BOUNDS ON NORMAL APPROXIMATIONS TO STUDENT'S AND THE CHI-SQUARE DISTRIBUTIONS[1]

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1. Summary. Formulas closely related to

$$u(t) = [n \log (1 + t^2/n)]^{\frac{1}{2}}$$
$$w(\chi^2) = [\chi^2 - n - n \log (\chi^2/n)]^{\frac{1}{2}}$$

are considered for converting upper tail values of Student's t or chi-square variates with n degrees of freedom to normal deviates. The chief object of the paper is to construct bounds on the deviation from the exact normal deviates such that the absolute deviation is bounded by $cn^{-\frac{1}{2}}$ uniformly in the entire tail. Two approximations for Student's t are suggested that are remarkably accurate and an improvement over other available approximations. The bounds and approximations for Student's t are given in Section 3 and those for chi-square in Section 4. Some of the methods used in obtaining bounds may be of value in other investigations. These are given in Section 2.

The development of the bounds was stimulated by the work of Teichroew [3]. He obtains expansions for the normal deviates corresponding to tail values of Student's t and chi-square and achieves spectacular accuracy even for small n. The idea and the construction of the expansion is set forth, briefly, in [4], p. 647. The first terms of these expansions are the $u(t)$ and $w(\chi^2)$ used here. The bounds of Theorems 3.1 and 4.2 show that these first approximations are correct to $O(n^{-\frac{1}{2}})$ uniformly for all $t > 0$ or $\chi^2 > n$. This fact can be used to show that the Teichroew expansions are valid asymptotic expansions.

2. Some results useful for obtaining bounds. Let F be an arbitrary, absolutely continuous distribution function with density function f, let Φ, ϕ be respectively the unit normal distribution and density functions, and let $x(t)$ be the root of $\Phi(x) = F(t)$ (i.e. $x(t)$ is the normal deviate corresponding to the argument t of F). The problem considered is that of finding bounds on $x(t)$ for a given F. Any numerical bound on $F(t)$ can be converted numerically to a bound on $x(t)$. Frequently, though, a simple analytic expression for the bound is useful. An inequality $F(t) \leqq \Phi(z(t))$ yields directly the bound $x(t) \leqq z(t)$. Two simple sufficient conditions for such inequalities are given in Theorems 2.1 and 2.2.

Often however, only a weaker inequality $1 - F(t) \leqq c[1 - \Phi(z(t))]$ can be

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obtained in which c is typically slightly greater than one (and may depend on t). A simple bound on $x(t)$ can still be obtained analytically, although it is not as strong as the one that could be obtained numerically. The bounds are obtained by using the normal tail inequalities and become relatively stronger as $z(t)$ increases. For two directions of inequality, results are given in Theorems 2.3 and 2.4.

Assume throughout that the density $f(t)$ is positive and continuous for $a < t < \infty$ and that any approximation $z(t)$ to $x(t)$ is a continuously differentiable, strictly increasing function for $a < t < \infty$. (a is any appropriately chosen constant, which can be $-\infty$ but need not be the lower boundary of the domain of f.)

Denote by g the function

$$g(t) = \frac{\phi(z(t))z'(t)}{f(t)}. \tag{2.1}$$

THEOREM 2.1. *If*
(a) $\lim_{t\to\infty} z(t) = \infty$
(b) $\lim_{t\to a} F(t) = \Phi(\lim_{t\to a} z(t))$
(c) sgn $[g(t) - 1]$ *is a monotonic function of t for $a < t < \infty$, then $x(t) \geqq z(t)$ or $x(t) \leqq z(t)$ for all $a < t < \infty$ according as the function in* (c) *is increasing or decreasing.*

THEOREM 2.2. *If $g(t) \geqq 1/c$ ($\leqq$) for all $a < t < \infty$, then*

$$1 - F(t) \leqq c[1 - \Phi(z(t))] \ (\geqq) \text{ for all } a < t < \infty.$$

If $c = 1$, then $x(t) \geqq z(t)$ ($\leqq$).

PROOF. Let $\delta(t) = F(t) - \Phi(z(t))$

$$\delta(t) = \int_{z(t)}^{\infty} \phi(u)\,du - \int_t^{\infty} f(s)\,ds.$$

In the first integral, make the substitution $u = z(s)$, so that

$$\delta(t) = \int_t^{\infty} f(s)[g(s) - 1]\,ds.$$

By (a) and (b), $\delta(a) = 0 = \delta(\infty)$ and if, by (c), sgn $[g(s) - 1]$ is, say, increasing in s, then $\delta(t) \leqq 0$ and $\Phi(z(t)) \geqq F(t) = \Phi(x(t))$ and $z(t) \geqq x(t)$ for all $a < t < \infty$. Theorem 2.2 follows directly from $\delta(t) \geqq [1 - F(t)](1 - c)/c$ ($\leqq$).

Both theorems clearly hold if Φ is any distribution function with continuous positive density on the entire real line.

THEOREM 2.3. *If, for some value of t such that $z_1(t) > 0$, $F(t)$ satisfies an inequality*

$$1 - F(t) \leqq c_1[1 - \Phi(z_1(t))] \tag{2.2}$$

with $c_1 \geqq 1$, and if, in addition either (a) $x(t) > -z_1(t)$ *or* (b) $[1 - \Phi(z_1(t))] \leqq 1/(1 + c_1)$ *holds, then*

$$x(t) \geqq z_1(t) - \frac{c_1 - 1}{z_1(t)}.$$

THEOREM 2.4. *If, for some value of t, such that* $z_2(t) > 0$, $F(t)$ *satisfies an inequality*

$$1 - F(t) \geqq c_2[1 - \Phi(z_2(t))] \tag{2.3}$$

with $0 < c_2 \leqq 1$, *then*

$$x(t) \leqq z_2(t) + \frac{1 - c_2}{c_2} \frac{1}{z_2(t)}.$$

If $c_1 \leqq 1$ in (2.2) it may be replaced by 1 and the bound $x(t) \geqq z_1(t)$ used. $c_2 \geqq 1$ in (2.3) can be handled similarly. Results taking advantage of these constants can be obtained but are rather poor.

PROOFS. By definition of $x(t)$, $1 - F(t) = 1 - \Phi(x(t))$. Henceforth the argument t in $x(t)$ and $z(t)$ will be dropped. The proofs use the Taylor expansion

$$\Phi(z) = \Phi(x) + (z - x)\phi(\theta z + (1 - \theta)x), \qquad 0 \leqq \theta \leqq 1,$$

and the normal tail inequality

$$1 - \Phi(u) < \frac{\phi(u)}{u}, \qquad u > 0. \tag{2.4}$$

Inequality (2.2) and condition (b) of Theorem 2.3 together imply condition (a) since $1 - \Phi(x) \leqq c_1[1 - \Phi(z_1)] \leqq c_1/(1 + c_1)$ so that $\Phi(x) \geqq 1/(1 + c_1) \geqq 1 - \Phi(z_1) = \Phi(-z_1)$ and hence $x \geqq -z_1$.

Eliminating $\Phi(x)$ between inequality (2.2) and the Taylor expansion and solving for x gives

$$x \geqq z_1 - \frac{(c_1 - 1)[1 - \Phi(z_1)]}{\phi(\theta z_1 + (1 - \theta)x)}. \tag{2.5}$$

Let $c_1 \geqq 1$ and assume first that $x \leqq z_1$ so that, with condition (a), $|x| \leqq z_1$ and hence $\phi(\theta z_1 + (1 - \theta)x) \geqq \phi(z_1)$. Using this and inequality (2.4) in inequality (2.5), $x \geqq z_1 - (c_1 - 1)/z_1$. But this holds trivially when $x \geqq z_1$, so that Theorem 2.3 is proved.

Eliminating $\Phi(z)$ between inequality (2.3) and the Taylor expansion and solving for x gives

$$x \leqq z_2 + \frac{(1 - c_2)}{c_2} \frac{(1 - \Phi(x))}{\phi(\theta z_2 + (1 - \theta)x)}. \tag{2.6}$$

Let $0 < c_2 \leqq 1$, and assume first that $x \geqq z_2$. Then $\phi(\theta z_2 + (1 - \theta)x) \geqq \phi(x)$ and with inequality (2.4),

$$x \leqq z_2 + \frac{(1 - c_2)}{c_2 \mid x \mid} \leqq z_2 + \frac{(1 - c_2)}{c_2 z_2}.$$

These inequalities hold trivially when $x \leqq z_2$ and Theorem 2.4 is proved.

Let $\{F_n(t)\}$ be a sequence of distribution functions and $\{x_n(t)\}$ the corresponding normal deviates. An approximate normal deviate $z_n(t)$ which is a close approximation to $x_n(t)$ in the entire tail of the distribution would often be useful.

The results of this section enable detailed boundings of the errors of such approximations from the corresponding distribution function approximations. The essential qualitative result is that the absolute deviate error will be of order $\rho(n)$ throughout the entire tail if the per cent error (relative to smaller tail) in the distribution function approximation is of order $\rho(n)$ throughout the tail. The result is not quite necessary.

3. Normal approximations to Student's t distribution. Let F_n be the distribution function of Student's t on n degrees of freedom.

$$1 - F_n(t) = a_n(2\pi)^{-\frac{1}{2}} \int_t^\infty \left(1 + \frac{s^2}{n}\right)^{-\frac{n+1}{2}} ds,$$

$$a_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{2}{n}\right)^{\frac{1}{2}}.$$

Denote by $x_n(t)$ the normal deviate corresponding to the deviate t of Student's distribution. Chu [1] has studied the normal approximation $\Phi(t)$ of $F(t)$. He was not concerned with approximations in the extreme tails of the distribution nor with quantile approximations; but methods similar to his can be used.

Bounds on the deviate $x_n(t)$ are given by

THEOREM 3.1. *For all $t > 0$ and with $u(t) = [n \log (1 + t^2/n)]^{\frac{1}{2}}$ and $k = .368$,*

(a) $x_n(t) \leqq u(t)$, $\quad n > 0$;

(b) $x_n(t) \geqq u(t)(1 - (1/2n))^{\frac{1}{2}} \equiv u_2(t)$, $\quad n > .50$;

(c) $x_n(t) \geqq u(t) - k/n^{\frac{1}{2}} \equiv u_3(t)$, $\quad n \geqq .50$.

COROLLARY. *Inequality* (b) *can be written as*

(b') $x_n(t) \geqq u(t)(1 - b_1/n))$, $\quad n \geqq n_0 > .50$,

with $b_1 = n_0[1 - (1 - 1/2n_0)^{\frac{1}{2}}]$. Three numerical values of b_1 which will suffice for almost all uses are: $n_0 = 1$, $b_1 = .293$; $n_0 = 3$, $b_1 = .262$; $n_0 = 10$, $b_1 = .254$.

The bounds show that $u(t)$, as an approximation to $x_n(t)$, has an absolute error not exceeding $.368n^{-\frac{1}{2}}$ and a relative error (relative to $u(t)$) not exceeding b_1/n. Except for very large values of t, the bound (c) is much poorer than the bound (b). The main interest in (c) is the rather remarkable fact that even as t and $x_n(t)$ increase indefinitely the error remains bounded and even of order $n^{-\frac{1}{2}}$. An interesting theoretical application will be noted in Section 6.

The derivations of the bounds and a few calculations suggest the following conjectures on the behavior of $x_n(t)$: that $x_n(t)/u_2(t) \to 1$ as $t \to 0$, and that $u(t) - x_n(t)$ as a function of t, increases monotonically to a maximum value slightly less than $.368n^{-\frac{1}{2}}$ and then decreases monotonically to zero, the maximum occurring for t and n for which $u^2(t)/n$ is substantial.

Calculations indicate that the error, $u(t) - x_n(t)$, is close to its maximum value unless t is very large, so that the maximum of the two bounds (b) and (c) is a good approximation. Two superior approximations that were obtained empirically are *approximations* $u_4(t)$ *and* $u_5(t)$,

$$u_4(t) = u(t)\left(\frac{8n+1}{8n+3}\right),$$

$$u_5(t) = u(t) - \frac{2u(t)}{8n+3}\,(1 - e^{-s^2})^{\frac{1}{2}}$$

with

$$s = \frac{.368(8n+3)}{2\sqrt{n}\,u(t)}.$$

For all $n > 1$ *and all* $t > 0$,

$$u_2(t) < u_4(t) < u(t),$$

$$\max\,(u_2(t), u_3(t)) < u_5(t) < u(t).$$

u_4 was chosen as slightly larger than u_2 to give a good fit for small t^2/n. u_5 was constructed to be larger than u_4 and u_3 and to so join them as to give excellent approximation over a wide range of values. Though the function is somewhat complicated, it is amenable to slide rule calculation. u_4 seems to be within .02 of x for t^2/n less than about 5 and u_5 within .02 of x for a much wider range.

The bounds $u(t)$, $u_2(t)$, $u_3(t)$, the approximations $u_4(t)$, $u_5(t)$, and the approximation $u_6(t)$ obtained from the Paulson approximation [2] to F are illustrated in Table 1 for $n = 1, 3, 10$, and selected values of t. The Paulson approximation gives a normal deviate corresponding to the double tail t probability and hence has to be converted to be comparable.

$$K_p(t) = \frac{\left[\frac{2}{9n}t^{2/3} - \frac{2}{9}\right]}{\left[\frac{2}{9n}t^{4/3} + \frac{2}{9}\right]^{\frac{1}{2}}}, \qquad 1 - \Phi(u_6(t)) = \frac{1}{2}[1 - \Phi(K_p(t))]$$

Polynomial approximations such as the Hotelling-Frankel approximations, are very poor for small n or for very large t.

All bounds and approximations except $u_5(t)$ can easily be inverted analytically to give bounds or approximations for the Student's deviate corresponding to a given normal deviate, i.e., for the quantiles of t.

The proof of the theorem will be preceded by two lemmas.

LEMMA 1. *For all* $x > 0$, $h_c(x) = (e^x - 1)/xe^{cx}$ *is monotone decreasing for* $c = 1$, *monotone increasing for* $c = \frac{1}{2}$ *and not monotonic for* $\frac{1}{2} < c < 1$.

PROOF. $h_c'(x) = (1/x^2e^{cx})[xe^x - (e^x - 1)(cx + 1)]$ and is $\geqq 0$ or $\leqq 0$ according as $xe^x/((e^x - 1)(cx + 1))$ is $\geqq 1$ or $\leqq 1$. The result follows from a termwise comparison of the Maclaurin expansions of the numerator and denominator.

TABLE 1

Bounds on the normal deviate $x_n(t)$ for Student's distribution

$1 - \Phi(x_n(t)) = 1 - F_n(t)$

n	t	Exact $x_n(t)$	Bounds from Theorem 3.1			Approximation		
			Upper $u(t)$	Lower $u_2(t)$	Lower $u_3(t)$	$u_4(t)$	$u_5(t)$	$u_6(t)$
1	0.3	.235	.294	.208	<0	.241	.241	.257
	1	.674	.832	.589	.465	.680	.681	.674
	2	1.047	1.269	.897	.901	1.038	1.048	1.031
	4	1.419	1.683	1.190	1.315	1.377	1.416	1.349
	8	1.756	2.043	1.445	1.675	1.672	1.750	1.576
	12	1.935	2.231	1.577	1.863	1.825	1.927	1.670
	10^2	2.729	3.035	2.146	2.667	2.177	2.704	1.896
	10^5	4.514	4.799	3.393	4.431	3.926	4.447	1.964
3	1	.858	.929	.848	.717	.860	.860	.855
	2	1.478	1.594	1.455	1.382	1.476	1.478	1.477
	4	2.197	2.353	2.148	2.141	2.179	2.197	2.160
	8	2.872	3.053	2.787	2.840	2.826	2.879	2.705
	12	3.228	3.417	3.119	3.204	3.164	3.237	2.935
	$\sqrt{3} \times 10^2$	5.057	5.256	4.797	5.044	4.866	5.058	3.493
10	1	.952	.976	.952	.860	.953	.953	.948
	2	1.790	1.834	1.788	1.718	1.790	1.790	1.805
	4	3.021	3.091	3.013	2.975	3.017	3.020	3.014
	8	4.382	4.474	4.361	4.357	4.366	4.384	4.279
	12	5.128	5.229	5.097	5.113	5.103	5.133	4.902
100	100	21.447	21.483	21.429	21.446	21.429	21.450	18.541

LEMMA 2. *For all* $x > 0$, $((e^x - 1)e^{2kx^{1/2}})/xe^x \geqq 1$ *with* $k = .368$.

PROOF. The desired inequality is equivalent to the inequality

$$Q(x) = e^x - 1 - xe^{x-2kx^{1/2}} \geqq 0.$$

Let T be defined by

$$Q'(x) = e^{x-2kx^{1/2}}[e^{2kx^{1/2}} - (1 - kx^{1/2} + x)] = e^{x-2kx^{1/2}}T(x).$$

The simultaneous equations in x and k: $Q(x) = 0$ and $T(x) = 0$ will have exactly one solution with positive x and the root for k is (to three decimals) the smallest value for which the inequality $Q(x) \geqq 0$ holds for all $x > 0$. The solution is $k = .368$ and $x = 7.312$.

PROOF OF THEOREM. Proceeding as in Theorem 2.1, set $z(t) = \lambda u(t) - \mu$ with $u(t) = [n \log (1 + t^2/n)]^{1/2}$ and with λ, μ constants to be chosen. Then form the function $g(t) = \phi(z(t))z'(t)/f_n(t)$ written as a function of $x = u^2/n$, which is monotonic in t,

$$g^2(t) = h(x) = \frac{\lambda^2 e^{-\mu^2}}{a_n^2} \frac{(e^x - 1)e^{2\lambda\mu(nx)^{1/2}}}{xe^{cx}}$$

where $c = 1 - n(1 - \lambda^2)$.

First, set $\mu = 0$. Then $z(t) = \lambda u(t)$ satisfies conditions (a), (b) of Theorem 2.1. Monotony of $g(t)$ and hence condition (c) of Theorem 2.1 follow from Lemma 1: decreasing for $c = 1$ or $\lambda = 1$, increasing for $c = \frac{1}{2}$ or $\lambda = (1 - (1/2n))^{\frac{1}{2}}$. Conclusions (a) and (b) follow.

Next set $\lambda = 1$ and $\mu = k/n^{\frac{1}{2}}$ with $k = .368$. Then, using Lemma 2, $g(t) \geqq 1$ for all $t > 0$, provided that $a_n e^{k^2/2n} \leqq 1$. Hence (c) follows from Theorem 2.

The proof that $a_n e^{k^2/2n} \leqq 1$ for all $n \geqq .50$ and that $(1 - (1/2n))^{\frac{1}{2}} \geqq 1 - b/n$ from which the Corollary follows are given in Section 5.

4. Normal approximations to the chi-square distribution. Let F_n be the distribution function of chi-square on n degrees of freedom (using t instead of χ^2 as argument).

$$1 - F_n(t) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} \int_t^\infty s^{\frac{n}{2}-1} e^{-s/2}\, ds.$$

Denote by $y_n(t)$ the normal deviate corresponding to the chi-square argument t. Only the upper tail with $t > n$ is treated in this paper. Bounds on $1 - F_n(t)$ and $y_n(t)$ are given by

THEOREM 4.1. *For all $t > n$, all $n > 0$, and with $w(t) = [t - n - n \log (t/n)]^{\frac{1}{2}}$, and $w_2(t) = w(t) + \frac{1}{3}(2/n)^{\frac{1}{2}}$*

(a) $1 - F_n(t) > d_n e^{1/9n}[1 - \Phi(w_2(t))]$

(b) $1 - F_n(t) < d_n[1 - \Phi(w(t))]$

in which

$$d_n = \frac{\left(\frac{n}{2}\right)^{\frac{n-1}{2}} e^{-\frac{n}{2}} (2\pi)^{\frac{1}{2}}}{\Gamma(n/2)}.$$

THEOREM 4.2. *For all $t > n$,*

(a) $y_n(t) \leqq w_2(t) + (1/w_2(t)) \max [0, d_n^{-1}e^{-1/9n} - 1]$, $\quad n > 0$,

(b) $y_n(t) \geqq w(t)$, $\quad n > .37$.

COROLLARY 1. *Inequality* (a) *can be written as*

(a′) $y_n(t) \leqq w_2(t) + b_2/nw_2(t)$, $\quad n \geqq n_0 > 0$,

with $b_2 = n_0(e^{1/18n_0} - 1)$. Numerical values of b_2 which will suffice for almost all uses are: $n_0 = .37$, $b_2 = .060$; $n_0 = 1$, $b_2 = .058$, $n_0 = 10$, $b_2 = .056$.

COROLLARY 2. *For all $t > n$ and all $n > .37$,*

$$w(t) \leqq y_n(t) \leqq w(t) + .60n^{\frac{1}{2}}.$$

The bounds on $y_n(t)$ are illustrated in Table 2 for $n = 8$ and selected values of

TABLE 2

Bounds on the normal deviate $y_n(t)$ for the chi-square distribution

$$1 - \Phi(y_n(t)) = 1 - F_n(t), \qquad n = 8$$

t	$\frac{t-n}{(2n)^{1/2}}$	Exact $y_n(t)$	Bounds from Theorem 4.2		Wilson-Hilferty $w_3(t)$
			Upper (a')	Lower (b)	
12	1	1.031	1.095	.869	1.035
16	2	1.724	1.769	1.566	1.726
20	3	2.314	2.354	2.160	2.310
24	4	2.835	2.874	2.685	2.820
32	6	3.737	3.776	3.593	3.691
40	8	4.512	4.553	4.373	4.427
72	16	6.940	6.989	6.813	6.647

t. Shown are bounds (a) and (b), the exact normal deviate $y_n(t)$ and the Wilson-Hilferty [6] approximate deviate

$$w_3(t) = \frac{(t/n)^{1/3} - 1 + \frac{2}{9n}}{\left(\frac{2}{9n}\right)^{\frac{1}{2}}}.$$

The Wilson-Hilferty approximation is much superior to the bounds as approximations except in the extreme tail and the chief value of the approximation is the uniform bound of order $n^{-\frac{1}{2}}$ on the error in the tail.

The proof of the theorem will be preceded by a lemma.

LEMMA 3. *For all* $x > 0$,

(a) $\lambda(x) < x$

(b) $e^{\frac{1}{3}\lambda(x)}\lambda(x) > x$

with $\lambda(x) = 2^{\frac{1}{2}}[x - \log(1 + x)]^{\frac{1}{2}}$.

Inequality (a) follows immediately from $\log(1 + x) > x - x^2/2$. It cannot be improved by any factor of the form $\exp(k\lambda(x))$.

Inequality (b) is sharp for small x and the coefficient in the exponent cannot be decreased. Let

$$y_1 = e^{2\lambda(x)/3}, \qquad y_2 = x^2/u, \qquad u = \lambda^2(x).$$

Denote derivatives with respect to x by primes. The proof consists in showing that $y_1' > \frac{2}{3}$ and $y_2' < \frac{2}{3}$ for all $x > 0$, from which it follows that $y_2 < 1 + 2x/3 < y_1$ and hence, inequality (b): $y_1^{\frac{1}{2}} > y_2^{\frac{1}{2}}$.

$$2/3 - y_2' = \frac{2}{3u^2}\left[u^2 - 3ux + \frac{3x^3}{1 + x}\right] = \frac{2}{3u^2}\beta(x),$$

$$\beta(x) = 4\left[\log(1 + x) - \frac{x}{4}\right]^2 + \frac{3x^2(x - 3)}{4(1 + x)}.$$

Hence $\beta(x) > 0$ for all $x \geqq 3$.

$$\beta'(x) = \frac{2(3-x)}{1+x}\left[\log(1+x) - \frac{x}{4}\right] + \frac{3x[x^2-3]}{2(1+x)^2}.$$

Let

$$\gamma(x) = \frac{(1+x)}{2(3-x)}\beta'(x) = \log(1+x) - x + \frac{3x^2}{2(1+x)(3-x)}.$$

$$\gamma'(x) = \frac{x^3(5-x)}{(1+x)^2(3-x)^2}.$$

Hence $\gamma'(x) > 0$ for all $0 < x < 5$. $\gamma(0) = 0$ so that $\gamma(x) > 0$ for all $0 < x < 5$. Then $\beta'(x) > 0$ for all $0 < x < 3$. Since $\beta(0) = 0$, $\beta(x) > 0$ for $0 < x < 3$, which, combined with the result for $x \geqq 3$ gives $y_2' < \frac{2}{3}$ and $y_2 < 1 + 2x/3$ for all $x > 0$.

Let

$$\delta(x) = \frac{3\lambda^3(1+x)^2 y_1''}{2y_1}.$$

Then $\delta(x) = \lambda^2 - x^2 + (\frac{2}{3})x^2\lambda = \lambda^2(1 - y_2) + (\frac{2}{3})x^2\lambda$. Using the inequalities $y_2 < 1 + 2x/3$ and $\lambda \leqq x$ gives $\delta(x) > 0$ for all $x > 0$. Since $y_1'(0) = \frac{2}{3}$ and $y_1(0) = 1$, the desired result $y_1 > 1 + 2x/3$ for $x > 0$ follows immediately and inequality (b) is proved.

Proof of Theorem. Set $z(t) = w(t) + c(2/n)^{\frac{1}{2}}$ and form the function $g(t)$ of (2.1), written as a function of $x = (t - n)/n$, then

$$g(t) = d_n^{-1} e^{-c^2/n} \frac{xe^{-c\lambda(x)}}{\lambda(x)}$$

with $\lambda(x) = [2(x - \log(1+x))]^{\frac{1}{2}}$. Using Lemma 3 and Theorem 2.2, with c equal to 0 and $\frac{1}{3}$, Theorem 4.1 follows.

The first part of Theorem 4.2 follows using Theorem 2.4 and the second part from the fact, proved in Section 5, that $d_n < 1$ for $n \geqq .37$. Corollary 1 follows from the fact, proved in Section 5, that $e^{-(1/9n)}d_n^{-1} < 1 + b_2/n$ for $n \geqq n_0 > 0$. Corollary 2 follows from Corollary 1 and the theorem.

5. Bounding of some simple functions. In this section four results, used in Sections 3 and 4, are derived. Specifically, with

$$a_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{2}{n}\right)^{\frac{1}{2}}, \qquad d_n = \frac{\left(\frac{n}{2}\right)^{\frac{n-1}{2}} e^{-\frac{n}{2}} (2\pi)^{\frac{1}{2}}}{\Gamma(n/2)}, \qquad k = .368,$$

$$b_1 = n_0\left[1 - \left(1 - \frac{1}{2n_0}\right)^{\frac{1}{2}}\right], \qquad b_2 = n_0\,[e^{1/18n_0} - 1],$$

$$a_n e^{k^2/2n} \leqq 1, \qquad n \geqq .50; \tag{5.1}$$

$$(5.2)\qquad \left(1-\frac{1}{2n}\right)^{\frac{1}{2}} \geqq 1-\frac{b_1}{n}, \qquad n \geqq n_0 > .50;$$

$$(5.3)\qquad d_n \leqq 1, \qquad n \geqq .37;$$

$$(5.4)\qquad e^{-\frac{1}{9n}} d_n^{-1} \leqq 1+\frac{b_2}{n}, \qquad n \geqq n_0 > 0.$$

An easily proved result that is used repeatedly (with $x = 1/n$) is the following:

LEMMA 4. *If $f(x)$ has a uniformly convergent Maclaurin series for $0 \leqq x \leqq x_0$ and if all derivatives of $f(x)$ at $x = 0$ of order greater than m are of constant sign, say positive, then for all $0 \leqq x \leqq x_0$,*

$$T_m(x) \leqq f(x) \leqq T_{m-1}(x) + x^m \left[\frac{f(x_0) - T_{m-1}(x_0)}{x_0^m}\right]$$

where $T_m(x)$ is the partial sum through order m of the Maclaurin series. (If sign is negative, the direction of the inequalities is reversed.)

(5.2) is a direct application of Lemma 4.

The Stirling expansion with argument $n/2$ is just the expansion of $-\log d_n$ and the first two partial sums bracket the value ([5], p. 253).

$$(5.5)\qquad \frac{1}{6n} - \frac{1}{45n^3} \leqq -\log d_n \leqq \frac{1}{6n}, \qquad n > 0.$$

By the duplication formula for the gamma function, $a_n = d_n^2/d_{2n}$ so that,

$$(5.6)\qquad -\frac{1}{4n} - \frac{1}{360n^3} \leqq \log a_n \leqq -\frac{1}{4n} + \frac{2}{45n^3}, \qquad n > 0.$$

Using (5.5), it follows that $\log d_n \leqq 0$ and hence (5.3) for $n^2 \geqq 2/15$ or $n \geqq .37$. Also, for all $n > 0$,

$$d_n^{-1} e^{-(1/9n)} - 1 \leqq e^{1/18n} - 1$$

and (5.4) follows by application of Lemma 4.

From (5.6) it follows that

$$a_n\, e^{k^2/2n} \leqq \exp\left[\frac{1}{n}\left(-\frac{1}{4} + \frac{k^2}{2} + \frac{2}{45n^2}\right)\right].$$

The exponent is negative if $n \geqq .494$ proving (5.1).

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