 0$), in the case where X_n has a moment generating function. (His formula differs in appearance from the present one, in that he defined $R_{n,k}^+ \equiv 0$ for $k > n$ and carried the inner summation to $k = \infty$, and he obtained the analogous exponent on the right-hand side of the equation as a contour integral. However, the two expressions are readily shown to agree.)

Two identities are stated in the next theorem. The first of these enables the joint distribution of an order statistic and the corresponding partial sum to be calculated from the joint distributions of extreme values and partial sums, while the second can be viewed as relating the conditional distribution of S_n, given that a specified number of partial sums are positive, to conditional distributions given that all (resp. none) are positive. (See also equations (5.6) below.)

THEOREM 2.2. *The relations* $a_{n,k} = a_{n-k,0}a_{k,k}$ *hold when either*

$$a_{n,k} = \text{cf } (R_{n,k} \ \& \ S_n) \tag{2.2a}$$

or

$$a_{n,k} = E(\exp (i\sigma S_n); N_n = k). \tag{2.2b}$$

The case (2.2a) [with $\sigma = 0$] was proved combinatorially by Bohnenblust, Spitzer and Welch, and was brought to the author's attention by Spitzer; see also Theorem 5.1 below. Sparre Andersen [5] proved (2.2b), also with $\sigma = 0$.

3. The integral equation. Let z be a complex number, $|z| < 1$, and let σ be real. Define

$$h_n(x) = h_n(x, z, \sigma) = E(z^{N_n(x)} \exp (i\sigma S_n)).$$

The functions $h_n(x)$ are left-continuous and have total variation on $-\infty < x < \infty$ not exceeding one. Then

$$g_n(x) = h_n(x) - h_n(-\infty) = h_n(x) - z^{n+1}\varphi(\sigma)^n$$

can be considered as the distribution functions of uniformly bounded complex-valued measures g_n on the real line.

A recurrence relation for h_n can be easily obtained by calculating $E(\cdots)$ as $E(E(\cdots \mid X_1))$, and using the facts that if $X_1 = y$ then

$$N_{n+1}(x) = 1 - e(x) + N_n^*(x - y), \qquad S_{n+1} = S_n^* + y,$$

where the starred quantities depend on $X_2, X_3, \cdots, X_{n+1}$ in the same fashion that the original ones do on $X_1, X_2, \cdots, X_n$. Setting

$$t(x) = z^{1-e(x)} = e(x) + (1 - e(x))z$$

we have the immediate relations $h_0(x) = t(x)$,

$$h_{n+1}(x) = t(x) \int_{-\infty}^{\infty} h_n(x - y) \exp (i\sigma y)\, df(y).$$

In terms of the g_n this becomes

$$g_0(x) = t(x) - z = (1 - z)e(x) \tag{3.1}$$

$$g_{n+1}(x) = (1 - z)z^{n+1}\varphi(\sigma)^{n+1} + t(x) \int_{-\infty}^{\infty} g_n(x - y) \cdot \exp (i\sigma y)\, df(y). \tag{3.2}$$

For $|w| < 1$ let $g(x) = \sum_{n=0}^{\infty} w^n g_n(x)$, the generating function of the g_n. It is again left-continuous and of bounded variation. Multiplying (3.1) by w^{n+1}, summing on $n = 0, 1, \cdots$ and using (3.1) we obtain

$$g(x) = ce(x) + wt(x) \int_{-\infty}^{\infty} g(x - y) \exp(i\sigma y)\, df(y) \tag{3.3}$$

where for brevity we have written $c = (1 - z)(1 - wz\varphi(\sigma))^{-1}$. This is the basic integral equation. (The discontinuous factor $t(x)$ gives it a kind of "Wiener-Hopf" flavor; however, we shall not discuss (3.3) in that context. The author has recently learned that Widom[7] has treated an equation similar to (3.3) by Wiener-Hopf techniques in order to obtain new probabilistic limit theorems.)

4. Algebraic considerations. Let $\mathfrak{X}$ be a commutative Banach algebra with identity e. For $f \in \mathfrak{X}$ the elements $\exp f$, $(e - f)^{-1}$ and $\log(e - f)$ are defined by their Maclaurin series, the latter two only when $\|f\| < 1$. Then

$$\exp(f_1 + f_2) = (\exp f_1)(\exp f_2)$$

and $(e - f)^{\pm 1} = \exp\{\pm \log(e - f)\}$. Let P be a bounded linear operator on $\mathfrak{X}$ and let f be a given element of $\mathfrak{X}$. In order to discuss operations of the form $P(fg)$, $P[f(P\{fg\})]$, $\cdots$, acting on $g \in \mathfrak{X}$, it is convenient to introduce the operator F, which sends g into fg. Then alternating multiplications by f and operations by P can be simply expressed as powers of the operator PF. If g is an unknown element of $\mathfrak{X}$ satisfying the equation $g = e + P(fg)$ then, leaving aside questions of uniqueness, existence and convergence for the moment, the solution ought to be given by $g = \sum (PF)^n e$, or, more compactly, by $g = (I - PF)^{-1}e$, where of course I is the identity operator. More generally, we are going to want to solve an equation of the form

$$g = ce + P(f_1 g) + (I - P)(f_2 g), \tag{4.1}$$

where c is a constant and f_1, f_2 are given. The following theorem gives a special set of sufficient conditions under which g can be found explicitly. As shown by Baxter, at least when $f_2 = 0$, the conditions are stronger than necessary, but adequate for our purposes.

THEOREM 4.1. *Let P be a projection on $\mathfrak{X}$, i.e. $P^2 = P$, such that $P\mathfrak{X}$ and $(I - P)\mathfrak{X}$ are subalgebras, then necessarily closed in norm. Suppose that the elements f_1, f_2 and the operator $PF_1 + (I - P)F_2$ have norms less than one. Then* (4.1) *has the unique solution*

$$g = c \exp[-P \log(e - f_1) - (I - P) \log(e - f_2)]. \tag{4.2}$$

If ψ is a continuous homomorphism of $\mathfrak{X}$ to complex numbers, i.e. a multiplicative linear functional on $\mathfrak{X}$, then

$$\psi(g) = c \exp \sum_{n=1}^{\infty} n^{-1} \psi(Pf_1^n + (I - P)f_2^n). \tag{4.3}$$

PROOF: It is plainly enough to consider the case $c = 1$. If Q is any operator such that $\| Q \| < 1$ it is well known that $(I - Q)^{-1}$ exists as a bounded operator. It follows that $g = (I - [PF_1 + (I - P)F_2])^{-1}e$ is the unique solution to (4.1) (with $c = 1$). Let h denote the right member of (4.2). Then $(e - f_1)h = h \exp \log (e - f_1)$ has the form $\exp (I - P)h_1$. Expanding the exponential function and using the fact that $(I - P)\mathfrak{X}$ is a closed subalgebra it follows that

$$(4.4) \qquad (e - f_1)h = e + (I - P)h_2$$

for some $h_2 \varepsilon \mathfrak{X}$. In a similar way we have

$$(4.5) \qquad (e - f_2)h = e + Ph_3$$

for some $h_3 \varepsilon \mathfrak{X}$. Apply P to both sides of (4.4) and $(I - P)$ to (4.5), then add the resulting equations. The result is

$$P[(e - f_1)h] + (I - P)[(e - f_2)h] = e$$

which on rearrangement shows that $g = h$ satisfies (4.1), thereby proving (4.2). Equation (4.3) follows at once, being included in the statement only for ease of reference.

The next theorem gives an explicit formula for Pg in the subcase that we will actually confront.

THEOREM 4.2. *If in addition to the hypotheses of the previous theorem we have* $Pe = e$ *and* $f_2 = zf_1$ *for a scalar* $z \neq 1$ *then*

$$(4.6) \quad Pg = c(1 - z)^{-1}\{\exp -P[\log (e - f_1) - \log (e - f_2)] - ze\};$$

if ψ *is a homomorphism on* $P\mathfrak{X}$ *then*

$$(4.7) \qquad \psi(Pg) = c(1 - z)^{-1}\left\{\exp\left[\sum_{n=1}^{\infty} n^{-1}(1 - z^n)\psi(Pf_1^n)\right] - z\right\}.$$

PROOF: We write (4.1) (with $c = 1$) in the form

$$(4.8) \qquad g = e + zf_1g + (1 - z)P(f_1g).$$

Applying P to (4.8) we obtain

$$(4.9) \qquad Pg = e + P(f_1g);$$

then eliminating $P(f_1g)$ between (4.8) and (4.9) yields (4.6). Relation (4.7) is immediate.

We shall now show that the integral equation (3.3) can be viewed as an instance of the algebraic equation (4.1), the conditions of Theorems 4.1 and 4.2 being satisfied; temporarily, however, we have to restrict the moduli of w and z somewhat more severely than required in the derivation of (3.3).

Let $\mathfrak{X}$ be the algebra of bounded complex-valued measures on the line, with convolution as product operation and norm equal to total variation. $\mathfrak{X}$ has identity element e, representing a unit mass at the origin; of course the distribution function corresponding to e is the function $e(x)$ defined in Section 2. P will be the operation that throws all mass lying to the left of the origin onto the origin; expressed in terms of distribution functions P sends $g(x)$ into $e(x)g(x)$

[ordinary multiplication!]. The measures of the form Pg are those whose support is contained in $0 \leqq x < \infty$; the convolution of two such measures is easily seen to be another of the same kind. Thus $P\mathfrak{X}$ is a sub-algebra. Similarly,

$$(I - P)\mathfrak{X}$$

is a subalgebra, for its elements g are characterized by having $-\infty < x \leqq 0$ as support and $g\{(-\infty, 0]\} = 0$. Clearly $Pe = e$ and $\| P \| = 1$. For a bounded measurable function $k(x)$ we note the formula

$$\int_{-\infty}^{\infty} k(x)\, d(Pg)(x) = \int_{-\infty}^{\infty} k(x^+)\, dg(x). \tag{4.10}$$

Let w and z be complex numbers such that $| z | < 1$ and

$$| w |\{| z | + | 1 - z |\} < 1.$$

Let f_1 be the measure whose distribution function is

$$f_1(x) = w \int_{-\infty}^{x-0} \exp(i\sigma y)\, df(y).$$

It is easy to see that the n-fold convolutions of f_1 and f are related by the equation

$$f_1^{(n)}(x) = w^n \int_{-\infty}^{x-0} \exp(i\sigma y)\, df^{(n)}(y). \tag{4.11}$$

Let $f_2 = zf_1$. Clearly $\| f_1 \| \leqq | w | < 1$, $\| f_1 \| \leqq | wz | < 1$ and $\| f_1 - f_2 \| \leqq | w || 1 - z |$. Then the operator $PF_1 + (I - P)F_2 = F_2 + P(F_1 - F_2)$ has norm not exceeding $| w || z | + | w || 1 - z | < 1$.

Recalling that $t(x) = e(x) + z(1 - e(x))$ we can write (3.3) in the form $g = ce + P(f_1 g) + z(I - P)(f_1 g)$, as claimed. Hence its solution is given by (4.2), and the projection Pg is given by (4.6). The solution is unsatisfactory in a certain sense, because its corresponding distribution function $g(x)$ is only determined after performing a rather large number of convolutions: g has the form $\exp h = e + h + h^2/2! + \cdots$, and each term h^n stands for an n-fold convolution. However, we only need the transforms (4.3) and (4.7), and these will be evaluated explicitly in the next section.

5. Proofs and remarks. We begin by deriving the equation (2.1a). The (random) function $N_n(x)$ is constant except at the points $x = x_k = R_{n,k}$; clearly $N_n(x_k - 0) = N_n(x_k) = k + 1$, while $N_n(x_k + 0) = k$. Then the quantity $z^{N_n(x)}$ jumps by an amount $z^k(1 - z)$ as x moves from left to right across x_k. Hence for the Fourier-Stieltjes transform we have

$$\int_{-\infty}^{\infty} \exp(i\rho x)\, d_x z^{N_n(x)} = (1 - z) \sum_{k=0}^{n} \exp(i\rho x_k) z^k, \tag{5.1}$$

a formula which persists even if several x_k happen to coincide.

Multiplying (5.1) by exp $(i\sigma S_n)$ and interchanging expectation and transform signs, as is clearly legitimate, we obtain

$$\int_{-\infty}^{\infty} \exp(i\rho x)\, dh_n(x) = \int_{-\infty}^{\infty} \exp(i\rho x)\, dg_n(x) = (\text{def.})\ \psi(g_n)$$
$$= (1 - z) \sum_{k=0}^{n} \text{cf}\,(R_{n,k}\ \&\ S_n) z^k;$$

it is of course well known that ψ is a continuous homomorphism on the algebra of bounded measures.

Then by (4.3) we have for the left side of (2.1*a*)

$$\sum_{n=0}^{\infty} w^n \sum_{k=0}^{n} \text{cf}\,(R_{n,k}\ \&\ S_n) z^k = (1 - z)^{-1} \psi(g)$$
$$(5.2) \qquad = c(1 - z)^{-1} \exp \sum_{n=1}^{\infty} n^{-1}\{z^n \psi(f_1^n) + (1 - z^n)\psi(Pf_1^n)\}$$
$$= \exp \sum_{n=1}^{\infty} n^{-1}\{(wz)^n \varphi(\sigma)^n + z^n \psi(f_1^n) + (1 - z^n)\psi(Pf_1^n)\},$$

since $c = (1 - wz\varphi(\sigma))^{-1}(1 - z)$. To evaluate $\psi(Pf_1^n)$ we have, by (4.10) and (4.11),

$$\psi(Pf_1^n) = \int_{-\infty}^{\infty} \exp(i\rho x^+)\, df_1^{(n)}(x) = w^n \int_{-\infty}^{\infty} \exp(i\rho x^+ + i\sigma x)\, df^{(n)}(x)$$
$$= w^n\ \text{cf}\,(S_n^+\ \&\ S_n).$$

Similarly, $\psi(f_1^n) = w^n$ cf $(S_n\ \&\ S_n)$. Then the expression in braces can be written

$$w^n\ \text{cf}\,(S_n^+\ \&\ S_n) + (wz)^n[\text{cf}\,(0\ \&\ S_n) + \text{cf}\,(S_n\ \&\ S_n) - \text{cf}\,(S_n^+\ \&\ S_n)]$$
$$= w^n\ \text{cf}\,(S_n^+\ \&\ S_n) + (wz)^n\ \text{cf}\,(S_n^-\ \&\ S_n),$$

which, inserted on the right side of (5.2), yields the equation (2.1a). Since both sides converge for $|w| < 1$, $|z| < 1$, analytic continuation shows that (2.1a) holds throughout that region, as well as in the smaller region of Section 4.

We remark that an analogous formula can easily be found for the order statistics $R_{n,k}^*$ of the first n partial sums *omitting* S_0. In fact, if X (with distribution function f) is independent of the pair $R_{n-1,k}$ & S_{n-1} then

$$X + R_{n-1,k}\ \&\ X + S_{n-1}$$

have the same joint distribution as $R_{n,k}^*$ & S_n. Hence

$$\text{cf}\,(R_{n,k}^*\ \&\ S_n) = \varphi(\rho + \sigma)\ \text{cf}\,(R_{n-1,k}\ \&\ S_{n-1}),$$

and the generating function can be quickly written down.

The relation (2.1b) follows from (4.7) by the same method applied to Pg.

In order to prove (2.1c) we observe that for measures of the form Pg the mass at the origin defines a continuous homomorphism ψ_0 on $P\mathfrak{X}$, given by $\psi_0(Pg) = (Pg)(0+0) = g(0+0)$. With this definition of ψ it follows easily that

$$\psi_0(Pg_n) = E(z^{N_n} \exp(i\sigma S_n)) - z^{n+1}\varphi(\sigma)^n$$

and therefore that

$$\psi_0(Pg) = \sum_{n=0}^{\infty} w^n E(z^{N_n} \exp(i\sigma S_n)) - z(1 - wz\varphi(\sigma))^{-1}. \tag{5.3}$$

Combining (5.3) with (4.7) we see that

$$\sum_{n=0}^{\infty} w^n E(z^{N_n} \exp i\sigma S_n) = \exp \sum_{n=1}^{\infty} n^{-1}\{(1 - z^n)\psi_0(Pf_1^n) + w^n z^n \varphi(\sigma)^n\}. \tag{5.4}$$

Clearly

$$\psi_0(Pf_1^n) = f_1^{(n)}(0+0) = w^n \int_{-\infty}^{0+0} \exp(i\sigma y)\, df^{(n)}(y)$$
$$= w^n E([1 - e(S_n)] \exp i\sigma S_n).$$

Then the expression in braces can be written as

$$w^n E([(1 - z^n)(1 - e(S_n)) + z^n] \exp i\sigma S_n) = w^n E(z^{ne(S_n)} \exp i\sigma S_n).$$

We substitute this on the right side of (5.4) and obtain

$$\sum_{n=0}^{\infty} w^n E(z^{N_n} \exp i\sigma S_n) = \exp \sum_{n=1}^{\infty} n^{-1} w^n E(z^{ne(S_n)} \exp i\sigma S_n). \tag{5.5}$$

Setting $z = r \exp i\rho$ and letting $r \to 1 - 0$ yields (2.1c).

We turn now to the proof of Theorem 2.2. Both parts follow from the simple observation that if a double power series $\sum_0^\infty w^n \sum_0^n a_{n,k} z^k$ can be written as a product $(\sum_0^\infty b_n w^n)(\sum_0^\infty c_n (wz)^n)$ with $a_{0,0} = b_0 = c_0 = 1$ then $a_{n,0} = b_n$, $a_{n,n} = c_n$ and $a_{n,k} = a_{n-k,0} a_{k,k}$. Applying this remark to (2.1a) we obtain (2.2a) immediately. The relation (2.2b) follows from (5.5) by noting that $E(z^{N_n} Y) = \sum_0^n z^k E(Y; N_n = k)$ and

$$E(z^{ne(S_n)} Y) = E(Y; S_n \leqq 0) + z^n E(Y; S_n > 0),$$

where $Y = \exp i\sigma S_n$. As by-products of the argument we have the relations

$$\sum_0^\infty w^n \operatorname{cf}(\bar{R}_n \,\&\, S_n) = \exp \sum_1^\infty n^{-1} w^n \operatorname{cf}(S_n^+ \,\&\, S_n)$$
$$\sum_0^\infty w^n E(\exp i\sigma S_n;\, N_n = n) = \exp \sum_1^\infty n^{-1} w^n E(\exp i\sigma S_n;\, S_n > 0) \tag{5.6}$$

and analogues with $\bar{R}_n$ replaced by $\underline{R}_n$, S_n^+ by S_n^-, $N_n = n$ by $N_n = 0$ and $S_n > 0$ by $S_n \leqq 0$. The first of these is of course Spitzer's formula, since

$$\bar{R}_n = \bar{R}_n^+.$$

In concluding this section we point out that various combinatorial identities can be derived from (2.1), (2.2) and (5.6), by a method sketched in [8]. To minimize the amount of new notation required we illustrate the idea on just one case.

Let $a_1, a_2, \cdots, a_n$ be arbitrary real numbers, and let π be a permutation of $1, 2, \cdots, n$. Applying π to the a's we obtain $a_{\pi(1)}, a_{\pi(2)}, \cdots, a_{\pi(n)}$. We set $S_k(\pi)$ equal to the sum of the first k of these, and $S_0(\pi)$ equal to zero. These sums arranged in descending order of magnitude are designated by $R_{n,k}(\pi)$.

Fix k, and let D stand for an arbitrary $n - k$-element subset of $1, 2, \cdots, n$; D' is its complement. Let $\pi_D, \pi_{D'}$ be arbitrary permutations of the sets D, D'. Let $\bar{R}_{n-k}(\pi_D)$ and $\underline{R}_k(\pi_{D'})$ be defined as the greatest and least of the obvious partial sums. Then we have the following combinatorial identity, presumably already known to Bohnenblust, Spitzer and Welch.

THEOREM 5.1. *There is a* $1 - 1$ *correspondence* $\pi \leftrightarrow (D, \pi_D, \pi_{D'})$ *carrying the numbers* $R_{n,k}(\pi)$ *into the sums* $\bar{R}_{n-k}(\pi_D) + \underline{R}_k(\pi_{D'})$.

PROOF: Let $p_1, p_2, \cdots, p_n$ be nonnegative numbers with sum one; let the common distribution of independent random variables X_n be specified by $\Pr\{X = a_i\} = p_i$. We form the partial sums S_n, the order statistics $R_{n,k}$, and apply the identity (2.2a) with $\sigma = 0$. In the resulting formula we equate the coefficients of the product $p_1 p_2 \cdots p_n$, and obtain

$$\sum_{\pi} \exp i\rho R_{n,k}(\pi) = \sum_{D,\pi_D,\pi_{D'}} \exp i\rho\{\bar{R}_{n-k}(\pi_D) + \underline{R}_k(\pi_{D'})\},$$

which proves the result.

6. Continuous case. Let x_t be a centered separable process with stationary independent increments, starting at the origin; $E(\exp i\sigma x_t) = \exp t\omega(\sigma)$. For a real number x let $L_t(x)$ be the occupation time of the half-line $[x, \infty)$, i.e. the measure of the set of τ, $0 \leqq \tau \leqq t$, for which $x_\tau \geqq x$, and let

$$L_t = L_t(0 + 0),$$

the length of time during which x_τ is positive.

Let X_n be independent, each having the distribution of x_Δ, $\Delta > 0$. Then the partial sum S_n has the distribution of $x_{n\Delta}$, and it is natural to expect that if $\Delta \to 0$, $n \to \infty$, $n\Delta \to t > 0$ then the behavior of the first n partial sums will closely approximate the behavior of the process on the interval $[0, t]$. This is of course a well-known idea that has been exploited by many authors; the treatment here is close in spirit to that of Baxter and Donsker [3], who studied $\bar{r}_t = \sup_{0 \leqq \tau \leqq t} x_\tau$. We have the following result, which in principle yields the joint distribution of L_t & x_t:

THEOREM 6.1: *The Laplace transform of* cf $(L_t$ & $x_t)$ *is given by the relation*

$$s \int_0^\infty e^{-st} \operatorname{cf}(L_t \,\&\, x_t)\, dt = \exp \int_0^\infty t^{-1} e^{-st} [\operatorname{cf}(te(x_t) \,\&\, x_t) - 1]\, dt. \tag{6.1}$$

PROOF: In the identity (2.1c) let $w = e^{-s\Delta}$, replace ρ by $\rho\Delta$, and multiply both sides by $1 - e^{-s\Delta}$. There results

$$(6.2)\quad \begin{aligned}(1 - e^{-s\Delta}) \sum_0^\infty e^{-sn\Delta} E\,(\exp i[\rho \Delta N_n + \sigma S_n]) \\ = \exp \sum_1^\infty n^{-1} e^{-sn} \{E\,(\exp i[\rho n \Delta e(S_n) + \sigma S_n]) - 1\}.\end{aligned}$$

It is not hard to show that as $\Delta \to 0$, $n\Delta \to t$, ΔN_n approaches L_t in the mean of order one; hence the joint distribution of ΔN_n & S_n tends to that of L_t & x_t. Then the left side of (6.2) approaches that of (6.1). The integral on the right side of (6.1) clearly exists, because the quantity in square brackets can be written

$$[\exp\,(i\rho t) - 1]E(\exp i\sigma x_t\,;\, x_t > 0) + \exp t\omega(\sigma) - 1$$

which is $O(t)$ as $t \to 0$. Hence the right side of (6.2) approaches that of (6.1) and the proof is complete.

We remark that in a special case formulas of the "arcsine law" variety are recovered. In fact, if we put $\sigma = 0$ in (6.1) and write $\Pr\{x_t > 0\} = p_t$ we obtain

$$(6.3)\quad s \int_0^\infty e^{-st} \operatorname{cf}\,(L_t)\, dt = \exp \int_0^\infty t^{-1} e^{-st} (\exp i\rho t - 1) p_t\, dt.$$

For some interesting processes p_t is constant, say $p_t = p$, $0 < p < 1$. Writing $q = 1 - p$ (6.3) then reduces to

$$(6.4)\quad \int_0^\infty e^{-st} \operatorname{cf}\,(L_t)\, dt = (s - i\rho)^{-p} s^{-q}.$$

It follows at once from (6.4) that $L = t^{-1} L_t$ has probability density $f(L)$ given by

$$f(L) = \pi^{-1} (\sin \pi p) L^{-q} (1 - L)^{-p}, \qquad 0 < L < 1;$$

this density appears in Spitzer [7] and Sparre Andersen [6], in connection with the limiting distribution of $n^{-1} N_n$.

One can also obtain a formula for the general case $L_t(x)$, or rather, for a transform on the variable x. The formula is

$$(6.5)\quad \begin{aligned} s \int_0^\infty e^{-st}\, dt \int_{-\infty}^\infty \exp\,(i\rho x)\, d_x E\,(\exp\,[-\lambda L_t(x) + i\sigma x_t]) \\ = \lambda(s + \lambda)^{-1} \exp\left\{ \int_0^\infty t^{-1} e^{-st} [(\operatorname{cf}\,(x_t^+ \,\&\, x_t) - 1) + e^{-\lambda t} (\operatorname{cf}\,(x_t^- \,\&\, x_t) - 1)] dt \right\}. \end{aligned}$$

We are led to this by writing the left side of (2.1a) in the form

$$(1 - z)^{-1} \sum_0^\infty w^n \int_{-\infty}^\infty \exp\,(i\rho x)\, d_x E(z^{N_n(x)} \exp i\sigma S_n),$$

then multiplying both sides by $(1 - w)(1 - z) = [(1 - z)(1 - wz)^{-1}] (1 - w)(1 - wz)$, setting $w = e^{-s\Delta}$, $z = e^{-\lambda\Delta}$, and letting $\Delta \to 0$. As before, the left side converges, to the left side of (6.5), and therefore the right side converges too. Its limit is the expression on the right of (6.5); the integral exists because the bracketed quantity can be written

$$(1 - e^{-\lambda t}) \text{ cf } (x_t^+ \ \& \ x_t) + e^{-\lambda t}[e^{t\omega(\rho+\sigma)} + e^{t\omega(\sigma)} - 1] - 1,$$

which is $O(t)$ as $t \to 0$.

Letting $\lambda \to \infty$ in (6.5) we obtain formally

$$s \int_0^\infty e^{-st} \text{ cf } (\bar{r}_t \ \& \ x_t)\, dt = \exp \int_0^\infty t^{-1}e^{-st} (\text{cf } (x_t^+ \ \& \ x_t) - 1)\, dt,$$

which in effect extends the result of Baxter and Donsker to the joint distribution.

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