NOTES

A NOTE ON A CLASS OF PROBLEMS IN 'NORMAL' MULTIVARIATE ANALYSIS OF VARIANCE[1]

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Summary. Let the columns of $X(p \times n)$ be independent non-singular p-dimensional normal variates with a common variance-covariance matrix and expectations given by

$$\mathcal{E}X' = A\xi,$$

where $A(n \times m)$ is a matrix of known constants and $\xi(m \times p)$ is a matrix of unknown parameters. This will be called the model. Under this model consider the hypothesis

$$\mathcal{H} : \xi = B\eta,$$

where $B(m \times k)$ is a given matrix of constants and $\eta(k \times p)$ is a matrix of unknown parameters.

It is shown that the hypothesis $\mathcal{H}$ is "completely testable" if and only if

$$\text{rank } A + \text{rank } B - \text{rank } AB = m.$$

Further, if rank $A \leqq n - p$, it is always possible to construct a testable hypothesis $\mathcal{H}^*$ which is implied by $\mathcal{H}$; the test-criterion proposed for $\mathcal{H}^*$ is based on the latent roots of the matrix $S_2S_1^{-1}$ where S_1 and $(S_1 + S_2)$ are the "error-matrices of sums of squares and products" under the model and under $\mathcal{H}$, respectively. It is further shown that the rank of the matrix S_2 is min [p, rank A − rank (AB)].

Let $X(p \times n)$ be a matrix of random variables, the columns of which are independent p-dimensional normal variates with the same positive-definite variance-covariance matrix $\Sigma(p \times p)$ and with expectations given by

$$\mathcal{E}X' = A\xi, \tag{1}$$

where $A(n \times m)$ is a matrix of known constants and $\xi(m \times p)$ is a matrix of unknown parameters.

Let the rank of the matrix A be r. We shall assume that $r \leqq \min(m, n - p)$. Without loss of generality, the first r columns of A may be taken to be linearly independent and so to form a basis of A. Then [2] we can partition and factorize A in the form:

$$\begin{aligned} A &= [A_1(n \times r) \vdots A_2(n \times (m - r))] \\ &= L'(n \times r)[T_1'(r \times r) \vdots T_2'(r \times (m - r))], \end{aligned} \tag{2}$$

Received December 3, 1957.

[1] This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 18(600)-83. Reproduction in whole or in part is permitted for any purpose of the United States Government.

where A_1, L and T_1 are matrices each of rank r, T_1 being triangular and L semi-orthogonal, that is

$$LL' = I(r \times r). \tag{3}$$

It is well known [2] that the error-matrix of sums of squares and products is given by

$$S_1 = XEX', \tag{4}$$

where

$$E = I - A_1(A_1'A_1)^{-1}A_1' = I - L'L, \tag{5}$$

and that this E is an $n \times n$ matrix of rank $n - r$. By our assumption that $r \leqq \min(m, n - p)$, the matrix S_1 is, a.e., non-singular.

Consider, now, the hypothesis that the parameters ξ can be expressed in terms of a smaller number of parameters $\eta(k \times p)$ in the form

$$\mathcal{H} : \xi = B\eta \qquad (k < m)$$

where $B(m \times k)$ is a given matrix. Under $\mathcal{H}$ the expectations are given by

$$\mathcal{E}X' \doteq AB\eta. \tag{6}$$

Let rank $AB = s$. Obviously $s \leqq \min(r, k)$. Here again, without any loss of generality, we can regard the first s columns of AB to be linearly independent. The rank of the matrix $[T_1' \vdots T_2']B$ must be s [2] and it can be factorized the same way as (2). Thus,

$$[T_1' \vdots T_2']B = M'(r \times s)[U_1(s \times s) \vdots U_2(s \times (k - s))], \tag{7}$$

where matrices M and U_1 are each of rank s, U_1 is triangular and M semi-orthogonal, that is,

$$MM' = I(s \times s). \tag{8}$$

We thus have

$$AB = (ML)'[U_1 \vdots U_2], \tag{9}$$

where $LM(n \times s)$ is seen to be semi-orthogonal. Using (5) it immediately follows that the error-matrix of sums of squares and products under the hypothesis is given by

$$X\, E_{\mathcal{H}}\, X', \tag{10}$$

where

$$E_{\mathcal{H}} = I - L'M'ML', \tag{11}$$

and that this $E_{\mathcal{H}}$ is an $n \times n$ matrix of rank $n - s$.

Let us choose a matrix $N((r - s) \times r)$ which is an orthogonal completion of M; that is,

$$NN' = I((r - s) \times (r - s)) \text{ and } NM' = 0. \tag{12}$$

The difference of the error-matrices (10) and (5) is the hypothesis-matrix of sums of squares and products, and is given by

$$S_2 = XHX', \tag{13}$$

where

$$H = E_{\mathcal{H}} - E = L'L - L'M'ML = L'N'NL. \tag{14}$$

Using (3) it is easily checked that,

$$EH = 0. \tag{15}$$

Thus the matrices E and H are orthogonal and S_2 is, a.e., of rank $= \min(p, r - s)$.

It will now be shown that the matrix S_2 is the appropriate hypothesis-matrix of sums of squares and products for testing a hypothesis $\mathcal{H}^*$ which is testable [2] and will be introduced presently. It will be shown that, in general, the hypotheses $\mathcal{H}$ and $\mathcal{H}^*$ are *not* identical; though $\mathcal{H}$ implies $\mathcal{H}^*$, the converse is not generally true.

Let the rank of the matrix B be t. Then we can find a matrix $C((m - t) \times m)$ of rank $(m - t)$ such that

$$CB = 0. \tag{16}$$

Since the row-vectors of C generate the vectorspace completely orthogonal to that generated by the column-vectors of B, it follows that, if C^* is any other matrix such that

$$C^*B = 0, \tag{17}$$

we can factorize C^* in the form

$$C^* = DC. \tag{18}$$

Define the matrix $C^*((r - s) \times m)$ by

$$C^* = N[T_1' : T_2'], \tag{19}$$

with T_1, T_2 defined by (2) and N defined by (12). Notice that this C^* is of rank $r - s$. Then

$$C^*B = N[T_1' : T_2']B = NM'[U_1 : U_2] = 0,$$

because of (12). Thus, for the matrix C^*, the relation (17) holds and consequently a matrix D exists which satisfies (18).

It is easily seen that on elimination of η by pre-multiplication by C the hypothesis $\mathcal{H}$ may be expressed in the equivalent form

$$\mathcal{H} : C\xi = 0. \tag{20}$$

Pre-multiplication by D gives,

$$\mathcal{H}^* : C^*\xi = 0. \tag{21}$$

Note that D is a matrix of the form $(r - s) \times (m - t)$ of rank $(r - s)$.

Obviously $\mathcal{H}$ implies $\mathcal{H}^*$ but the converse is not true unless D is a non-singular matrix of form $(m - t) \times (m - t)$. A necessary and sufficient condition for this is that $r - s = m - t$ or, in words, that

$$\text{rank } (A) + \text{rank } (B) - \text{rank } (AB) = m. \tag{22}$$

Now partition C^* in the form $C^* = [C_1 \vdots C_2]$, where

$$C_1 = NT_1' \text{ and } C_2 = NT_2', \tag{23}$$

so that

$$C_2 = C_1 T_1^{-1\prime} T_2'. \tag{24}$$

Note that (23) is precisely the condition that the hypothesis $\mathcal{H}^*$ is testable [2].

The hypothesis-matrix of sums of squares and products for testing $\mathcal{H}^*$ (which is testable) computed directly from the formula given in [2] turns out to be

$$S^* = XH^*X', \tag{25}$$

where

$$\begin{aligned} H^* &= A_1(A_1'A_1)^{-1}C_1'[C_1(A_1'A_1)^{-1}C_1']^{-1}C_1(A_1'A_1)^{-1}A_1' \\ &= L'T_1^{-1}C_1'[C_1(T_1T_1')^{-1}C_1']^{-1}C_1T_1'^{-1}L && \text{(using (2))} \\ &= L'N'(NN')^{-1}NL' && \text{(using (23))} \\ &= L'N'NL && \text{(using (12))} \\ &= H && \text{(from (14)).} \end{aligned}$$

Thus, we have

$$S^* = S_2. \tag{26}$$

An important special case is where we have $n > m > k$ and rank $A = m$ and rank $B = k$. In this case, rank $AB = k$. Consequently the condition (22) is satisfied and the hypotheses $\mathcal{H}$ and $\mathcal{H}^*$ are identical.

The statistical criterion for testing the hypothesis $\mathcal{H}^*$ would be based on the latent roots of the matrix $S_2S_1^{-1}$, the particular critical region proposed here [1, 2] being given by

$$C_{\max}[S_2S_1^{-1}] \geqq \lambda_\alpha(t^*, r - s, n - r), \tag{27}$$

where $C_{\max}[S_2S_1^{-1}]$ denotes the largest characteristic root of the matrix $[S_2S_1^{-1}]$ all of whose roots are non-negative and, a.e., t^* roots are positive, $t^* = \min(p, r - s)$, and $\lambda_\alpha(t^*, r - s, n - r)$ is a constant, depending upon t^*, $r - s$, $n - r$ and the size of the critical region α, which can be obtained, since the distribution is known and the percentage points are being tabulated.

If $p = 1$ we have the univariate problem, in which case (27) is replaced by a β-critical region or, after a little transformation, by an F-critical region.

REFERENCES

[1] Roy, S. N., "On a heuristic method of test construction and its use in multivariate analysis," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 220–238.

[2] Roy, S. N., "A report on some aspects of multivariate analysis," *North Carolina Institute of Statistics Mimeograph Series No.* 121 (1954).

NOTE ON AN APPLICATION OF THE SCHUMANN-BRADLEY TABLE

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Summary. In a recent paper [1] Schumann and Bradley present a table of the ratio of central variance ratios for use in comparing the sensitivity of experiments. The purpose of this note is to point out some other applications of the Schumann-Bradley Table I [1] which stem from a variance components model.

Model. The model employed in [1] is a "fixed effects" analysis of variance schema (henceforth abbreviated FEM). In this note a "random effects" model (REM) will be used. Let s_{ei}^2 and s_{ti}^2 be the mean squares for "error" and "treatments" in the ith experiment ($i = 1, 2$). Let n_{ei} and n_{ti} be the respective degrees of freedom, let σ_{ei}^2 and σ_{ti}^2 be the variance components, and let K_i be a design constant. Finally let χ_{ei}^2 and χ_{ti}^2 be central chi-squares with n_{ei} and n_{ti} degrees of freedom respectively. Then the usual REM is

$$n_{ei}s_{ei}^2 = \sigma_{ei}^2\chi_{ei}^2, \qquad n_{ti}s_{ti}^2 = (\sigma_{ei}^2 + K_i\sigma_{ti}^2)\chi_{ti}^2. \tag{1.01}$$

Let F_i be the ratio of the mean squares and let F_{ci} be a central F ratio with degrees of freedom corresponding to F_i (i.e., n_{ti}, n_{ei}). Then for independent mean squares,

$$F_i = \left(\frac{\sigma_{ei}^2 + K_i\,\sigma_{ti}^2}{\sigma_{ei}^2}\right) F_{ci}. \tag{1.02}$$

Let w be the ratio of F_1 to F_2, let w_c be the ratio of central variance ratios (i.e., F_{c1}/F_{c2}), and let

$$\psi = \frac{\sigma_{e1}^2 + K_1\,\sigma_{t1}^2}{\sigma_{e1}^2} \cdot \frac{\sigma_{e2}^2}{\sigma_{e2}^2 + K_2\,\sigma_{t2}^2}, \tag{1.03}$$

then

$$w = \psi w_c. \tag{1.04}$$

For the special case (C_0) where the two experiments have the same structure (i.e., $n_{e1} = n_{e2}$, $n_{t1} = n_{t2}$, $K_1 = K_2$) equation (1.04) leads immediately to exact significance tests based on Schumann-Bradley Table I [1]. In the nota-

Received February 4, 1958.