OPTIMUM INVARIANT TESTS[1]

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Summary. The standard (likelihood ratio) test of the general linear hypothesis has been shown to possess numerous different optimum properties. A brief survey of these was included in a recent paper by Kiefer [2]. In the present note it is shown that all of these, and in fact a wide class of optimum properties of which the above are special cases, are consequences of the fact that the test is uniformly most powerful invariant.

1. Order relations among tests. Let X be a random variable with possible distributions $\mathcal{P} = \{P_\theta, \theta \varepsilon \Omega\}$ and consider the hypothesis $H: \theta \varepsilon \omega$ where ω is a subset of Ω. Suppose that the problem of testing H against the alternatives $K: \theta \varepsilon \Omega - \omega$ remains invariant under a group G of transformations of the sample space. Let $\mathfrak{J}$ be a class of tests φ of H, for example the class of all level α tests or of all unbiased level α tests, which is invariant under G in the sense that $\varphi \epsilon \mathfrak{J}$ implies $\varphi g \varepsilon \mathfrak{J}$ for all $g \varepsilon G$. Here φg denotes the critical function defined by

$$\varphi g(x) = \varphi(gx).$$

Suppose that a relation $\precsim$ has been defined among the tests of $\mathfrak{J}$ such that every pair $\varphi, \varphi' \varepsilon \mathfrak{J}$ satisfies either $\varphi \precsim \varphi'$ or $\varphi' \precsim \varphi$. When both of these relations hold, we write $\varphi \approx \varphi'$. Let the (weak) ordering $\precsim$ satisfy the following conditions:

(i) If φ' is uniformly at least as powerful as φ, then $\varphi \precsim \varphi'$.

(ii) If $\varphi_\gamma, \gamma \varepsilon \Gamma$ is any family of tests belonging to $\mathfrak{J}$ and ν any probability measure over the label space Γ, then $\varphi \precsim \varphi_\gamma$ for all $\gamma \varepsilon \Gamma$ implies $\varphi \precsim \int \varphi_\gamma \, d\nu(\gamma)$.

(iii) If $\varphi_0 \precsim \varphi_n$ for $n = 1, 2, \cdots$ and if φ is a critical function such that the power-functions $\beta_{\varphi_n}(\theta) \to \beta_\varphi(\theta)$ for all $\theta \varepsilon \Omega$ as $n \to \infty$, then $\varphi_0 \precsim \varphi$.

(iv) If $\varphi \precsim \varphi'$ then $\varphi g \precsim \varphi' g$ for all $g \varepsilon G$.

A test $\varphi_0 \varepsilon \mathfrak{J}$ will be called *optimum* within $\mathfrak{J}$ according to this ordering if $\varphi \precsim \varphi_0$ for all $\varphi \varepsilon \mathfrak{J}$.

The following are some examples of such orderings, which have been considered in the literature. Throughout, β_φ denotes the power function of φ.

Example 1. Let $a(\theta) \geqq 0$ and $b(\theta)$ be functions which are invariant under the transformations $\bar{G}$ induced by G in the parameter space, and let $\varphi \precsim \varphi'$ if

$$\inf_{\Omega-\omega} [a(\theta)\beta_\varphi(\theta) + b(\theta)] \leqq \inf_{\Omega-\omega} [a(\theta)\beta_{\varphi'}(\theta) + b(\theta)].$$

Then conditions (i) to (iv) are clearly satisfied. A particular case is obtained by putting $b(\theta) = 0$; $a(\theta) = 1$ if $\theta \varepsilon \omega'$ and $a(\theta) = 0$ otherwise, where ω' is

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an invariant subset of $\Omega - \omega$. Then $\varphi \precsim \varphi'$ if

$$\inf_{\omega'} \beta_\varphi(\theta) \leqq \inf_{\omega'} \beta_{\varphi'}(\theta).$$

A test is optimum according to this ordering if it maximizes the minimum power over ω'.

Example 2. Let the tests be ordered according to $-s(\varphi)$ where $s(\varphi)$ is the stringency of φ defined by

$$s(\varphi) = \sup_{\Omega-\omega} [\beta^*(\theta) - \beta_\varphi(\theta)]$$

with β^* denoting the envelope power function. Then $\varphi \precsim \varphi'$ if $s(\varphi) \geqq s(\varphi')$ and the four conditions are again easily verified.

Example 3. Let ω' be an invariant subset of $\Omega - \omega$ and suppose that there exists a probability distribution λ over ω' which is invariant under the group $\bar{G}$ induced by G in the parameters face. Then the relation $\varphi \precsim \varphi'$ if

$$\int_{\omega'} \beta_\varphi(\theta) d\lambda(\theta) \leqq \int_{\omega'} \beta_{\varphi'}(\theta)\, d\lambda(\theta)$$

also satisfies conditions (i) to (iv).

Example 4. Suppose that $\theta = (\theta_1, \cdots, \theta_r)$ and that ω consists of the single point $\theta^0 = (\theta_1^0, \cdots, \theta_r^0)$. We shall assume that the power function $\beta_\varphi(\theta)$ of any test φ possesses continuous second derivatives $\partial^2\beta/\partial\theta_i\theta_j$ for all i and j at θ^0. Let $\mathfrak{J}$ be the class of all level α tests that are strictly unbiased in the neighborhood of θ^0 and let $\Delta(\varphi)$ denote the Gaussian curvature of the power surface at θ^0, which is given by the determinant of the positive definite matrix $(\partial^2\beta/\partial\theta_i\partial\theta_j)\,|_{\theta^0}$. The relation $\varphi \precsim \varphi'$ if $\Delta(\varphi) \leqq \Delta(\varphi')$ clearly satisfies (i) and (iii). It follows from a remark of Isaacson [1] that the relation is invariant provided the transformations $\bar{g}$ of the parameter space possess continuous second partial derivatives at θ^0, which (under this restriction) verifies (iv). Condition (ii), finally, is easily verified. Optimum tests according to the present ordering correspond to the type D tests of Isaacson.

2. Consequences of the Hunt-Stein theorem. Under the assumptions of the preceding section we shall now show that if G satisfies the conditions of the Hunt-Stein theorem (cf. [3], p. 336) and if there exists test ψ_0 which is optimum according to the ordering $\precsim$, then there exists an almost invariant test which is optimum. Here we require of $\mathfrak{J}$ that it be closed under convex combinations and under weak limits.

The proof is completely analogous to and essentially follows from that of the Hunt-Stein theorem, and can be indicated very briefly. If ν_n is the sequence of almost invariant probability measures over G postulated in the theorem, consider the sequence of tests

$$\psi_n = \int \psi_0 g\, d\nu_n(g)$$

Let ψ be the weak limit of a subsequence ψ_{n_i}. Then it is shown in the proof of the Hunt-Stein theorem that ψ is almost invariant, and it remains only to show that ψ is optimum. By conditions (iv) and (ii) it follows for any $\varphi \varepsilon \mathfrak{J}$ that $\varphi \precsim \psi_n$ for all n. Hence by condition (iii) also $\varphi \precsim \psi$ for all $\varphi \varepsilon \mathfrak{J}$ as was to be proved.

Under the above assumptions, whenever there exists a UMP almost invariant test, this will be optimum with respect to any ordering $\precsim$ satisfying conditions (i)–(iv). This explains the great varietv of optimum properties possessed by certain tests and makes it unnecessary to prove each of them separately.

3. Applications. Consider a sequence of n independent trials and let $X_i = 1$ or 0 as the ith trial is or is not successful. Let $P(X_i = 1) = p_i$ and consider the hypothesis $H: p_1 = \cdots = p_n = \frac{1}{2}$ against the alternatives

$$p_i > \tfrac{1}{2} \ (i = 1, \cdots, n).$$

The problem is invariant under any permutation of the variables and the sign test, which rejects when $\sum X_i > C$, is uniformly most powerful almost invariant (cf. [3], p. 219). This test therefore maximizes the minimum power over the alternatives ω': $\min p_i \geqq \frac{1}{2} + \Delta$ or ω: $\max p_i \geqq \frac{1}{2} + \Delta$ for any $\Delta > 0$; it is most stringent and of type D.

As a second application, consider the general univariate linear hypothesis in the canonical form according to which the variables $X_1, \cdots, X_r; Y_1, \cdots, Y_s; Z_1, \cdots, Z_m$ are independently normally distributed with common variance σ^2 and means $E(X_i) = \xi_i$, $E(Y_j) = \eta_j$, $E(Z_k) = 0$. The hypothesis to be tested is $H: \xi_1 = \cdots = \xi_r = 0$. This problem remains invariant under the three groups

G_1: $Y_j' = Y_j + c_j(-\infty < c_j < \infty); X_i' = X_i; Z_k' = Z_k$.

G_2: Orthogonal transformations of $X_1, \cdots, X_r$; $Y_j' = Y_j; Z_k' = Z_k$.

G_3: $X_i' = aX_i; Y_j' = aY_j; Z_k' = aZ_k \ (a \neq 0)$.

The standard test has the following two basic optimum properties:

(a) It is uniformly most powerful among level α tests which are almost invariant with respect to G_1, G_2, G_3.

(b) It is uniformly most powerful among all unbiased (or similar) level α tests which are almost invariant with respect to G_2.

The first of these is well known; the second is easily shown by a standard argument.

Since the groups $G_1 - G_3$ satisfy the conditions of the Hunt-Stein theorem, it follows from (a), for example that the standard test is most stringent and that it maximizes the minimum power against the class of alternatives

$$\omega': \sum \xi_i^2/\sigma^2 \geqq \Delta.$$

To apply (b), consider fixed values of $\eta_1, \cdots, \eta_s$ and σ, so that the power becomes a function only of $\xi_1, \cdots, \xi_r$. It then follows that for any $\eta_1, \cdots, \eta_s$ and σ the standard test maximizes (among all unbiased level α tests), for example the minimum power over the sets $\omega'(\eta_1, \cdots, \eta_s, \sigma): \sum \xi_i^2 \geqq \Delta$ and the average power over the spheres $\sum \xi_i^2 = \Delta$. This was first proved by Wald [4]. It follows further that the test maximizes the Gaussian curvature of the power surface, considered for fixed $\eta_1, \cdots, \eta_s, \sigma$ as a function of the ξ's, and hence is of Isaacson's type E. This has been shown previously by Kiefer [2], who deduced it as a consequence of the test maximizing the average power over the spheres $\sum \xi_i^2 = \Delta$.

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