OPTIMUM TOLERANCE REGIONS AND POWER WHEN SAMPLING FROM SOME NON-NORMAL UNIVERSES[1]

By Irwin Guttman

McGill University

1. Introduction and summary. We assume familiarity with the concepts defined in [1] and [2], where optimum β-expectation tolerance regions and their power functions were found for k-variate normal distributions. The method used is to reduce this problem to that of solving an equivalent hypothesis testing problem. It is the purpose of this paper to find optimum β-expectation tolerance regions for the single and double exponential distributions, and to exhibit the corresponding power functions.

Let $X = (X_1, \cdots, X_n)$ be a random sample point in n dimensions, where each X_i is an independent observation, distributed by some continuous probability distribution function. It is often desirable to estimate on the basis of such a sample point a region, say $S(X_1, \cdots, X_n)$, which contains a given fraction β of the parent distribution. We usually seek to estimate the center 100 β% of the distribution and/or one of the 100 β% tails of the parent distribution.

2. The single exponential distribution. The probability density function of the single exponential is given by

$$f(x)\,dx = \frac{1}{\sigma}\,e^{-\frac{1}{\sigma}(x-\mu)}\,dx, \qquad x \geqq \mu \tag{2.1}$$

If we wish to construct tolerance regions $S(x_1, \cdots, X_n)$ which have the ability to pick up sets on the right hand tail of (2.1), then a reasonable choice of "the measure of desirability" Q is

$$dQ_{\mu,\sigma} = \frac{1}{\alpha\sigma}\,e^{-\frac{1}{\alpha\sigma}(y-\mu)}\,dy, \qquad y \geqq \mu \tag{2.2}$$

where $\alpha > 1$. This clearly gives more measure to sets on the right hand tail of (2.1). The problem now separates itself into three cases.

Case I. μ known, σ unknown. Without loss of generality, put $\mu = 0$. We consider the analogous hypothesis testing problem. [see p. 171 [1]]. Let $X_1, \cdots, X_n$, Y be independent, each X_i having the distribution (2.1), and let Y have the distribution (2.2), all with $\mu = 0$. If a tolerance region is desired which tends to cover the right hand tail of (2.1), then the hypothesis testing problem has the form

$$\text{Hypothesis: } \alpha = 1; \text{ Alternative: } \alpha = \alpha_1 > 1. \tag{2.3}$$

If $\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$, then it can easily be verified that $(\bar{x}, y)$ is a sufficient statistic

Received October 7, 1958.

[1] Prepared in connection with research sponsored by the Office of Naval Research, while the author was at Princeton University.

for this problem. We now apply the invariance method expressed in terms of this sufficient statistic. Consider the group G of transformations given by

$$(2.4)\qquad G = \left\{ \begin{matrix} \bar{x}^1 = c\bar{x} \\ y^1 = cy \end{matrix} \middle| c \varepsilon (0, \infty) \right\}.$$

The function $W = y/\bar{x}$ is invariant under this group, and is in fact the maximal invariant function. It is shown in Appendix 1 that the density element of W is[2]

$$(2.5)\qquad g(w; \alpha)\, dw = \alpha^n n^{n+1} (n\alpha + w)^{-(n+1)}\, dw.$$

In terms of W, the hypothesis and alternative of (2.3) are simple, and we now apply the Neyman-Pearson fundamental Lemma. Then, the most powerful test function $\phi(w)$ is based on the probability ratio

$$\frac{\alpha_1^n n^{n+1} (n\alpha_1 + w)^{-(n+1)}}{n^{n+1}(n + w)^{-(n+1)}},$$

or, as this ratio is a monotone increasing function of w, $\phi(w)$ is based on W. Hence, the most powerful invariant test function is

$$(2.6)\qquad \phi(W) = \begin{cases} 1 & \text{if } W > a_\beta \\ 0 & \text{if } W < a_\beta \end{cases}$$

where the a_β are chosen to give the test size β, that is

$$(2.7)\qquad \int_{a_\beta}^{\infty} g(w{:}1)\, dw = \beta.$$

Because the test does not depend on α_1, provided it is greater than 1, and because it is based on the maximal invariant function, our most powerful invariant test function is minimax, most stringent and similar of size β. From the definition of W and following [1], we have that the β-expectation tolerance region which is minimax and most stringent is given by

$$(2.8)\qquad S(x_1, \cdots, x_n) = [a_\beta \bar{x}, \infty).$$

Values of a_β for $n = 1(1)20$, 40 and 60 are given in Table I, for $\beta = .99$, .95, .90 and .75. The power of the procedure summarized by (2.8) is discussed in Section 4.

Case II. μ unknown, σ known. Let the known value of σ be σ_0. The sufficient statistic is $(x_{(1)}, y)$, where $x_{(1)} = \min_{i=1}^n x_i$, each X_i has distribution (2.1) with $\sigma = \sigma_0$, and Y has the distribution (2.2) with $\sigma = \sigma_0$. Under the group of transformations

$$(2.9)\qquad G = \left\{ \begin{matrix} x_{(1)}^1 = x_{(1)} + a \\ y^1 = y + a \end{matrix} \middle| a \varepsilon R_1 \right\}$$

[2] Inspection of $g(w; \alpha)$ will show that it is related to Snedecor's F distribution with (2, 2n) degrees of freedom, where $W = \alpha F$.

TABLE I

Tolerance Factors a_β for single exponential distributions (2.1), μ known, σ unknown; sample size n.

n \ β	.75	.90	.95	.99
1	.333333	.111111	.052631	.010101
2	.309401	.108185	.051957	.010076
3	.301927	.107232	.051734	.010067
4	.298280	.106760	.051624	.010063
5	.296119	.106478	.051557	.010061
6	.294690	.106291	.051513	.010059
7	.293675	.106158	.051482	.010058
8	.292917	.106057	.051458	.010057
9	.292329	.105980	.051440	.010056
10	.291860	.105918	.051425	.010055
11	.291476	.105867	.051413	.010055
12	.291158	.105824	.051403	.010055
13	.290889	.105789	.051395	.010054
14	.290658	.105758	.051387	.010054
15	.290458	.105731	.051381	.010054
16	.290284	.105708	.051376	.010054
17	.290131	.105688	.051371	.010053
18	.289993	.105670	.051366	.010053
19	.289871	.105653	.051363	.010053
20	.289761	.105638	.051359	.010053
30	.289066	.105546	.051337	.010052
40	.288719	.105499	.051326	.010052
60	.288373	.105453	.051315	.010051

the statistic $W = (x_{(1)} - y)/\sigma_0$ is clearly a maximal invariant for the problem (2.3), and its distribution is given by

$$h(w;\alpha)\,dw = \begin{cases} \dfrac{n}{n\alpha+1}\, e^{-nw}\,dw & \text{if} \quad w > 0 \\[2ex] \dfrac{n}{n\alpha+1}\, e^{w/\alpha}\,dw & \text{if} \quad w < 0. \end{cases} \tag{2.10}$$

(This is proved in appendix 2)[3]. An analysis similar to that above shows that, for ability to pick up the right hand tail of (2.1), a minimax and most stringent tolerance region of β-expectation is

$$S(x_1, \cdots, x_n) = [x_{(1)} - b_\beta \sigma_0, \infty), \tag{2.11}$$

[3] Inspection of $h(w;\alpha)$ will show that it is a weighted combination of two densities that are simply related to χ^2 with 2 degrees of freedom, where $\chi_2^2 = \alpha n W$ for $W > 0$, and $\alpha\chi_2^2 = -2W$ for $W < 0$.

TABLE II

Tolerance Factors b_β for single exponential distribution μ unknown, σ known, sample size n

n \ β	.75	.90	.95	.99
1	.693147	1.60944	2.30258	3.91202
2	.143841	.601986	.948560	1.75328
3	.000000	.305430	.536479	1.07296
4	−.064538	.173287	.346574	.748932
5	−.105360	.102165	.240794	.562681
6	−.133531	.059446	.174970	.443209
7	−.154151	.031878	.130899	.360818
8	−.169899	.013170	.099813	.300993
9	−.182321	.000000	.077016	.255842
10	−.192372	−.010050	.059784	.220727
11	−.200671	−.018349	.046439	.192751
12	−.207639	−.025318	.035899	.170018
13	−.213574	−.031253	.027437	.151239
14	−.218689	−.036368	.020549	.135508
15	−.223143	−.040822	.014876	.122172
16	−.227057	−.044736	.010157	.110747
17	−.230524	−.048202	.006198	.100870
18	−.233615	−.051293	.002850	.092263
19	−.236389	−.054067	.000000	.084707
20	−.238892	−.056570	−.002503	.078032
30	−.254892	−.072571	−.018503	.039039
40	−.262989	−.080668	−.026601	.022290
60	−.271152	−.088831	−.034764	.008238

where the b_β are chosen to give the region size β, that is the b_β are such that

$$\int_{-\infty}^{b_\beta} h(w; 1)\, dw = \beta. \tag{2.12}$$

Values of b_β for $n = 1(1)20$, 40 and 60 are given in Table II for $\beta = .99, .95, .90$ and $.75$. The power of the procedure as summarized by (2.11) is discussed in Section 4.

Case III. μ and σ unknown. The sufficient statistic is given by $(x_{(1)}, s, y)$, where $x_{(1)} = \min_{i=1}^n x_i$, y is the random variable with density (2.2), and s is given by

$$s = (n-1)^{-1} \sum_{i=1}^n (x_i - x_{(1)}). \tag{2.13}$$

Under the group of transformations

$$G = \left\{ \begin{array}{l} y^1 = cy + a \\ s^1 = cs \\ x^1_{(1)} = cx_{(1)} + a \end{array} \middle| \begin{array}{l} a \, \varepsilon \, R^1 \\ c \, \varepsilon \, (0, \infty) \end{array} \right\} \tag{2.14}$$

a maximal invariant is found to be

$$W = \frac{x_{(1)} - y}{s(n^{-1} + 1)}. \tag{2.15}$$

The density element of W is

$$k(w; \alpha)\, dw = \begin{cases} \dfrac{n+1}{n\alpha + 1} \dfrac{dw}{[1 + (n+1)(n-1)^{-1}w]^n}, & w > 0 \\[2ex] \dfrac{n+1}{n\alpha + 1} \dfrac{dw}{[1 - (n+1)^{-1}(n-1)^{-1}\alpha^{-1}w]^n}, & w < 0. \end{cases} \tag{2.16}$$

(This is proved in Appendix 3)[4]. An analysis similar to that above shows that the minimax most stringent tolerance region of β-expectations, having ability to pick up the right hand tail of (2.1), is

$$S(x_1, \cdots, x_n) = [x_{(1)} - c_\beta s, \infty], \tag{2.17}$$

where $c_\beta = (n^{-1} + 1)c^1_\beta$, and the c^1_β are such that

$$\int_{-\infty}^{c^1_\beta} k(w; 1)\, dw = \beta.$$

The values of c_β are given in Table III for $n = 1(1)20$, 40 and 60 for $\beta = .75$, .90, .95 and .99, while the power function for (2.17) is discussed in Section 4.

3. The double exponential distribution. The density of this function is given by

$$\frac{1}{2\sigma} e^{-\frac{1}{\sigma}|x-\mu|} dx, \qquad -\infty < x < \infty \tag{3.1}$$

We discuss the case of μ known, say μ_0. It is easily shown that if a sample of n independent observations be drawn from (3.1), that the sampling distribution of the statistic

$$T = \sum_{i=1}^{n} |X_i - \mu_0| \tag{3.2}$$

has the density

$$\frac{1}{\sigma^n \Gamma(n)} t^{n-1} e^{-t/\sigma} dt \tag{3.3}$$

[4] Inspection of $k(w; \alpha)$ will show that it is a weighted combination of two densities that are simply related to an F distribution with 2, $2(n-1)$ degrees of freedom, where $(n+1)W = F$ if $W > 0$, and $n\alpha F = -(n+1)W$ if $W < 0$.

TABLE III

Tolerance Factors c_β for single exponential distribution μ and σ unknown, sample size n

n \ β	.75	.90	.95	.99
2	.166667	1.16667	2.83333	16.1666
3	.000000	.387426	.824045	2.66666
4	−.065238	.194941	.440551	1.28581
5	−.106760	.108976	.280960	.816410
6	−.135330	.061617	.194695	.585067
7	−.156148	.032478	.141423	.448641
8	−.171978	.013270	.105729	.359246
9	−.184415	.000000	.080451	.296463
10	−.194424	−.010056	.061814	.250149
11	−.202698	−.018366	.047645	.214709
12	−.209611	−.025349	.036611	.186807
13	−.215486	−.031293	.027848	.164334
14	−.220539	−.036418	.020778	.145895
15	−.224931	−.040881	.014995	.130528
16	−.228784	−.044802	.010213	.117554
17	−.232192	−.048275	.006218	.106474
18	−.235227	−.051371	.002854	.096920
19	−.237948	−.054148	.000000	.088609
20	−.240400	−.056654	−.002503	.081326
30	−.256016	−.072661	−.018509	.039838
40	−.263878	−.080751	−.026609	.022547
60	−.271776	−.088898	−.034774	.008273

Further, T is sufficient for σ. If the tolerance region is constructed so that it has ability to pick up the center part of (3.1), a reasonable choice for the 'measure of desirability' is the measure Q, defined by

$$dQ = \frac{1}{2\alpha\sigma} e^{-\frac{1}{\alpha\sigma}|y-\mu_0|} dy, \tag{3.4}$$

where $-\infty < y < \infty$ and α is such that $0 < \alpha < 1$. The analogous hypothesis testing problem can now be put in the form

(3.5) Hypothesis: $\alpha = 1$ Alternative: $\alpha = \alpha_1$, $0 < \alpha_1 < 1$.

We use the principle of invariance. The maximal invariant under the group of transformations

$$G = \left\{ \begin{matrix} t' = ct \\ (y - \mu_0)' = c(y - \mu_0) \end{matrix} \middle| c \,\varepsilon\, (0, \infty) \right\} \tag{3.6}$$

is the statistic $W = |y - \mu_0|/t$, and its density element is given by

$$p(w; \alpha)\, dw = \frac{n\alpha^n}{(\alpha + w)^{n+1}}\, dw. \tag{3.7}$$

(This is proved in Appendix 4)[5]. In terms of W the problem (3.5) is a simple hypothesis versus a simple hypothesis and clearly (t, y) is sufficient. Applying the Neyman-Pearson Fundamental Lemma, the most powerful invariant test is

$$\phi(W) = \begin{cases} \text{if } W \leqq d_\beta \\ 0 \text{ otherwise} \end{cases} \tag{3.8}$$

The test does not depend on α_1 (so long as $0 < \alpha_1 < 1$), and, because the test is based on the maximal invariant, it is minimax, most stringent, and similar of size β. The d_β are chosen to give the test size β. Again following [1], we have the minimax most stringent tolerance regions of β-expectation with ability to put up the center 100 β% of (3.1) is

$$S(x_1, \cdots, x_n) = [\mu_0 - d_\beta t, \mu_0 + d_\beta t], \tag{3.9}$$

where the d_β are such that

$$\int_0^{d_\beta} p(w; 1)\, dw = \beta. \tag{3.10}$$

Values of d_β for $n = 1(1)20$, 40 and 60 for $\beta = .75, .90, .95$ and .99 are given in Table IV. The power of (3.9.) is discussed in the next section.

4. Formulation of the power functions. Suppose sampling from (2.1), where

A. *Case* 1. μ known, σ unknown. For this case, the solution of the corresponding hypothesis testing problem is given by (2.6). The power of ϕ, P_ϕ, (see p. 170 of [1] and p. 774 of [2]) and hence of S is determined by the distribution of W under the alternative of (2.3). That is, we have

$$P_\phi = P_{\text{Alt.}}(W \geqq a_\beta) = \int_{a_\beta}^{\infty} g(w; \alpha_1)\, dw, \tag{4.1}$$

where $g(w; \alpha)$ is defined by (2.5), a_β is given in Table I, and $\alpha_1 > 1$. The power measures the 'degree of confidence' we have that $S(X_1, \cdots, X_n)$ covers the right hand 100 β% of (2.1) when the desirability of covering this set is given by

$$Q_\sigma(S) = \int_s \frac{1}{\alpha\sigma} e^{-\frac{1}{\alpha\sigma}(x-\mu)}\, dx, \qquad 1 < \alpha.$$

For example, if it is 99.5% desirable to cover the right hand 90% of (2.1), then $\alpha_1 = 21.01938$ and the power is found by (4.1) using this value of α_1. Values of the power for the regions S (as given by (2.8)) are given in Table V when the desirability of the right hand 100 β% sets is .995.

[5] Inspection of $p(w; \alpha)$ will show that it is simply related to the F distribution with $(2, 2n)$ degrees of freedom, where $nW = \alpha F$.

TABLE IV

Tolerance Factors d_β for the double exponential distributions mean and variance unknown; sample size n

n \ β	.75	.90	.95	.99
1	3.00000	9.00000	19.0000	98.9995
2	1.00000	2.16228	3.47214	8.99998
3	.587401	1.15443	1.71442	3.64158
4	.414213	.778279	1.11474	2.16227
5	.319508	.584893	.820564	1.51188
6	.259921	.467799	.647549	1.15443
7	.219014	.389495	.534127	.930696
8	.189207	.333521	.454215	.778278
9	.166529	.291550	.394951	.668070
10	.148698	.258925	.349283	.584892
11	.134312	.232847	.313032	.519910
12	.122462	.211528	.283569	.467799
13	.112531	.193777	.259155	.425102
14	.104090	.178769	.238599	.389495
15	.096825	.165914	.221055	.359356
16	.090507	.154782	.205908	.333521
17	.084964	.145048	.192700	.311134
18	.080060	.136464	.181080	.291549
19	.075691	.128838	.170780	.274275
20	.071773	.122018	.161586	.258925
30	.047294	.079775	.105014	.165914
40	.035265	.059254	.077770	.122018
60	.023374	.039122	.051196	.079775

TABLE V

Power of β-expectation tolerance regions, $[a_\beta \bar{x}, \infty)$, when sampling from the single exponential distribution, sample size n

	Measure of Desirability = .995			
α_1	57.39245356	21.01937897	10.23299086	2.005037823
n \ β	.75	.90	.95	.99
1	.9942255	.9947417	.9948830	.9949873
3	.9947577	.9949156	.9949614	.9949958
5	.9948565	.9949496	.9949769	.9949975
7	.9948982	.9949642	.9949837	.9949982
10	.9949289	.9949751	.9949885	.9949989
15	.9949527	.9949839	.9949928	.9949994
30	.9949772	.9949924	.9949968	.9950000
60	.9949897	.9949968	.9950000	.9950000

TABLE VI

Power of β-expectation tolerance regions, $[x_{(1)} - b_\beta\sigma_0, \infty)$, when sampling from the single exponential distribution, sample size n

	Measure of Desirability = .995			
α_1	57.39245356	21.01937897	10.23299086	2.005037823
n \ β	.75	.90	.95	.99
1	.9914372	.9909171	.9910976	.9933444
3	.9942255	.9937556	.9936906	.9942980
5	.9946996	.9943447	.9942490	.9945578
7	.9948414	.9945995	.9944926	.9946791
10	.9949202	.9947892	.9946772	.9947744
15	.9949637	.9949042	.9948218	.9948512
30	.9949907	.9949755	.9949524	.9949305
60	.9949977	.9949938	.9949880	.9949712

TABLE VII

Power of β-expectation tolerance regions, $[x_{(1)} - c_\beta s, \infty)$ when sampling from the single exponential distribution, sample size n

	Measure of Desirability = .995			
α_1	57.39245356	21.01937897	10.23299086	2.005037823
n \ β	.75	.90	.95	.99
2	.9935224	.9930295	.9930122	.9940120
4	.9945321	.9941230	.9940379	.9944568
6	.9947566	.9944932	.9943908	.9946278
8	.9948420	.9946794	.9945693	.9947184
10	.9948851	.9947891	.9946772	.9947744
15	.9949337	.9949021	.9948218	.9948512
30	.9949724	.9949719	.9949525	.9949305
60	.9949912	.9949912	.9949881	.9949712

Case 2. μ unknown, σ known. An analysis similar to the above shows that the power of (2.11) is given by

$$P_\phi = P_{\text{Alt.}}(W \leqq b_\beta) = \int_{-\infty}^{b_\beta} h(w; \alpha_1)\, dw, \tag{4.2}$$

where $h(w; \alpha)$ is given by (2.10) and b_β is given in Table II. Values of (4.2) for the regions (2.11) are given in Table VI.

TABLE VIII

Power of β-expectation tolerance regions, $[\mu_0 - d_\beta t, \mu_0 + d_\beta t]$ when sampling from the double exponential distribution, sample size n

	Measure of Desirability = .995			
α_1	.261648041	.434587989	.565411999	.869175979
n \ β	.75	.90	.95	.99
1	.9197804	.9539367	.9711014	.9912967
3	.9707346	.9795429	.9847458	.9928455
5	.9815020	.9859235	.9887008	.9935183
7	.9858373	.9886565	.9904921	.9938755
10	.9888911	.9906625	.9929161	.9944304
15	.9911096	.9921757	.9929161	.9944304
30	.9931575	.9936285	.9939683	.9947047
60	.9941067	.9943259	.9944876	.9948496

Case 3. μ and σ unknown. Proceeding as above, one finds that

$$(4.3) \qquad P_\phi = P_{\text{Alt.}}(W \leqq c'_\beta) = \int_{-\infty}^{c'_\beta} k(w; \alpha_1)\, dw,$$

where $k(w; \alpha)$ is given by (2.16) and the values of c'_β can be found from Table III using the relationship $c_\beta = (n^{-1} + 1)c_\beta^{\frac{1}{2}}$. Values of (4.3) for 99.5% desirability of the right hand 100 β% sets are given in Table VII.

B. The Double Exponential Distribution. As before, the power of the regions (3.9) is given by the power of the test (3.8) under the alternative hypothesis of (3.5), that is by

$$(4.4) \qquad P_\phi = P_{\text{Alt.}}(W \leqq d_\beta) = \int_0^{d_\beta} p(w; \alpha_1)\, dw$$

where $p(w; \alpha)$ is given by (3.7) and d_β is tabulated in Table IV. Values of (4.4) are given in Table VIII.

5. Acknowledgement. The author wishes to express his thanks to Professor D. A. S. Fraser of the University of Toronto for valuable discussion and encouragement, and to Mr. A. Cseuz of the University of Alberta for doing the computations.

APPENDIX

A. 1. Derivation of (2.5). To restate, the distribution of Y is given by (2.2) with $\mu = 0$. Define $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$, where the X_i are independent observations from (2.1), with $\mu = 0$. It is well known that the density element of $\bar{X}$ is

$$\frac{1}{\sigma}\frac{n^n}{\Gamma(n)}\, e^{-\frac{n}{\sigma}\bar{x}}\, \bar{x}^{n-1}\, d\bar{x}.$$

Hence the joint density element of $\bar{x}$ and y is

$$\frac{n^n}{\alpha\sigma^{n+1}\Gamma(n)} e^{-\frac{n}{\sigma}\bar{x}-\frac{y}{\alpha\sigma}} \bar{x}^{n-1}\, d\bar{x}\, dy.$$

We make the transformation $w = y/\bar{x}$, $z = y$. (The absolute value of the Jacobian is z/w^2.) The joint density element of W and Z is

$$g(w, z)\, dw\, dz = \frac{n^n}{\alpha\sigma^{n+1}\Gamma(n)} e^{-\frac{nz}{\sigma w}} e^{-\frac{z}{\alpha\sigma}} \frac{z^n}{w^{n+1}}\, dw\, dz.$$

On integrating out z we have $g(w; \alpha)\, dw = \alpha^n n^{n+1}(n\alpha + w)^{-(n+1)}\, dw$. It is easily verified that $g(w; \alpha)$ is a probability density.

A. 2. Derivation of (2.10). Here the distribution of Y is given by (2.2) with $\sigma = \sigma_0$. Define $X_{(1)} = \min_{i=1}^n X_i$, where X_i are n independent observations from (2.1) with $\sigma = \sigma_0$. It is well known that the density element of $X_{(1)}$ is given by

$$\frac{n}{\sigma_0} e^{-\frac{n}{\sigma_0}(x_{(1)}-\mu)}\, dx_{(1)}.$$

Let $s = n/\sigma_0(x_{(1)} - \mu)$ and $z = (y - \mu)/\alpha\sigma_0$. Then the density elements of s and z are respectively $e^{-s}\, ds$ and $e^{-z}\, dz$, and their joint density element is $e^{-s-z}\, ds\, dz$. Make the transformation

$$w = \frac{s}{n} - \alpha z \quad \text{and} \quad t = \frac{s}{n} + \alpha z.$$

Note that $w = (x_{(1)} - y)/\sigma_0$. The absolute value of the Jacobian is $n/2\alpha$. Hence

$$h(w, t)\, dw\, dt = \frac{n}{2\alpha} e^{-w\left(\frac{n}{2}-\frac{1}{2\alpha}\right)} e^{-t\left(\frac{n}{2}+\frac{1}{2\alpha}\right)}.$$

Integrating out t,

$$h(w; \alpha)\, dw = \begin{cases} \dfrac{n}{n\alpha + 1} e^{-nw}\, dw \text{ if } w > 0 \\ \dfrac{n}{n\alpha + 1} e^{\frac{w}{\alpha}}\, dw \text{ if } w < 0, \end{cases}$$

and it is easily verified that $h(w; \alpha)$ is a density.

A. 3. Derivation of (2.16). Using A. 2., it is easily seen that the density element of $z = (x_{(1)} - y)/(1 + n^{-1})$ is

$$\frac{n+1}{\sigma(n\alpha + 1)} e^{-\frac{n+1}{\sigma}z}\, dz \text{ if } z > 0$$

$$\frac{n+1}{\sigma(n\alpha + 1)} e^{\frac{n+1}{n\alpha\sigma}}\, dz \text{ if } z < 0,$$

where σ is now unknown. The density element of

$$s = (n - 1)^{-1}\textstyle\sum_{i=1}^n (x_i - x_{(1)})$$

is given by

$$\left(\frac{n-1}{\sigma}\right)^{n-1} \frac{1}{\Gamma(n-1)} s^{n-2} e^{-\frac{(n-1)s}{\sigma}} ds$$

([3], p. 54). Hence the joint density element of z and s is

$$\frac{n+1}{\sigma(n\alpha+1)} \left(\frac{n-1}{\sigma}\right)^{n-1} \frac{s^{n-2}}{\Gamma(n-1)} e^{-\frac{(n-1)s}{\sigma}-\left(\frac{n+1}{\sigma}\right)z} ds\, dz \text{ if } z > 0$$

and

$$\frac{n+1}{\sigma(n\alpha+1)} \left(\frac{n-1}{\sigma}\right)^{n-1} \frac{s^{n-2}}{\Gamma(n-1)} e^{-\frac{(n-1)s}{\sigma}} e^{\frac{n+1}{n\sigma\alpha}} ds\, dz \text{ if } z < 0.$$

Making the transformation $w = z/s$ and $r = z$ (the absolute value of the Jacobian is r/w^2), the joint distribution of w and r becomes

$$k(w, r)\, dw\, dr = \begin{cases} \dfrac{n+1}{\sigma(n\alpha+1)} \left(\dfrac{n-1}{\sigma}\right)^{n-1} \dfrac{r^{n-1}}{w^n\Gamma(n-1)} e^{-\left(\frac{n-1}{\sigma}\right)\frac{r}{w}} e^{-\left(\frac{n+1}{\sigma}\right)r} dw\, dr & \text{if } w > 0 \\ \dfrac{n+1}{\sigma(n\alpha+1)} \left(\dfrac{n-1}{\sigma}\right)^{n-1} \dfrac{r^{n-1}}{w^n\Gamma(n-1)} e^{-\frac{(n-1)r}{\sigma w}} e^{\frac{(n+1)r}{\alpha n\sigma}} dw\, dr & \text{if } w < 0. \end{cases}$$

Integrating out r

$$k(w;\alpha)\, dw = \begin{cases} \dfrac{n+1}{n\alpha+1} \dfrac{dw}{[1+(n+1)(n-1)^{-1}w]^n} & \text{if } w > 0 \\ \dfrac{n+1}{n\alpha+1} \dfrac{dw}{[1-(n+1)n^{-1}(n-1)^{-1}\alpha^{-1}w]^n} & \text{if } w < 0, \end{cases}$$

and it is readily seen that $k(w;\alpha)$ is a density.

A. 4. Derivation of (3.7). Let Y have the distribution (3.4) and define $T = \sum_{i=1}^{n} |X_i - \mu_0|$, $V = |Y - \mu_0|$, where each X_i is distributed by (3.1), and so T has the density (3.3). It is easily shown that V has the density element

$$\frac{1}{\alpha\sigma} e^{-\frac{v}{\alpha\sigma}} dv, \qquad v \geqq 0.$$

The joint density element of V and T is then

$$\frac{1}{\alpha\sigma^{n+1}} \frac{t^{n-1}}{\Gamma(n)} e^{-t/\sigma} e^{-v/\alpha\sigma} dt\, dv.$$

If we let $w = v/t$ and $z = t$ (the absolute value of the Jacobian is z), the joint density element is

$$p(w, z)\, dw\, dz = \frac{1}{\alpha\sigma^{n+1}} \frac{z^n}{\Gamma(n)} e^{-z/\sigma} e^{-\frac{zw}{\alpha}} dw\, dz.$$

Integrating over z

$$p(w; \alpha)\, dw = \frac{n\alpha^n}{(\alpha + w)^{n+1}}\, dw, \qquad w > 0,$$

and it is easily verified that $p(w; \alpha)$ integrates to 1.

REFERENCES

[1] D. A. S. Fraser and Irwin Guttman, "Tolerance regions," *Ann. Math. Stat.*, Vol. 27 (1956), p. 162.

[2] Irwin Guttman, "On the power of optimum tolerance regions when sampling from normal distributions," *Ann. Math. Stat.*, Vol. 28 (1957), p. 773.

[3] P. V. Sukhatme, "Tests of signifince for samples of the χ^2-population with two degrees of freedom," *Ann. Eugenics*, Vol. 8 (1937), p. 52.