ON THE INTEGRODIFFERENTIAL EQUATION OF TAKÁCS. II[1]

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1. Introduction. This paper continues (Cf [1]) the study of the properties of the function, $F(t, 0)$, $0 \leqq F(t, 0) = F(t) \leqq 1$, where $F(t, x) = \Pr\{\eta(t) \leqq x\}$, $t \geqq 0$, $x \geqq 0$, satisfies[2]

$$\frac{\partial F(t, x)}{\partial t} = \frac{\partial F(t, x)}{\partial x} - \lambda(t)F(t, x) + \lambda(t) \int_{0-}^{x} H(x - y)\, d_y F(t, y) . \tag{1.1}$$

The functions $\Phi(s) = \int_0^\infty e^{-xs}\, dF(0, x)$, $H(x) = \int_0^x h(\xi)\, d\xi$,

$$(h(\xi) \geqq 0, \int_0^\infty h(\xi)\, d\xi = 1),$$

and $\lambda(t) = \Lambda'(t) \geqq 0$ are given. It is assumed that there exists a $c > 0$ such that $e^{-cx} h(x) \, \varepsilon \, L^2(0, \infty)$. The moment $\int_0^\infty x^k h(x)\, dx$, if it exists, is denoted by μ_k. We put $\psi(s) = \int_0^\infty h(x) e^{-sx}\, dx$. Furthermore, we suppose that $\int_0^T [\lambda(t)]^2\, dt$ exists as a possibly improper Riemann integral for all $T > 0$.

The stochastic process $\eta(t)$ represents the waiting time of a customer arriving at time t in a queue with Poisson arrivals of variable density $\lambda(t)$, with $H(x)$ the distribution of service times. $F(t)$ is the probability that the counter is unoccupied at time t. Our present purpose is to study the behavior of $F(t)$ for large t, especially under conditions that turn out to guarantee that $F(t)$ does not approach zero. Previous knowledge ([2], [3], [4]) in this direction appears to be restricted essentially to the case $\lambda(t) = \text{const.}$

The following was proved in [1], although it does not appear as an explicit statement there.

Theorem 1. *There is only one distribution-solution, $F(t, x)$, of* (1.1). *Moreover, $F(t)$ is the unique continuous solution of the Volterra equation of the first kind*

$$\int_0^t G(t, u)F(u)\, du = g(t), \qquad (\textit{almost all } t \geqq 0), \tag{1.2}$$

where

$$\begin{cases} G(t, u) = \text{P.V.} \dfrac{1}{2\pi i} \displaystyle\int_{x-i\infty}^{x+i\infty} e^{(t-u)s-[\Lambda(t)-\Lambda(u)][1-\psi(s)]} \dfrac{ds}{s}, \; x > c, \\ g(t) = \dfrac{1}{2\pi i} \displaystyle\int_{x-i\infty}^{x+i\infty} \Phi(s) e^{ts-\Lambda(t)[1-\psi(s)]} \dfrac{ds}{s^2}. \end{cases}$$

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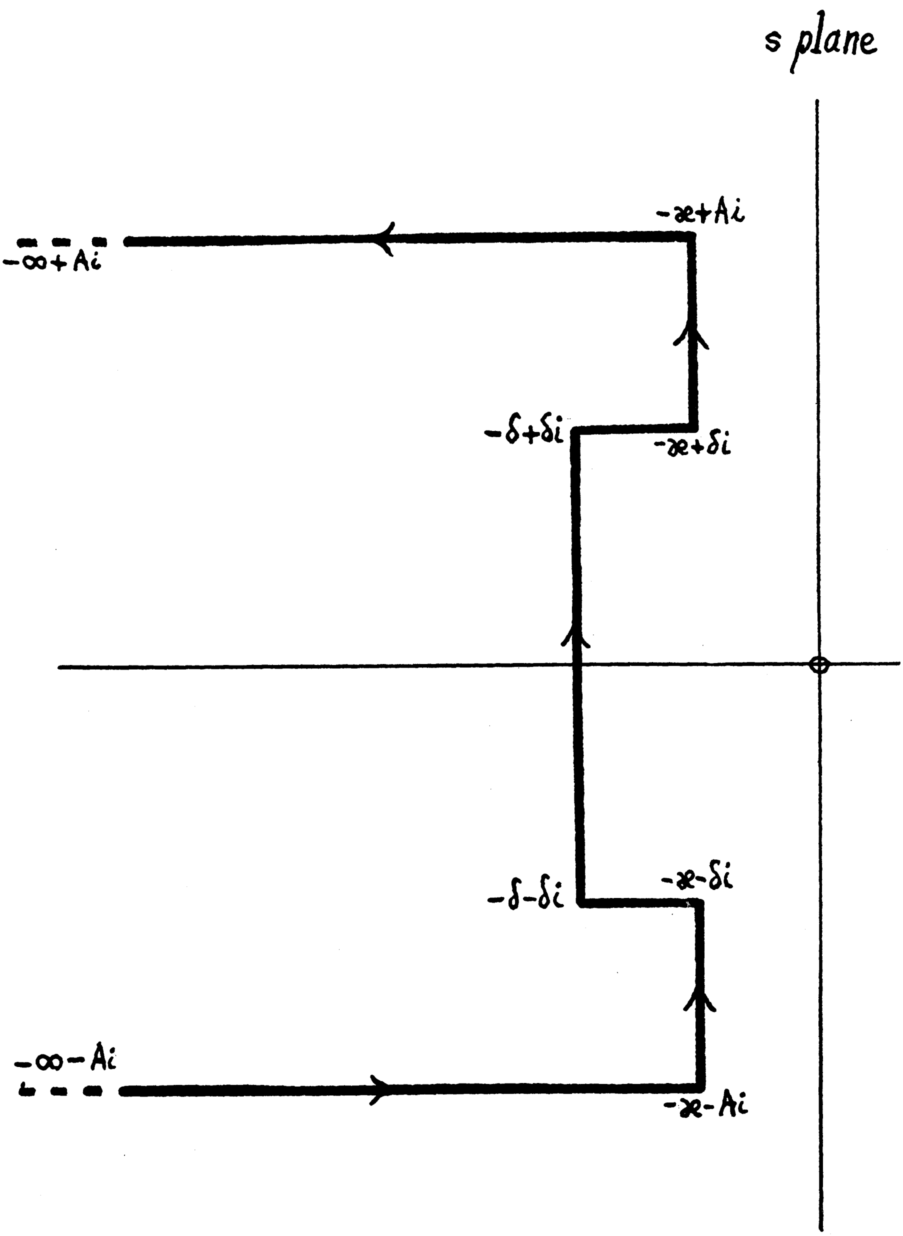
[2] As the referee points out, the derivation of (1.1) in [2] implicitly *assumes* that $F(t, x)$ is differentiable in x, $x > 0$. However, it seems possible to *prove* the differentiability by an argument based on that of [2], p. 108.

Our approach to the asymptotic behavior of $F(t)$ will be through (1.2). This will, of course, not be as simple as in the case, $\lambda(t) = \text{const}$, when (1.2) can be solved by Laplace transforms. The main results shall be based on the restrictive assumption that $\psi(s)$ is regular at infinity, and in a neighborhood of the imaginary axis. The class of such $\psi(s)$ is, however, still large enough to include as a proper subclass the important class of rational $\psi(s)$.



2. Some Abelian lemmas.

LEMMA 2.1. *Suppose $\psi(s)$ is regular at infinity and in a neighborhood of the imaginary axis, and $\beta\mu_1/\alpha \leqq \rho < 1$. Then there exist functions $k_j(\rho) > 0, j = 1, 2,$ such that*

$$\left| \text{P.V.} \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\alpha s - \beta[1-\psi(s)]} \frac{ds}{s} - 1 \right| \leqq k_1(\rho) e^{-k_2(\rho)\alpha}, \alpha \geqq 0, \beta \geqq 0.$$

PROOF. If $A > 0$ and $\delta > 0$ are chosen so that ψ is regular for $|\mathcal{I}s| \geqq A$ and $\mathcal{R}s \geqq -\delta$, we can write the quantity inside the absolute value signs as $H(\alpha, \beta) = (1/2\pi i) \int_c$, where $C = C(A, \delta, \varkappa)$ is the contour shown in the figure.

Since $\psi(\infty) = 0$ we may choose A sufficiently large so that $|\psi(x \pm iA)| < 1$. The function $\psi(s)$ is of the form $\psi(s) = 1 - \mu_1 s + \mu_2/2s^2 + O(s^3)$, $\mu_i > 0$. Recalling that $\mu_2 > 0$, let us choose δ sufficiently small so that

$$\mathcal{I} \frac{\psi - 1 + \mu_1 s}{s} > 0, \tag{2.1}$$

and < 0, on the segments $[-\delta + \delta i, \delta i]$, and $[-\delta - \delta i, -\delta i]$, respectively. Furthermore, we choose $\delta = \delta(\rho)$ sufficiently small so that, also,

$$\max_{|s| \leqq \sqrt{2}\delta} \left| \frac{\psi - 1 + \mu_1 s}{\mu_1 s} \right| \leqq \frac{1}{\sqrt{2}} (\rho^{-1/2} - 1).$$

Finally, we select the number $\varkappa = \varkappa(\rho)$, $0 < \varkappa < \delta$, so that

$$\mathcal{R}\psi(-\varkappa + iy) < 1 \qquad \text{when } \delta \leqq |y| \leqq A.$$

(This is possible since $\mathcal{R}\psi(iy) = \int_0^\infty \cos(y\xi)h(\xi)\,d\xi < 1$ when $|y| > 0$.) In the proof of the lemma we may assume that $\alpha \geqq 1$, because in the opposite case both α and β are bounded from above, and the truth of the lemma follows from the fact that, because of the continuity of $H(\alpha, \beta)$, at most an adjustment of $k_i(\rho)$ is necessary.

On the segment $[-\delta - \delta i, -\delta + \delta i]$,

$$\begin{aligned} \mathcal{R}\{\alpha s - \beta[1 - \psi(s)]\} &= \mathcal{R}\left\{ \left[\alpha - \beta\mu_1 \left(1 - \frac{\psi - 1 + \mu_1 s}{\mu_1 s} \right) \right] s \right\} \\ &\leqq (\alpha - \beta\mu_1)(-\delta) + \beta\mu_1 |s| \left| \frac{\psi - 1 + \mu_1 s}{\mu_1 s} \right| \leqq (\alpha - \beta\mu_1)(-\delta) \\ &\quad + \beta\mu_1\sqrt{2}\delta \frac{1}{\sqrt{2}} (\rho^{-1/2} - 1) \leqq -\alpha\delta(1 - \rho^{1/2}). \end{aligned}$$

Thus

$$\left| \int_{-\delta-\delta i}^{-\delta+\delta i} \right| \leqq \frac{2\delta}{\delta} e^{-\alpha\delta(1-\rho^{1/2})} = \mathcal{C}_1(\rho) \exp[-\mathcal{C}_2(\rho)\alpha], \mathcal{C}_2(\rho) > 0.$$

On the segments $[-\delta \pm \delta i, -\varkappa \pm \delta i]$ we find, by (2.1), that

$$\begin{aligned} \mathcal{R}\{\alpha s - \beta[1 - \psi(s)]\} &= (\alpha - \beta\mu_1)\mathcal{R}(s) + \beta\mu_1 \left[\mathcal{R}(s)\mathcal{R}\left(\frac{\psi - 1 + \mu_1 s}{\mu_1 s} \right) \right. \\ &\quad \left. \mp \delta\mathcal{I}\left(\frac{\psi - 1 + \mu_1 s}{\mu_1 s} \right) \right] \leqq \alpha \left\{ 1 - \frac{\beta\mu_1}{\alpha} \left[1 - \mathcal{R}\left(\frac{\psi - 1 + \mu_1 s}{\mu_1 s} \right) \right] \right\} \mathcal{R}s. \end{aligned}$$

Since $\Re s < 0$, the above

$$\leq \alpha\left\{1 - \frac{\beta\mu_1}{\alpha}\left[1 + \left|\frac{\psi - 1 + \mu_1 s}{\mu_1 s}\right|\right]\right\}\Re s \leq \alpha\{1 - \rho[1 + \rho^{-1/2} - 1]\}\,\Re s \leq -\alpha(1 - \rho^{1/2})x.$$

Hence

$$\left|\int_{-x-\delta i}^{-\delta-\delta i} + \int_{-\delta+\delta i}^{-x+\delta i}\right| \leq \frac{2(\delta - x)}{\delta} e^{-x(1-\rho^{1/2})\alpha} = \mathcal{C}_3(\rho)e^{-\mathcal{C}_4(\rho)\alpha}, \qquad \mathcal{C}_4(\rho) > 0.$$

In view of the choice of x,

$$\left|\int_{-x-Ai}^{-x-\delta i} + \int_{-x+\delta i}^{-x+Ai}\right| \leq \frac{2A}{\delta} e^{-x\alpha} = \mathcal{C}_5(\rho)e^{-\mathcal{C}_6(\rho)\alpha}, \quad \mathcal{C}_6(\rho) > 0.$$

Finally,

$$\left|\int_{-\infty-Ai}^{-x-Ai} + \int_{-x+Ai}^{-\infty+Ai}\right| \leq \frac{2}{A}\int_x^\infty e^{-\alpha x}dx = \mathcal{C}_7 e^{-\mathcal{C}_8(\rho)\alpha}, \mathcal{C}_8(\rho) > 0, \text{if } \alpha \geq 1.$$

Putting all the above inequalities together, we obtain the lemma.

COROLLARY. *Suppose $\psi(s)$ is regular at infinity and in a neighborhood of the imaginary axis, and*

$$\limsup_{t-u\to\infty}\left[\mu_1\frac{\Lambda(t) - \Lambda(u)}{t - u}\right] < 1.$$

Then

$$\int_0^t G(t,u)F(u)\,du = \int_0^t F(u)\,du + O(1), \qquad \textit{as } t \to \infty. \tag{2.2}$$

PROOF. There exists a $T > 0$ such that

$$\mu_1\frac{\Lambda(t) - \Lambda(u)}{t - u} \leq \rho < 1 \text{ whenever } t - u \geq T.$$

If $|G(t, u)| \leq M$ for $|t - u| \leq T$ then, for $t > T$,

$$\left|\int_0^t G(t,u)F(u)\,du - \int_0^t F(u)\,du\right| \leq \left|\int_0^{t-T}[G(t,u) - 1]F(u)\,du\right| + (M+1)T$$

$$\leq \int_0^{t-T} k_1(\rho)e^{-k_2(\rho)(t-u)}du + (M+1)T \leq \frac{k_1(\rho)}{k_2(\rho)} + (M+1)T = O(1).$$

LEMMA 2.2. *Suppose $\psi(s) = 1 - \mu_1 s + o(s)$, as $s \to 0$, uniformly with respect to* arg s, $|\arg s| \leq \pi/2$, *and $\mu_1\Lambda(t)/t \leq \rho < 1$. Then $g(t) = t - \mu_1\Lambda(t) + o(t)$, as $t \to \infty$.*

PROOF. Let C_r denote a contour along the imaginary axis, with a semicircular detour of radius r, center 0, to the right. Since $|\Phi(s)| \leq |$ for $\Re s \geq 0$, we have, according to a known Abelian theorem ([5], pp. 494–5),

$$g_1(t) = \frac{1}{2\pi i}\int_{C_r} \Phi(s)e^{[t-\mu_1\Lambda(t)]s}\frac{ds}{s^2} = t - \mu_1\Lambda(t) + o(t), \qquad \text{as } t \to \infty.$$

It therefore suffices to show that

$$D(t) = g(t) - g_1(t) = \frac{1}{2\pi i}\int_{C_r} \Phi(s)e^{(t-\mu_1\Lambda)s}[e^{(\psi-1+\mu_1 s)\Lambda} - 1]\frac{ds}{s^2} = o(t), \text{ as } t \to \infty.$$

Choose $\epsilon > 0$. Then, if $|s| \leqq \delta(\epsilon)$, $|\psi - 1 + \mu_1 s| \leqq \epsilon|s|$, and therefore

$$|e^{(\psi-1+\mu_1 s)\Lambda} - 1| \leqq \Lambda\epsilon|s|e^{\Lambda\epsilon|s|}.$$

Consider points $\pm iA$, $0 < r < A \leqq \delta(\epsilon)$ on C_r. Evidently,

$$\left|\int_{-iA}^{iA}\right| \leqq e^{(t-\mu_1\Lambda)r}\Lambda\epsilon e^{\Lambda\epsilon A}\int_{-iA}^{iA}\frac{|ds|}{|s|} = \Lambda\epsilon e^{(t-\mu_1\Lambda)r+\Lambda\epsilon A}\left(\pi + 2\log\frac{A}{r}\right).$$

Also,

$$\left|\int_{-i\infty}^{-iA} + \int_{iA}^{i\infty}\right| \leqq 2\int_A^\infty |e^{ts-(1-\psi)\Lambda} - e^{(t-\mu_1\Lambda)s}|\frac{dy}{y^2} \leqq 4\int_A^\infty\frac{dy}{y^2} = \frac{4}{A}.$$

Hence,

$$D(t) \leqq \frac{4}{A} + \Lambda\epsilon e^{(t-\mu_1\Lambda)r+\Lambda\epsilon A}\left(\pi + 2\log\frac{A}{r}\right).$$

Putting $r = t^{-1}$, $A = \epsilon^{-1}t^{-1}$, where $t \geqq T(\epsilon) = [\epsilon\delta(\epsilon)]^{-1}$, we find that

$$D(t) \leqq Kt\epsilon\log(1/\epsilon) \text{ for } t \geqq T(\epsilon).$$

We shall state the following without proof.

LEMMA 2.3. *If* $\psi(s)$ and $\Phi(s)$ *are regular in a neighborhood of* $s = 0$, *and if* $\mu_1\Lambda(t)/t \leqq \rho < 1$ *then* $g(t) = t - \mu_1\Lambda(t) + O(1)$, *as* $t \to \infty$.

3. Asymptotic average of $F(t)$. In the special case $\lambda(t) = \lambda =$ const it is known [4] that, for the distribution solution $F(t, x)$ of (1.1), $\mu_1\lambda < 1$ implies $\lim_{t\to\infty}F(t, 0) = 1 - \mu_1\lambda$. In view of our above results we are able to make the following statements in the general case.

THEOREM 2. *Suppose* $\psi(s)$ *is regular at infinity and in a neighborhood of the imaginary axis, and*

$$\limsup_{t-u\to\infty} \mu_1\frac{\Lambda(t) - \Lambda(u)}{t - u} < 1.$$

Then, if $F(t, x)$ *is the distribution solution of* (1.1),

$$\frac{1}{T}\int_0^T F(t, 0)\,dt = 1 - \mu_1\frac{\Lambda(T)}{T} + o(1), \text{ as } T \to \infty.$$

THEOREM 3. *If we also assume that* $\Phi(s)$ *is regular in a neighborhood of* $s = 0$ *then "*$o(1)$*" in the conclusion of Theorem 2 can be replaced by "*$O(T^{-1})$*."*

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