nificant at the 5% level. For the FEM test recommended in [3] the critical value is less than 2.87 so that *w is* significant. In general the critical values for the FEM test will be smaller but there is an "ultraconservative" [3] FEM test which is the same as the REM test.

Whether a FEM or a REM test is appropriate is a tricky question involving the *context* of the study. Thus since the *same* set of treatments was used in the comparison of judges a FEM test would seem to be implied. However, as noted in [3], there is a possible judge × treatment interaction to consider. In other words the judges might make *self-consistent* subjective ratings but there might be little correlation between the two sets of ratings. If this happened the ratings would behave more or less *as if* two *different* sets of treatments were used (implying a REM test). Actually the judge × treatment interaction does not appear to be serious so the FEM test seems more appropriate. However, this is an issue which must be carefully considered in each specific application.

REFERENCES

[1] D. E. W. Schumann and R. A. Bradley, "The comparison of the sensitivities of similar experiments: Theory," *Ann. Math. Stat.*, Vol. 28 (1957), p. 902.
[2] R. L. Anderson and T. Bancroft, *Statistical Theory in Research*, McGraw-Hill Book Co., New York, 1952, p. 313.
[3] D. E. W. Schumann and R. A. Bradley, "The comparison of the sensitivities of similar experiments: Applications," *Biometrics*, Vol. 13 (1957), p. 496.

ON A THEOREM OF LÉVY-RAIKOV

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Theorem of Lévy-Raikov. *If ϕ_1, ϕ_2 are characteristic functions and $\phi = \phi_1\phi_2$ is analytic, then so are ϕ_1 and ϕ_2 and the strips in the complex plane where ϕ_1 and ϕ_2 may be extended analytically are at least as large as the strip where ϕ may be extended analytically.*

This theorem was originally proved by P. Lévy [2, 3] for the case where ϕ may be extended analytically over the entire plane, and by Raikov [5]. Another very simple proof may be found in [1].

The purpose of this note is to give a sharpened version of this result.

Theorem. *If ϕ_1, ϕ_2 are characteristic functions and $\phi = \phi_1\phi_2$ is differentiable $2n$ times, then so are ϕ_1 and ϕ_2. For any real a let $\psi_a(x) = e^{iax}\phi(x)$; then there exist numbers a_j, m_j such that*

$$|\phi_j^{(2k)}(0)| \leqq m_j\,|\psi_{a_j}^{(2k)}(0)|, \qquad j = 1, 2, 0 \leqq k \leqq n. \tag{1}$$

If ϕ is infinitely differentiable and the Hamburger moment sequence

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$\{(-i)^k\phi^{(k)}(0)\}_0^\infty$ *is determined, then so are the Hamburger moment sequences* $\{(-i)^k\phi_j^{(k)}(0)\}_0^\infty, j = 1, 2.$

PROOF. Let

$$\phi_j(x) = \int e^{ixt}\, dF_j(t), \qquad j = 1, 2;$$

then

$$\phi(x) = \iint e^{ix(t+\tau)}\, dF_1(t)\, dF_2(\tau).$$

We shall suppose that the intervals $(-\infty, 0]$ and $[0, \infty)$ both have non-zero measure with respect to dF_2. If this is not the case, then by a suitable choice of a real number a, the measure corresponding to the characteristic function $e^{iax}\phi_2(x)$ has this property.[2] We would then work with the functions $\psi_a(x)$, $\phi_1(x)$ and $e^{iax}\phi_2(x)$.

Let us set (see [4])

$$\Delta_h^1\phi(x) = \phi(x+h) - \phi(x-h),\ \Delta_h^k\phi(x) = \Delta_h^1\,\Delta_h^{k-1}\phi(x).$$

We get

$$\frac{1}{(2h)^{2n}}\,\Delta_h^{2n}\phi(0) = (-1)^n \iint \left[\frac{\sin h(t+\tau)}{h}\right]^{2n} dF_1(t)\, dF_2(\tau).$$

If $\phi^{(2n)}(0)$ exists, then the left side approaches this value as $h \to 0$ and hence the integral on the right is uniformly bounded as $h \to 0$. This gives

$$(-1)^n\phi^{(2n)}(0) = \iint (t+\tau)^{2n}\, dF_1(t)\, dF_2(\tau),$$

and from this it follows immediately that

$$(2) \qquad (-i)^k\phi^{(k)}(0) = \iint (t+\tau)^k\, dF_1(t)\, dF_2(\tau), \qquad 0 \leqq k \leqq 2n.$$

If we integrate over a closed rectangle $0 \leqq t, \tau \leqq a$, we get

$$\int_0^a\int_0^a (t+\tau)^{2k} dF_1(t)\, dF_2(\tau) \leqq |\phi^{(2k)}(0)|.$$

Expanding by the binomial theorem gives

$$\lim_{a\to\infty}\int_0^a . dF_2(t)\int_0^a t^{2k}\, dF_1(t) \leqq |\phi^{(2k)}(0)|,$$

since each term in the binomial expansion is non-negative. By hypothesis, the dF_2 measure of $[0, \infty)$ is not zero and hence $\int_0^\infty t^{2k} dF_1(t)$ is bounded by a constant times $|\phi^{(2k)}(0)|$. If we repeat this process for the closed rectangle $-a \leqq t, \tau \leqq 0$, and then repeat the whole process, interchanging the role of ϕ_1 and ϕ_2, we get the first two statements of the theorem.

[2] We could, in fact, choose a so that the corresponding measures of $(-\infty, 0]$ and $[0, \infty)$ are both $\geqq 1/2$. As our proof will show, we could then take $m_j \leqq 2$.

To prove the last statement of the theorem we shall suppose that ϕ is infinitely differentiable, the Hamburger moment sequence $\mu_n = (-i)^n\phi^{(n)}(0)$ is determined, but the moment sequent $\nu_n = (-i)^n\phi_1^{(n)}(0)$ is not determined. By expanding the integrand in (2) by the binomial formula we see that the left hand side remains fixed if dF_1 is replaced by any solution of the ν_n-moment problem.

By (2) it is clear that the unique solution dF of the μ_n-moment problem is given by

$$dF = dG_1 * G_2,$$

where $dG_1 * G_2$ is the convolution of the measures dG_1 and dG_2 for any solutions of the corresponding moment problems. The characteristic function of dF is the product of the characteristic functions of dG_1 and dG_2 respectively. If dG_2 is fixed and dG_1 changes, then the characteristic function of dF must change, and hence dF must change, which contradicts the initial hypothesis. Hence the assumption that the ν_n-moment problem is not determined is untenable.

COROLLARY. *The previous theorem includes the Lévy-Raikov theorem. Moreover, if* $\mu_n = (-i)^n\phi^{(n)}(0)$ *and*

$$\sum_{n=0}^{\infty} 1/(\mu_{2n})^{1/2n} = \infty,$$

the same is true for the moments corresponding to ϕ_1 *and* ϕ_2.

PROOF. The first statement follows by the formula (1) and the fact that if ϕ is analytic so is ψ_a for any a.

To prove the second statement we note that if we set $\mu_n(a) = (-i)^n\psi_a^{(n)}(0)$, then for n even

$$\mu_n(a) = \sum_{k=0}^{n} \binom{n}{k} \mu_{n-k}\, a^k$$

$$\leqq \sum_{k=0}^{n} \binom{n}{k} \mu_n^{(n-k)/n}|a|^k = [\mu_n^{1/n} + |a|]^n$$

Hence,

$$\sum_{n=0}^{\infty} 1/(\mu_{2n}(a))^{1/2n} = \infty$$

and our assertions follow from the inequalities (1). This proves the corollary.

Finally, we remark that the inequalities in (1) are the best possible in the sense that there exist ϕ_j, $j = 1, 2$ such that $(-i)^n\phi_j^{(n)}(0) = (-i)^n\psi_{a_j}^{(n)}(0)$ for even n and some real a_j. For example, let $\phi_1(x) = e^{iax}$, $\phi_2(x) = e^{-iax}$

REFERENCES

[1] D. DUGUÉ, "Analyticité et convexité des fonctions caracteristiques, *Ann. Inst. Henri Poincaré*, Vol. 12 (1951), pp. 45–56.

[2] PAUL LÉVY, "L'arithmétique des lois de probabilité", *C. R. Acad. Sci. Paris*, Vol. 204 (1937), pp. 80–82.

[3] PAUL LÉVY, "L'arithmétique des lois de probabilité", *J. Math. Pures Appl.*, Vol. 103 (1938), pp. 17–40.
[4] EUGENE LUKACS AND OTTO SZÁSZ, "On analytic characteristic functions, *Pacific J. Math.*, Vol. II, No. 4 (1952), pp. 615–625.
[5] D. A. RAIKOV, "On the decomposition of Gauss and Poisson laws", *Izvestiya Akad. Nauk SSSR Ser. Mat.* (1938), pp. 91–124 (Russian).

NOTE ON THE FACTORIAL MOMENTS OF THE DISTRIBUTION OF LOCALLY MAXIMAL ELEMENTS IN A RANDOM SAMPLE

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0. Summary. The results reported by T. Austin, R. Fagen, T. Lehrer, and W. Penney [1] are extended to include a general recurrence relation for the factorial moments of the distribution. This recurrence relation is solved for the mean and second factorial moments, and it is shown that the method applied may also be used to obtain a general solution for any desired factorial moment of higher order.

1. Introduction. Austin, Fagen, Lehrer, and Penney [1] have discussed the distribution of locally maximal elements in a random sample. Among other results, the authors defined certain elements in an ordered random sample of n distinct real numbers to be locally k-maximal, provided such an element is the greatest of some set of k consecutive elements of the sample. Denoting by $f_k(n, t)$ the number of sequences of the first n positive integers which have exactly t elements which are locally k-maximal, and defining a generating function, $v_k(x, y)$,

$$v_k(x, y) \equiv \sum_{\alpha,\beta} f_k(\alpha, \beta)x^\alpha y^\beta/\alpha!, \tag{1.1}$$

a recurrence relation and partial differential equation were then derived:

$$f_k(n + 1, r + 1) = \sum_{m,t} \binom{n}{m} f_k(m, t)f_k(n - m, r - t), \qquad n \geqq k - 1 \tag{1.2}$$

$$\partial v_k/\partial x = yv_k^2 + (1 - y)\sum_{t=0}^{k-2} (t + 1)x^t. \tag{1.3}$$

Unless specified otherwise, the range of a summation variable may be taken as $(0, +\infty)$ in these and the subsequent sums.

The relations (1.1) and (1.3) may be employed to obtain a general recurrence relation for the factorial moments of the distribution. Information on such moments would be useful in any application of the distribution as a non-parametric test, and would generally be of value in characterizing the distribution.

2. Recurrence relation for the factorial moments of the distribution. Let the r-th factorial of β be defined as $\beta^{(r)} \equiv \beta(\beta - 1)\cdots(\beta - r + 1)$, with

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