DENSITIES FOR STOCHASTIC PROCESSES[1]

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1. Introduction and summary. Let $\{x_\theta(t),\ \theta \varepsilon \Omega\}$ be a family of stochastic processes defined by their finite dimensional distributions; that is, $\{F_\theta[x(t_1), \cdots, x(t_n)];\ \theta \varepsilon \Omega\}$ is given for all finite sets of time points $t_1, \cdots, t_n$. A general procedure for treating a statistical problem concerning this family has been to solve the problem for the finite dimensional families and then see what happens to the solution when limits are taken over suitably selected sets of time points. For example, if $\hat{\theta}[x(t_1), \cdots, x(t_n)]$ is an estimate of θ based on the finite dimensional family and it can be shown that the limit

$$\hat{\theta}[x(t_1), \cdots, x(t_n)] \rightarrow \hat{\theta}[x(t)]$$

exists in some sense and is independent of the defining set $(t_1, t_2, \cdots)$, then this limit will usually provide an adequate estimate of θ for the process. Frequently the properties of the estimates $\hat{\theta}[x(t_1), \cdots, x(t_n)]$ can be extended to $\hat{\theta}[x(t)]$.

An alternative approach to the problem is proposed by Grenander [1]. He introduces the likelihood ratio of two processes P and Q restricted to a finite number of time points, shows that it converges to a limit as the number of points goes to infinity and that this limit is the density of P with respect to Q if this density exists. He uses these results to derive numerous statistical results.

The only criterion which he gives for the existence of the density is that the limit of the likelihood ratio be finite a.s. P. In applying this criterion he must always make use of some additional knowledge of the processes such as a.s. existence of certain integrals. In Section 2 these results are established very simply using the theory of martingales, and a criterion for the existence of the density is given which proves convenient in several applications. A condition also is given under which a density computed for a countable number of time points is valid for the continuous parameter process. Once the existence of the density is established standard statistical techniques can be applied directly. For example, sufficient statistics can often be found by inspection, or maximum likelihood methods can be used.

Densities for a normal process with continuous covariance and unknown mean value function are derived in Section 3. Minimum variance unbiased estimates of regression coefficients are obtained.

In [2] Cameron and Martin consider processes which are obtained from a Wiener process by linear transformations. They state conditions under which

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such a process is absolutely continuous with respect to the Wiener process, and they give a formula for computing the density. These results can be applied to the Ornstein Uhlenbeck process with covariance $\sigma^2 e^{-\beta|s-t|}$ to obtain a family of densities for $2\beta\sigma^2$ = constant. In Section 4 this family of densities is derived using the methods of Section 2. Using the same techniques it is shown that this family of Ornstein Uhlenbeck processes is mutually absolutely continuous. The maximum likelihood estimate of the correlation parameter β is computed.

2. Existence and computation of densities. Let $\mathfrak{X}^\infty = \mathfrak{X}_1 \times \mathfrak{X}_2 \times \ldots$ be a countably dimensional Euclidean space and $\mathfrak{A}^\infty$ the Borel σ-field over this space. Denote by $\mathfrak{A}^n$ the subfield of $\mathfrak{A}^\infty$ consisting of all cylinder sets with bases in $\mathfrak{X}_1 \times \mathfrak{X}_2 \times \cdots \times \mathfrak{X}_n$. If P is a measure over $\{\mathfrak{X}^\infty, \mathfrak{A}^\infty\}$, P^n will denote this measure restricted to $\mathfrak{A}^n$.

LEMMA 1. *Let P and Q be two probability measures on $\{\mathfrak{X}^\infty, \mathfrak{A}^\infty\}$ such that P^n is absolutely continuous with respect to Q^n for $n = 1, 2, \cdots$. Let $dP^n/dQ^n = f^n$. Then $\{f^n, \mathfrak{A}^n, n = 1, 2, \cdots\}$ is a martingale on the probability space $\{\mathfrak{X}^\infty, \mathfrak{A}^\infty, Q\}$.*

PROOF. For $m > n$, $\mathfrak{A}^m \supset \mathfrak{A}^n$, and hence for all $A^n \in \mathfrak{A}^n$

$$\int_{A^n} f^n \, dQ = P(A^n) = \int_{A^n} f^m \, dQ.$$

Since f^n is certainly $\mathfrak{A}^n$-measurable, it follows that $E[f^m \mid \mathfrak{A}^n] = f^n$.

THEOREM 1. *Under the assumptions of Lemma* 1:

(i) $f^n \to f$ *a.s.* Q.

(ii) $E \mid f^n - f \mid \to 0$ *if and only if P is absolutely continuous with respect to Q on $\mathfrak{A}^\infty$; and then $dP/dQ = f$ a.s. Q.*

(iii) *For* $r \geqq 1$, $E(f^n)^r \uparrow c_r$.

(iv) *If $c_r < \infty$ for $r > 1$, then the conditions of* (ii) *hold.*

PROOF. Since $E \mid f^n \mid = E(f^n) = 1, n = 1, 2, \cdots$, (i) follows from the martingale convergence theorem. If $dP/dQ = f'$ exists, then by the argument used in the proof of Lemma 1, f' closes the martingale. Since $E(f') = 1$, it follows from the martingale closure theorem that $f^n \to f'$ in the first mean and a.s. Q. Thus by (i) $f = f'$ a.s. Q. If $E \mid f^n - f \mid \to 0$, again by the closure theorem f closes the martingale. That is, $E[f \mid \mathfrak{A}^n] = f^n$ a.s. $Q, n = 1, 2, \cdots$. Thus for $A^m \in \bigcup_{n=1}^\infty \mathfrak{A}^n$

$$\int_{A^m} f \, dQ = \int_{A^m} f^m \, dQ = P(A^m). \tag{1}$$

The extension from $\bigcup \mathfrak{A}^n$ to $\mathfrak{A}^\infty$ is unique so that (1) also holds over $\mathfrak{A}^\infty$. Thus $dP/dQ = f$ a.s. Q. (iii) is a consequence of the convexity property of conditional expectations, and (iv) follows from the closure theorem.

The theorems on martingales used for this and the preceding lemma are Theorem 4.1 (i), (ii), and (iii) of Doob [3], Chapter 7. In Section 8 of this chapter, Doob obtains results of which Lemma 1 and Theorem 2 are easy extensions.

In order to apply these results to find densities for stochastic processes with a continuous time parameter, the step must be taken from $\mathfrak{X}^\infty$ to $\mathfrak{X}(T)$, the space of real-valued functions of t. The set T is assumed to be an interval, infinite or not, of the real line. In the space $\mathfrak{X}(T)$, let $\mathfrak{A}$ be the Borel σ-field over cylinder sets with bases Borel in finite dimensional subspaces. For some specified set $D = (t_1, t_2, \cdots)$, $\mathfrak{A}^\infty$ and $\mathfrak{A}^n$ will denote the subfields of cylinder sets with bases in $\prod_{t \varepsilon D} \mathfrak{X}_t$ and $\prod_{i=1}^n \mathfrak{X}_{t_i}$ respectively.

THEOREM 2. *Let* $\{\mathfrak{X}(T), \mathfrak{A}, P\}$ *and* $\{\mathfrak{X}(T), \mathfrak{A}, Q\}$ *be two stochastic processes such that* P *is absolutely continuous with respect to* Q *over* $\mathfrak{A}$ *and the* Q *process is continuous in probability. Then for any set* $D = (t_1, t_2, \cdots)$ *dense in* T, *the derivatives* dP^∞/dQ^∞ *and* dP/dQ *coincide a.s.* Q.

PROOF. Let g be a version of the derivative with respect to $\mathfrak{A}$. Since it is integrable, there exists a sequence of simple functions $g_n[x(t_1'), x(t_2'), \cdots]$ such that $g_n \to g$ a.s. Q. Let $D = (t_1, t_2, \cdots)$ be an arbitrary dense set in T. Since the process is continuous in probability for Q, for each t_i' there exists a subsequence $\{t_{i_j}\} \varepsilon D$ such that $x(t_{i_j}) \to x(t_i')$ a.s. Q. Thus each g_n and hence g is a.s. Q equal to an $\mathfrak{A}^\infty$-measurable function. This implies that $g = dP^\infty/dQ^\infty$ a.s. Q.

Now consider a family $\{P_\theta, \theta \varepsilon \Omega\}$ of probability measures over $\{\mathfrak{X}(T), \mathfrak{A}\}$. For a set dense in T, define

$$\frac{dP_\theta^n}{dP_\theta^n + dP_{\theta'}^n} = f_{\theta,\theta'}^n$$

for all $\theta, \theta' \varepsilon \Omega$.

COROLLARY 1. *If* $\{\mathfrak{X}(T), \mathfrak{A}, P_\theta\}$ *is continuous in probability for each* $\theta \varepsilon \Omega$, *then*

$$f_{\theta,\theta'}^n \to f_{\theta,\theta'} = \frac{dP_\theta}{dP_\theta + dP_{\theta'}} \text{ a.s. } P_\theta + P_{\theta'}.$$

A statistic $S[x(t)]$ *is pairwise sufficient for the family* $\{P_\theta, \theta \varepsilon \Omega\}$ *if and only if for all* $\theta, \theta' \varepsilon \Omega$, $f_{\theta,\theta'}$ *is a.s.* $(P_\theta + P_{\theta'})$ *a function of* $S[x(t)]$.

This result is immediate from Theorems 2 and 3 and Theorem 2 of Halmos and Savage [4].

When the processes P_θ are defined by their finite dimensional distributions, $f_{\theta,\theta'}$ can easily be computed. If it can be shown that the family is dominated by a σ-finite measure, then the statistic found in this manner is also sufficient.

3. Regression parameters for a normal process. Let $x(t)$ be a normal process with nonsingular continuous covariance $C(u, v)$ and mean value function

$$m(t) = \sum_{i=1}^s k_i \phi_i(t),$$

where the ϕ_i are known continuous functions and the k_i are parameters. Let P_0 be the process for which $k_1 = \cdots = k_s = 0$. For each parameter point

$k = (k_1, \cdots, k_s)$ define $dP_k^n/dP_0^n = f_k^n$. For a fixed set $D = (t_1, t_2, \cdots)$ dense in T, let

$$\Phi^n(\phi_i, \phi_j) = \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \phi_i(t_\alpha)\phi_j(t_\beta)C^{-1}(t_\alpha, t_\beta).$$

LEMMA 2.

$$E_0[(f_k^n)^2] = \exp \sum_i^s \sum_i^s k_i k_j \Phi^n(\phi_i, \phi_j).$$

PROOF.

$$f_k^n = \exp - \frac{1}{2}\sum_\alpha^n \sum_\beta^n \{[x(t_\alpha) - m(t_\alpha)][x(t_\beta) - m(t_\beta)] - x(t_\alpha)x(t_\beta)\}C^{-1}(t_\alpha, t_\beta)$$

$$E_0[(f_k^n)^2] = \int \cdots \int \frac{1}{(\sqrt{2\pi})^n(\mid C(t_\alpha, t_\beta)\mid)^{\frac{1}{2}}} \exp - \frac{1}{2}\sum_\alpha^n\sum_\beta^n \{[x(t_\alpha) - 2m(t_\alpha)]$$

$$\cdot [x(t_\beta) - 2m(t_\beta)] - 2m(t_\alpha)m(t_\beta)\} \, C^{-1}(t_\alpha, t_\beta)\, dx(t_1) \cdots dx(t_n)$$

$$= \exp \sum_\alpha^n \sum_\beta^n m(t_\alpha)m(t_\beta)\, C^{-1}(t_\alpha, t_\beta) = \exp \sum_i^s \sum_j^s k_i k_j \, \Phi^n(\phi_i, \phi_j).$$

From Theorem 1 (iii) $E_0[(f_k^n)^2] \uparrow c_2^k$ for all k. This implies that

$$\Phi^n(\phi_i, \phi_j) \rightarrow \Phi(\phi_i, \phi_j) \qquad i, j = 1, \cdots, n.$$

THEOREM 3. *If $\Phi(\phi_i, \phi_j)$ is finite for all i, j, and D, then P_k is absolutely continuous with respect to P_0 and has exponential density*

$$\exp\left[\sum_i^s k_i\,\Phi(x, \phi_i) - \frac{1}{2}\sum_i^s\sum_j^s k_i k_j\,\Phi(\phi_i, \phi_j)\right], \tag{2}$$

where $\Phi(x, \phi_i)$ is $\lim \Phi^n(x, \phi_i)$ if it exists and zero otherwise.

PROOF. From Theorem 1(i) $f_k^n \rightarrow f_k$ a.s. P_0. By assumption, $\lim E(f_k^n)^2 < \infty$ and hence by Theorem 1(iv) and (ii) $dP_k^\infty/dP_0^\infty = f_k$. This argument holds for any countable dense set D in T. Thus, over any $\mathcal{a}^\infty$, depending on a set of dense coordinates, P_k^∞ is absolutely continuous with respect to P_0^∞. It follows that P_k is absolutely continuous with respect to P_0 over $\mathcal{a}$. Continuity of $C(u, v)$ and the $\phi_i(t)$ implies continuity in quadratic mean and hence in probability. Thus by Theorem 2, $dP_k/dP_0 = f_k$ a.s. P_0.

This shows that, a.s. P_0, (2) is independent of D, and hence that $\Phi(\phi_i, \phi_j)$ is independent of D. It has been pointed out by E. Parzen that this is an immediate consequence of continuity of C and the ϕ_i. Thus the assumptions of the theorem can be weakened to require $\Phi(\phi_i, \phi_j)$ $i, j = 1, \cdots, s$ finite only for one dense set D.

COROLLARY 2. *The estimates*

$$\hat{k}_i = \sum_{j=1}^s \Phi^{-1}(\phi_i, \phi_j)\Phi(x, \phi_j) \qquad i = 1, 2, \cdots, s, \tag{3}$$

minimize $E[\hat{m}(t) - m(t)]^2$ among all unbiased estimates of $m(t)$ for each $t \in T$.

PROOF. The statistic $[\Phi(x, \phi_1), \cdots, \Phi(x, \phi_s)]$ is sufficient and complete for the exponential family (2). This result then is immediate from Theorems 1 and 2 of [5].

For the particular case that $x(t)$ is an Ornstein Uhlenbeck process with $0 \leqq t \leqq T$,

$$C(u, v) = \sigma^2 e^{-\beta|u-v|} \tag{4}$$

and

$$\Phi(\phi_i, \phi_j) = \frac{1}{2}\left[\phi_i(0)\phi_j(0) + \phi_i(T)\phi_j(T) + \frac{1}{\beta}\int_0^T \phi_i'(t)\phi_j'(t)\,dt + \beta\int_0^T \phi_i(t)\phi_j(t)\,dt\right].$$

In Theorem 1 of [6] it is shown that the estimates (3) minimize

$$\int_0^T E[\hat{m}(t) - m(t)]^2\,dt$$

among all linear unbiased estimates. Corollary 2 considerably strengthens this result.

4. Correlation parameter in the Ornstein Uhlenbeck process. Let $x(t)$ and $y(t)$ be normal processes with mean zero and nonsingular continuous covariances $C_0(u, v)$ and $C_1(u, v)$. As in the preceding section, application of Theorem 1(iv) and 2 to show that the $y(t)$ process is absolutely continuous with respect to the $x(t)$ process requires the computation of

$$E(f^n)^r = \int_{\mathfrak{X}^n}\left[\frac{f_1^n(x)}{f_0^n(x)}\right]^r f_0^n(x)\,d\mu^n(x).$$

In this expression μ^n is n-dimensional Lebesgue measure and $f_i^n(x)$ is the normal density with mean zero and covariance matrix

$$C_i^n(\alpha, \beta) = C_i^n(t_\alpha, t_\beta) \quad \alpha, \beta = 1, \cdots, n; i = 0, 1$$

with respect to a preassigned dense set $D = (t_1, t_2, \cdots)$. If the matrix $r(C_1^n)^{-1} - (r-1)(C_0^n)^{-1}$ is positive definite, this can easily be computed.

$$\begin{aligned}\int\left[\frac{f_1^n(x)}{f_0^n(x)}\right]^r f_0^n(x)\,d\mu^n(x) &= \int \frac{[(2\pi)^n \mid C_0^n \mid]^{\frac{r-1}{2}}}{[(2\pi)^n \mid C_1^n \mid]^{r/2}} \exp - \frac{1}{2} x[r(C_1^n)^{-1} \\ &\qquad - (r-1)(C_0^n)^{-1}]\, x'\,d\mu^n(x) \\ &= \frac{\mid C_0^n \mid^{\frac{r-1}{2}}}{\mid C_1^n \mid^{r/2}} \mid r(C_1^n)^{-1} - (r-1)(C_0^n)^{-1}\mid^{-1/2}.\end{aligned} \tag{5}$$

It must then be shown that for some $r > 1$, the limit of this expression is finite as $n \to \infty$.

Consider first $x(t)$ a Wiener process with parameter K and $y(t)$ determined by the linear transformation

$$Kx(t) = y(t) + \beta\int_0^t y(s)\,ds.$$

This satisfies the conditions of Cameron and Martin [2] so that the density of $y(t)$ with respect to $x(t)$ can be computed from their paper. The process determined by this transformation is $y(t) = z(t) - e^{-\beta t}z(0)$, where $z(t)$ is an Ornstein Uhlenbeck process (4) with parameters β and $\sigma^2 = K/2\beta$. The measures on $\mathfrak{X}_T$ for these processes will be denoted by P_K and $P_{\beta,\sigma}$. Ordering time points of the form $iT/2^m$, $i = 1, \cdots, 2^m$ in the obvious way and extracting appropriate subsequences, it is sufficient to consider (5) for $t_i = i\tau; i = 1, \cdots, n$; $\tau = T/n$ and $n \to \infty$. Then for the processes just described

$$r(C_1^n)^{-1} - (r-1)(C_0^n)^{-1}$$

$$= \frac{2\beta r}{K(1-e^{-2\beta\tau})}\begin{bmatrix} (1+e^{-2\beta\tau}) & -e^{-\beta\tau} & 0 & \cdots & 0 \\ -e^{-\beta\tau} & (1+e^{-2\beta\tau}) & -e^{-\beta\tau} & \ddots & \vdots \\ 0 & -e^{-\beta\tau} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & (1+e^{-2\beta\tau}) & -e^{-\beta\tau} \\ 0 & \cdots & 0 & -e^{-\beta\tau} & 1 \end{bmatrix}$$

$$- \frac{(r-1)}{K\tau}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & 0 \\ 0 & -1 & \ddots & 2 & -1 \\ \vdots & \cdots & 0 & -1 & 1 \end{bmatrix}$$

For the case of two Ornstein Uhlenbeck processes P_{β_0,σ_0} and P_{β_1,σ_1} with parameters β_0, $\sigma_0^2 = K/2\beta_0$ and β_1, $\sigma_1^2 = K/2\beta_1$ respectively,

$$r(C_1^n)^{-1} - (r-1)(C_0^n)^{-1}$$

$$= \frac{2r\beta_1}{K(1-e^{-2\beta_1\tau})}\begin{bmatrix} 1 & -e^{-\beta_1\tau} & 0 & \cdots & 0 \\ -e^{-\beta_1\tau} & 1+e^{-2\beta_1\tau} & -e^{-\beta_1\tau} & \ddots & 0 \\ 0 & -e^{-\beta_1\tau} & \ddots & 1+e^{-\beta_1\tau} & -e^{-\beta_1\tau} \\ \vdots & \cdots & 0 & -e^{-\beta_1\tau} & 1 \end{bmatrix}$$

$$- \frac{2(r-1)\beta_0}{K(1-e^{-2\beta_0\tau})}\begin{bmatrix} 1 & -e^{-\beta_0\tau} & 0 & \cdots & 0 \\ -e^{-\beta_0\tau} & 1+e^{-2\beta_0\tau} & e^{-\beta_0\tau} & \ddots & 0 \\ 0 & -e^{-\beta_0\tau} & \ddots & 1+e^{-2\beta_0\tau} & -e^{-\beta_0\tau} \\ \vdots & \cdots & 0 & -e^{-\beta_0\tau} & 1 \end{bmatrix}$$

In both cases all principal minors have the form

$$\left.\begin{bmatrix} \alpha & \lambda & 0 & \cdots & 0 \\ \lambda & \eta & \lambda & \ddots & \vdots \\ 0 & \lambda & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \eta & \lambda \\ 0 & \cdots & 0 & \lambda & \gamma \end{bmatrix}\right\} i.$$

The determinant of such a matrix of order $i \geqq 3$ is given by

$$\frac{1}{2}\left((r_1^{i-2} - r_2^{i-2})\left(\frac{\alpha\gamma\eta}{2} - \alpha\lambda^2 - \lambda^2\gamma + \frac{\lambda^2\eta}{2}\right)\Big/\left(\left(\frac{\eta}{2}\right)^2 - \lambda^2\right)^{1/2} + \frac{1}{2}\left(r_1^{i-2} + r_2^{i-2}\right)\left(\alpha\gamma - \lambda^2\right)\right),$$

where

$$r_1 = \frac{\eta}{2} + \left(\left(\frac{\eta}{2}\right)^2 - \lambda^2\right)^{1/2}, \qquad r_2 = \frac{\eta}{2} - \left(\left(\frac{\eta}{2}\right)^2 - \lambda^2\right)^{1/2}$$

In the first case take $r = 2$ and in the second case take $r > 1$ and satisfying $(\beta_1/\beta_0)^2 > (r-1)/r$. Then for τ sufficiently small it can be shown that $0 < -\lambda < \eta/2$ and $\alpha, \gamma > \eta/2$. These inequalities imply

$$\alpha\gamma - \lambda^2 > 0$$

$$\frac{\alpha\gamma\eta}{2} - \alpha\lambda^2 - \lambda^2\gamma + \frac{\lambda^2\eta}{2} > -\lambda\alpha\gamma - \alpha\lambda^2 - \lambda^2\gamma - \lambda^3$$

$$= (-\lambda)(\alpha + \lambda)(\gamma + \lambda) > 0,$$

and hence that for τ sufficiently small the matrices in question are positive definite. It is easily verified that minors for $i = 1, 2$ are positive. Thus (5) is valid, and using the formula above for the determinants involved, a routine computation shows that the limit of (5) as $\tau \to 0$ is finite in both cases.

Thus it has been established that in each case, $f_1^n(x)/f_0^n(x)$ converges a.s. to the desired density. The form of this limit is easily seen to be ce^{-Y} where c is a constant and Y is the a.s. limit of $\frac{1}{2}x[(C_1^n)^{-1} - (C_0^n)^{-1}]x'$. In the two cases considered this limit can be expressed in a more convenient form by expanding the terms of the sequence as follows:

$$\tfrac{1}{2}x[(C_1^n)^{-1} - (C_0^n)^{-1}]x' = A(\tau)x_0^2 + B(\tau)x_T^2 + C(\tau)\sum_{i=1}^{n} x_i^2\tau + D(\tau)\sum_{i=1}^{n}(x_i^2 - x_i x_{i-1}).$$

It has been shown [7] for the two processes concerned—the Wiener process and the Ornstein Uhlenbeck process—that

$$\sum_{i=1}^{n} x_i^2\,\tau \to \int_0^T x_t^2\,dt \quad \text{and} \quad \sum_{i=1}^{n}(x_i^2 - x_i x_{i-1}) \to K$$

where convergence is in quadratic mean. If subsequences are selected so that convergence is a.s., then

$$Y = Ax_0^T + Bx_T^2 + C \int_0^T x_t^2\, dt + DK$$

where $A(\tau) \rightarrow A$, etc. Routine computation of the coefficients then gives the final results.

$$\frac{dP_{\sigma,\beta}}{dP_K} = \exp - \frac{1}{2K}\left[\beta(x_T^2 - KT) + \beta^2 \int_0^T x_t^2\, dt\right]$$

$$\frac{dP_{\sigma_1,\beta_1}}{dP_{\sigma_0,\beta_0}} = \sqrt{\frac{\beta_1}{\beta_0}} \exp - \frac{1}{2K}\left[(\beta_1 - \beta_0)(x_0^2 + x_T^2 - KT) + (\beta_1^2 - \beta_0^2) \int_0^T x_t^2\, dt\right]$$

for $2\sigma^2\beta = 2\sigma_1^2\beta_1 = 2\sigma_0^2\beta_0 = K$.

From the second density, it is seen that

$$\left[(x_0^2 + x_T^2), \int_0^T x_t^2\, dt\right]$$

is a sufficient statistic for β. However, since this family is not complete no minimum variance unbiased estimate of β can be found as in the previous section. The maximum likelihood estimate of β is given by

$$\hat{\beta} = \frac{-(x_0^2 + x_T^2 - KT) + \left((x_0^2 + x_T^2 - KT)^2 + 8K \int_0^T x_t^2\, dt\right)^{\frac{1}{2}}}{4 \int_0^T x_t^2\, dt}.$$

For T large, this is approximated by

$$\hat{\beta} \sim \frac{K}{2 \frac{1}{T} \int_0^T x_t^2\, dt}$$

and for T small by

$$\hat{\beta} \sim \frac{K}{x_0^2 + x_T^2}.$$

If the continuous process is observed K can always be considered as known since

$$\sum_{i=1}^{n} (x_i^2 - x_i x_{i-1}) \rightarrow K$$

in probability as $\tau \rightarrow 0$.

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