LINEAR ESTIMATION FROM CENSORED DATA

By R. L. Plackett

University of Liverpool

1. Introduction. Suppose that a sample of n random variables is taken from a continuous probability distribution, whose density function is $f\,[(y - \mu)/\sigma]/\sigma$, where μ and σ are unknown. Arrange the variables in order of magnitude, and denote them by $y_1, y_2, \cdots, y_n$, where

$$y_1 < y_2 < \cdots < y_n .$$

We shall discuss the problem of estimating μ and σ from the k successive variables $y_u, y_{u+1}, \cdots, y_v$, where $v = u + k - 1$. This problem arises, for example, in life-testing, and some applications are described by Gupta [7].

When using the principal results derived here, the expected values of ordered variables are essential, but tables of these quantities for normal samples are, at present somewhat limited. However, recent studies by Berkson [1] have shown the importance of the logistic distribution, which closely resembles the normal, and some properties of ordered logistic variables are given in Section 2. We now turn to the main problem. If u and v are fixed, the best linear unbiased estimates of μ and σ can be calculated by least squares, given the expected value and dispersion matrix of the vector of ordered variables (Godwin [6], Lloyd [11], Gupta [7]. In general, special tables become necessary, and it seems desirable to obtain simple formulae when samples are moderate or large in size. This is achieved in Section 3, where asymptotic values of the coefficients of $y_u, y_{u+1}, \cdots, y_v$ are derived. An examination of the conditions involved is supplied in Section 4, by considering the limiting form of the maximum likelihood equations. Several illustrative numerical tables complete the paper.

2. Ordered logistic variables. The logistic distribution is defined by

$$L = \log\{p/(1 - p)\}, \tag{1}$$

where p is the probability of a value less than L. Suppose that $L(i; n)$ is the ith variable in ascending order in a sample of size n from this distribution. Then

$$\begin{aligned} \mathcal{E} \exp\{wL(i; n)\} &= \frac{n!}{(i-1)!(n-i)!} \int_0^1 \left(\frac{p}{1-p}\right)^w p^{i-1}(1-p)^{n-i}\,dp \\ &= \frac{(i-1+w)!(n-i-w)!}{(i-1)!(n-i)!}. \end{aligned} \tag{2}$$

Take logarithms, differentiate with respect to w, and put $w = 0$. The cumulants of $L(i; n)$ are

$$\kappa_j(i; n) = \frac{d^j}{dw^j} \log (i-1)! + (-1)^j \frac{d^j}{dw^j} \log (n-i)! \tag{3}$$

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and are thus expressible in terms of polygamma functions, tabulated for $j = 1, 2, 3, 4$ in [2]. For $(i - 1) > (n - i)$, we obtain

$$(4) \quad \kappa_1(i; n) = \frac{1}{(n-i+1)} + \frac{1}{(n-i+2)} + \cdots + \frac{1}{(i-1)},$$

$$(5) \quad \kappa_2(i; n) = \frac{\pi^2}{3} - \left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(i-1)^2}\right\} - \left\{1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{(n-i)^2}\right\},$$

$$(6) \quad \kappa_3(i; n) = 2\left\{\frac{1}{(n-i+1)^3} + \frac{1}{(n-i+2)^3} + \cdots + \frac{1}{(i-1)^3}\right\},$$

$$(7) \quad \kappa_4(i; n) = \frac{2\pi^4}{15} - 6\left\{1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{(i-1)^4}\right\} - 6\left\{1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots + \frac{1}{(n-i)^4}\right\}.$$

Suppose that $x(i; n)$ is the ith variable in ascending order in a sample of size n from the probability distribution whose density function is $f(x)$ and distribution function $F(x)$. Let α be fixed, $0 < \alpha < 1$, and define t by

$$(8) \quad \alpha = F(t).$$

We require the two following results. As $n \to \infty$, with $i = [n\alpha]$

$$(9) \quad \mathcal{E}x(i; n) = t + O(n^{-1})$$

and

$$(10) \quad F\{\mathcal{E}x(i+1; n)\} - F\{\mathcal{E}x(i; n)\} = 1/n + O(n^{-2}).$$

The proofs are based on the Taylor expansion of x_r considered as a function of L, about the value $L = \kappa_1(i; n)$. This, after expectation, gives

$$(11) \quad \mathcal{E}x(i; n) = x^{(0)} + \tfrac{1}{2}x^{(2)}\kappa_2 + \tfrac{1}{6}x^{(3)}\kappa_3 + \tfrac{1}{24}x^{(4)}(\kappa_4 + 3\kappa_2^2) + \cdots,$$

where $x^{(j)}$ is the value at $L = \kappa_1(i; n)$ of the jth derivative of x with respect to L. Now

$$(12) \quad \kappa_1(i; n) = \tfrac{1}{2}\log\{(i-1)i/(n-i)(n-i+1)\} + O(n^{-2})$$

whence

$$(13) \quad \kappa_1(i; n) = \lambda + O(n^{-1}),$$

where

$$(14) \quad \lambda = \log\{\alpha/(1-\alpha)\}.$$

Also

$$(15) \quad \kappa_j(i; n) = O(n^{1-j}) \qquad (j = 2, 3, \cdots).$$

Assuming $x^{(2)}$ to be bounded, we can substitute (13) and (15) in (11) to obtain (9). As regards (10), we suppose that $x^{(2)}$ and $x^{(3)}$ are bounded, in which case

$$(16) \qquad \mathcal{E}x(i+1;n) - \mathcal{E}x(i;n) = \left\{\frac{1}{i} + \frac{1}{(n-i)}\right\} x^{(1)} + O(n^{-2}).$$

On the further assumption that df/dx is bounded, (10) results.

We shall now consider the standard normal distribution in more detail. Denote its density function by $\phi(x)$ and distribution function by $\Phi(x)$. Here

$$(17) \quad x^{(1)} = \Phi(1-\Phi)/\phi,$$

$$(18) \quad x^{(2)} = x^{(1)}\{xx^{(1)} - (2\Phi - 1)\},$$

$$(19) \quad x^{(3)} = (x^{(1)})^3 + 2xx^{(1)}x^{(2)} + x^{(2)}(1-2\Phi) - 2x^{(1)}\Phi(1-\Phi),$$

$$(20) \quad x^{(4)} = 5(x^{(1)})^2x^{(2)} + x^{(3)}\{2xx^{(1)} - (2\Phi - 1)\} + 2x^{(2)}\{xx^{(2)} - 2\Phi(1-\Phi)\} + 2x^{(1)}(2\Phi - 1)\Phi(1-\Phi).$$

These derivatives are all bounded, their maximum absolute values being given below.

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$
0.62666	0.07376	0.06724	0.04597

The absolute value of the remainder after $(j-1)$ terms of the series (11) is at most $\beta_j \max |x^{(j)}|/j!$, where β_j is the jth absolute moment about the mean of the ith ordered logistic variable in a sample of n. Since β_j is known when j is even and the inequality $(\beta_j)^{1/j} \leqq (\beta_{j+1})^{1/(j+1)}$ is available when j is odd, we can thus assign bounds to $\mathcal{E}x(i;n)$ for all values of j. As an illustration, take $\mathcal{E}x(19;25)$.

j	Series (11) to j terms	Absolute maximum error
1	0.642835	0.007656
2	0.636781	0.002521
3	0.636656	0.000262

David and Johnson [5] express x as a function of Φ, and the value for $\mathcal{E}x(19;25)$ from the first four terms of the series on p. 236 of their paper is 0.636904. However, their formula is arranged as a power series in $(n+2)^{-1}$, and a similar rearrangement of (11) would be necessary before a full comparison of the two approaches can be made. This will be undertaken on another occasion.

3. Least squares estimation. Let t_i denote the expectation of $(y_i - \mu)/\sigma$. Write

$$(21) \qquad f_i = f(t_i),$$

$$(22) \qquad p_i = F(t_i),$$

and

(23) $$q_i = 1 - p_i \qquad (i = u, u + 1, \cdots, v).$$

Let m be the vector of $(y_i - \mu)/\sigma$ for $i = u, u + 1, \cdots, v$ and put

(24) $$t = \mathcal{E}m$$

and

(25) $$V = \mathfrak{D}m = \mathcal{E}\{(m - \mathcal{E}m)(m - \mathcal{E}m)'\}.$$

In principle, t and V can be computed from the known function $f(x)$. The estimate of

(26) $$\theta = \begin{bmatrix} \mu \\ \sigma \end{bmatrix}$$

given by generalized least squares is

(27) $$\theta^* = (A'V^{-1}A)^{-1}A'V^{-1}y,$$

where

(28) $$A = [1 \quad t].$$

As V is difficult to handle analytically, we replace it by W, a symmetric matrix whose elements are $\{a_i b_j\}$ for $i \leqq j$, where

(29) $$a_i = p_i/f_i \qquad (i = u, u + 1, \cdots, v)$$

and

(30) $$b_j = q_j/f_j \qquad (j = u, u + 1, \cdots, v).$$

Since $\mathfrak{D}y \sim W\sigma^2/n$, the unbiased estimate

(31) $$\theta^+ = (A'W^{-1}A)^{-1}A'W^{-1}y$$

may be presumed to have the same asymptotic properties as θ^*. We therefore consider the limiting form of θ^+.

The inverse of W has been given by Hammersley and Morton [9]. Put

(32) $$a_{u-1} = 0, \qquad a_{v+1} = 1, \qquad b_{u-1} = 1, \qquad b_{v+1} = 0,$$

Then

(33) $$W^{-1} = \begin{bmatrix} c_u & d_u & 0 & 0 & \cdots & 0 \\ d_u & c_{u+1} & d_{u+1} & 0 & \cdots & 0 \\ 0 & d_{u+1} & c_{u+2} & d_{u+2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & & c_{v-1} & d_{v-1} \\ 0 & 0 & \cdots & & d_{v-1} & c_v \end{bmatrix},$$

where

(34) $$c_i = (a_{i+1}b_{i-1} - a_{i-1}b_{i+1}) \,/\, (a_i b_{i+1} - a_{i+1}b_i)(a_{i-1}b_i - a_i b_{i-1})$$

and

$$d_i = 1/(a_i b_{i+1} - a_{i+1} b_i). \tag{35}$$

Denote $A'W^{-1}$ by G, and define

$$h_s = \tfrac{1}{2}(p_{s+1} - p_{s-1}) \qquad (s = u+1, u+2, \cdots, v-1), \tag{36}$$

$$h_u = p_{u+1} - p_u , \tag{37}$$

and

$$h_v = p_v - p_{v-1} . \tag{38}$$

If the elements of G are considered as functions of p_u, p_{u+1}, $\cdots$, p_v then g_{1u} depends on p_u and p_{u+1}; g_{1s} on p_{s-1}, p_s, and p_{s+1}; and g_{1v} on p_{v-1} and p_v. In g_{1u}, put $p_{u+1} = p_u + h_u$; in g_{1s}, replace p_{s-1} by $p_s - h_s$ and p_{s+1} by $p_s + h_s$; and in g_{1v}, put $p_{v-1} = p_v - h_v$. The first and third substitutions are exact; the second one is approximate, but if $n \to \infty$ with $u = [n\alpha]$ and $v = [n\beta]$, the values of p_i tend to become equally spaced between α and β, as (10) shows. The elements in the second row of G are treated similarly, so that both elements in the ith column are now expressed as functions of p_i and h_i. Expanding by Taylor series as far as h_s^3 in the numerators of g_{1s} and g_{2s} and as far as h^2 elsewhere, the elements of G finally reduce, after a good deal of straightforward algebra, to the expressions given below. The primes signify differentiation with respect to p, so that

$$f' = \frac{d \log f}{dx} \tag{39}$$

and

$$ff'' = \frac{d^2 \log f}{dx^2}. \tag{40}$$

In calculating the elements of $A'W^{-1}A$, we pass from sums involving h to integrals involving dp.

4. Maximum likelihood estimation. The likelihood of y_u, y_{u+1}, $\cdots$, y_v is

$$\frac{n!}{(u-1)!(n-v)!}\left\{F\left(\frac{y_u-\mu}{\sigma}\right)\right\}^{u-1}\prod_{i=u}^{v}\frac{1}{\sigma}f\left(\frac{y_i-\mu}{\sigma}\right)\left\{1-F\left(\frac{y_v-\mu}{\sigma}\right)\right\}^{n-v} \tag{41}$$

Denote by $\hat{\mu}$ and $\hat{\sigma}$ the maximum likelihood estimates of μ and σ, respectively. They satisfy the equations

$$-\frac{(u-1)\hat{\sigma}f\left(\dfrac{y_u-\hat{\mu}}{\hat{\sigma}}\right)}{nF\left(\dfrac{y_u-\hat{\mu}}{\hat{\sigma}}\right)} - \frac{\hat{\sigma}}{n}\sum_{i=u}^{v}\frac{d\log f}{dx}\left(\frac{y_i-\hat{\mu}}{\hat{\sigma}}\right) + \frac{(n-v)\hat{\sigma}f\left(\dfrac{y_v-\hat{\mu}}{\hat{\sigma}}\right)}{n\left\{1-F\left(\dfrac{y_v-\hat{\mu}}{\hat{\sigma}}\right)\right\}} = 0, \tag{42}$$

TABLE 1

Asymptotic value of $A' W^{-1}$

Row—Column	General Density	Normal Density
$(1, u)$	$f_u^2/p_u - f_u' f_u - \frac{1}{2} h_u f_u f_u''$	$f_u^2/p_u + t_u f_u + \frac{1}{2} h_u$
$(1, s)$	$-h_s f_s f_s''$	h_s
$(1, v)$	$f_v^2/q_v + f_v' f_v - \frac{1}{2} h_v f_v f_v''$	$f_v^2/q_v - t_v f_v + \frac{1}{2} h_v$
$(2, u)$	$t_u f_u^2/p_u - t_u f_u' f_u - f_u - \frac{1}{2} h_u (f_u' + f_u f_u'' t_u)$	$t_u f_u^2/p_u + t_u^2 f_u - f_u + h_u t_u$
$(2, s)$	$-h_s (f_s' + f_s f_s'' t_s)$	$2 h_s t_s$
$(2, v)$	$t_v f_v^2/q_v + t_v f_v' f_v + f_v - \frac{1}{2} h_v (f_v' + f_v f_v'' t_v)$	$t_v f_v^2/q_v - t_v^2 f_v + f_v + h_v t_v$

TABLE 2

Asymptotic value of $A' W^{-1} A$

Row—Column	General Density	Normal Density
$(1, 1)$	$-\int_{p_u}^{p_v} f f'' \, dp + f_v^2/q_v + f_v f_v' + f_u^2/p_u - f_u f_u'$	$f_v^2/q_v - t_v f_v + p_v + f_u^2/p_u + t_u f_u - p_u$
$(1, 2)$ and $(2, 1)$	$-\int_{p_u}^{p_v} t f f'' \, dp + t_v f_v^2/q_v + t_v f_v f_v' + t_u f_u^2/p_u - t_u f_u f_u'$	$t_v f_v^2/q_v - t_v^2 f_v - f_v + t_u f_u^2/p_u + t_u^2 f_u + f_u$
$(2,2)$	$-\int_{p_u}^{p_v} t^2 f f'' \, dp + t_v^2 f_v^2/q_v + t_v^2 f_v f_v' + p_v + t_u^2 f_u^2/p_u - t_u^2 f_u f_u' - p_u$	$t_v^2 f_v^2/q_v - t_v^3 f_v - t_v f_v + 2p_v + t_u^2 f_u^2/p_u + t_u^3 f_u + t_u f_u - 2p_u$

$$
(43) \quad -\frac{(u-1)(y_u - \hat{\mu}) f\left(\dfrac{y_u - \hat{\mu}}{\hat{\sigma}}\right)}{nF\left(\dfrac{y_u - \hat{\mu}}{\hat{\sigma}}\right)} - \frac{k\hat{\sigma}}{n} - \frac{1}{n} \sum_{i=u}^{v} (y_i - \hat{\mu}) \frac{d \log f}{dx} \left(\frac{y_i - \hat{\mu}}{\hat{\sigma}}\right) + \frac{(n-v)(y_v - \hat{\mu}) f\left(\dfrac{y_v - \hat{\mu}}{\hat{\sigma}}\right)}{n\left\{1 - F\left(\dfrac{y_v - \hat{\mu}}{\hat{\sigma}}\right)\right\}} = 0,
$$

where

$$
\frac{d \log f}{dx} \left(\frac{y_i - \hat{\mu}}{\hat{\sigma}}\right)
$$

means the value at $(y_i - \hat{\mu})/\hat{\sigma}$ of the function $d \log f/dx$. The direct solution of (42) and (43) for normal samples has been described by Cohen [3], who used successive approximation; and, when $u = 1$, by Gupta [7], who calculated a special table which shortens the work. Halperin [8] has indicated conditions under which

(a) the maximum likelihood equations have a consistent set of solutions $\hat{\mu}, \hat{\sigma}$;

(b) $\sqrt{n}(\hat{\mu} - \mu)$ and $\sqrt{n}(\hat{\sigma} - \sigma)$ have a bivariate normal limit distribution;
(c) the dispersion matrix of the limit distribution is best in the sense of Cramér [4], §32.6.

The necessary assumptions involve derivatives of $f[(y - \mu)/\sigma]/\sigma$ with respect to μ and σ, and we shall suppose henceforth that they are satisfied.

We expand $(y_i - \hat{\mu})/\hat{\sigma}$ in a Taylor series about t_i and obtain

$$(44)\qquad \begin{aligned} &-\frac{(u-1)\hat{\sigma}}{n}\left\{\frac{f_u}{p_u} + \left(\frac{y_u - \hat{\mu}}{\hat{\sigma}} - t_u\right)\left(\frac{f_u f_u'}{p_u} - \frac{f_u^2}{p_u^2}\right) + \frac{1}{2}\left(\frac{y_u - \hat{\mu}}{\hat{\sigma}} - t_u\right)^2 C\right\} \\ &-\frac{\hat{\sigma}}{n}\sum_{i=u}^{v}\left\{f_i' + \left(\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right) f_i f_i'' + \frac{1}{2}\left(\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right)^2 D_i\right\} \\ &+\frac{(n-v)\hat{\sigma}}{n}\left\{\frac{f_v}{q_v} + \left(\frac{y_v - \hat{\mu}}{\hat{\sigma}} - t_v\right)\left(\frac{f_v f_v'}{q_v} + \frac{f_v^2}{q_v^2}\right) + \frac{1}{2}\left(\frac{y_v - \hat{\mu}}{\hat{\sigma}} - t_v\right)^2 E\right\} = 0, \end{aligned}$$

$$(45)\qquad \begin{aligned} &-\frac{(u-1)\hat{\sigma}}{n}\left\{\frac{t_u f_u}{p_u} + \left(\frac{y_u - \hat{\mu}}{\hat{\sigma}} - t_u\right)\left(\frac{f_u}{p_u} + \frac{t_u f_u f_u'}{p_u} - \frac{t_u f_u^2}{p_u^2}\right)\right. \\ &\left.+\frac{1}{2}\left(\frac{y_u - \hat{\mu}}{\hat{\sigma}} - t_u\right)^2 R\right\} - \frac{k\hat{\sigma}}{n} - \frac{\hat{\sigma}}{n}\sum_{i=u}^{v}\left\{t_i f_i'\right. \\ &\left.+\left(\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right)(t_i f_i f_i'' + f_i') + \frac{1}{2}\left(\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right)^2 S_i\right\} + \frac{(n-v)\hat{\sigma}}{n} \\ &\left\{\frac{t_v f_v}{q_v} + \left(\frac{y_v - \hat{\mu}}{\hat{\sigma}} - t_v\right)\left(\frac{f_v}{q_v} + \frac{t_v f_v f_v'}{q_v} + \frac{t_v f_v^2}{q_v^2}\right) + \frac{1}{2}\left(\frac{y_v - \hat{\mu}}{\hat{\sigma}} - t_v\right)^2 T\right\} = 0. \end{aligned}$$

Here C, D_i, E, R, S_i and T are second-order derivatives with respect to x evaluated at points intermediate between $(y_i - \hat{\mu})\hat{\sigma}$ and t_i; and the primes have their previous meaning.

We assume that the second-order derivatives of

$$\frac{d\log f}{dx},\quad x\frac{d\log f}{dx},\quad \frac{f}{p},\quad \frac{f}{q},\quad \frac{xf}{p},\quad \frac{xf}{q},$$

with respect to x, are functions of bounded variation. This condition is not satisfied if $F(x) = 0$ at a finite value of x, since then

$$\left|\frac{d^2}{dx^2}\left(\frac{f}{p}\right)\right| \to \infty$$

at the lower terminus of the distribution; nor if $F(x) = 1$ for finite x, since

$$\left|\frac{d^2}{dx^2}\left(\frac{f}{q}\right)\right| \to \infty$$

there. However, all is well for the normal and logistic distributions, as the following table shows.

Maximum absolute values of second-order derivatives

Distribution	$\frac{d \log f}{dx}$	$x\frac{d \log f}{dx}$	$\frac{f}{p}$	$\frac{f}{q}$	$\frac{xf}{p}$	$\frac{xf}{q}$
Normal.............	0	2.00	0.30	0.30	2.00	2.00
Logistic............	0.19	1.00	0.10	0.10	0.50	0.50

Let α and β be fixed such that $0 < \alpha < \beta < 1$. We assume that $f(t) \geqq c > 0$ wherever $F^{-1}(\alpha) \leqq t \leqq F^{-1}(\beta)$. For any such t, let f, p and q be defined in accordance with (21), (22), and (23). Then

$$(46) \qquad z^2 = pq/f^2$$

has a finite maximum, which we denote by z_1^2. We proceed to derive the form taken by the maximum likelihood equations when $u = [n\alpha]$, $v = [n\beta]$, and $n \to \infty$.

Consider the variable

$$(47) \qquad \left(\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right) = \frac{(y_i - \mu - t_i\sigma) - (\hat{\mu} - \mu) - t_i(\hat{\sigma} - \sigma)}{\hat{\sigma}}$$

Given $\epsilon_1 > 0$, $\epsilon_2 > 0$, and ϵ_3 such that $\sigma > \epsilon_3 > 0$,

$$(48) \qquad \begin{aligned} \Pr\left\{\left|\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right| > \frac{\epsilon_1 + \epsilon_2}{\hat{\sigma} - \epsilon_3}\right\} &< \Pr\{|y_i - \mu - t_i\sigma| > \epsilon_1\} \\ &+ \Pr\{|(\hat{\mu} - \mu) + t_i(\hat{\sigma} - \sigma)| > \epsilon_2\} + \Pr\{|\hat{\sigma} - \sigma| > \epsilon_3\}. \end{aligned}$$

Typically, $i = [np]$, and as $n \to \infty$, $\frac{\sqrt{n}}{z}\left(\frac{y_i - \mu}{\sigma} - t\right)$ is asymptotically normal with zero mean and unit variance (Cramér [4], §28.5). According to (9), $(t_i - t)$ is $O(n^{-1})$ and so $\frac{\sqrt{n}}{z}\left(\frac{y_i - \mu}{\sigma} - t_i\right)$ has the same limit distribution. Hence

$$(49) \qquad \Pr\{|y_i - \mu - t_i\sigma| > \epsilon_1\} \sim 2\Phi(-n^{1/2}\epsilon_1/\sigma z) \sim 2(\sigma z/n^{1/2}\epsilon_1)\phi(n^{1/2}\epsilon_1/\sigma z).$$

Similarly, by the asymptotic properties of $\hat{\mu}$ and $\hat{\sigma}$, there exist finite quantities z_2 and z_3 such that

$$(50) \qquad \Pr\{|(\hat{\mu} - \mu) + t_i(\hat{\sigma} - \sigma)| > \epsilon_2\} \sim 2(\sigma z_2/n^{1/2}\epsilon_2)\phi(n^{1/2}\epsilon_2/\sigma z_2)$$

and

$$(51) \qquad \Pr\{|\hat{\sigma} - \sigma| > \epsilon_3\} \sim 2(\sigma z_3/n^{1/2}\epsilon_3)\phi(n^{1/2}\epsilon_3/\sigma z_3).$$

Consequently, as $n \to \infty$,

$$(52) \qquad \sum_{i=u}^{v} \Pr\left\{\left|\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right| > \frac{\epsilon_1 + \epsilon_2}{\sigma - \epsilon_3}\right\} < 2(\beta - \alpha)n^{1/2}\sigma \sum_{j=1}^{3} (z_j/\epsilon_j)\phi(n^{1/2}\epsilon_j/\sigma z_j).$$

Thus, given $\epsilon > 0$ and $\delta > 0$, arbitrarily small, we can find N such that

$$\sum_{i=u}^{v} \Pr\left\{\left|\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right| > \epsilon\right\} < \delta \qquad \text{for } n \geqq N. \tag{53}$$

Therefore, by Boole's inequality,

$$\Pr\left\{\left|\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right| \leqq \epsilon \quad \text{for} \quad i = u, u+1, \cdots, v\right\} \geqq 1 - \delta \quad \text{for} \quad n \geqq N. \tag{54}$$

When this event occurs, the terms in (44) and (45) which involve $\left(\frac{y_i - \hat{\mu}}{\hat{\sigma}} - t_i\right)^2$ are negligible compared with the remaining terms, because C, D_i, E, R, S_i and T are all bounded. We therefore omit the squared terms, and replace sums by integrals as before (cf. Hoeffding [10]), at the same time making an Euler-Maclaurin adjustment on the coefficients of y_u and y_v, so as to correct for bias in the estimates and bring the results into line with those previously obtained. The linearized form taken asymptotically by the maximum likelihood equations is then given by the coefficients in Tables 1 and 2 on replacing h_i by $1/n$ throughout. We shall use μ^0 and σ^0 to denote the corresponding linearized estimates. The asymptotic dispersion matrix of the maximum likelihood estimates is σ^2/n times the inverse of the equations for μ^0 and σ^0.

These results show, not only that the maximum likelihood estimates of μ and σ are asymptotically linear, but also that the best linear unbiased estimates

TABLE 3A

Coefficients of ordered variables when estimating the mean. θ^* above, θ° below

k	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
2	−1.8634	2.8634								
	−2.1547	3.0554								
3	−0.6596	−0.2138	1.8734							
	−0.7487	−0.2248	1.9309							
4	−0.2923	−0.0709	0.0305	1.3327						
	−0.3304	−0.0780	0.0362	1.3543						
5	−0.1240	−0.0016	0.0549	0.0990	0.9718					
	−0.1418	−0.0055	0.0567	0.1071	0.9797					
6	−0.0316	0.0383	0.0707	0.0962	0.1185	0.7078				
	−0.0394	0.0366	0.0718	0.1003	0.1261	0.7100				
7	0.0244	0.0636	0.0818	0.0962	0.1089	0.1207	0.5045			
	0.0222	0.0633	0.0829	0.0988	0.1131	0.1270	0.5045			
8	0.0605	0.0804	0.0898	0.0972	0.1037	0.1099	0.1161	0.3424		
	0.0616	0.0812	0.0911	0.0992	0.1066	0.1137	0.1210	0.3423		
9	0.0843	0.0921	0.0957	0.0986	0.1011	0.1036	0.1060	0.1085	0.2101	
	0.0877	0.0934	0.0973	0.1004	0.1033	0.1060	0.1089	0.1120	0.2113	
10	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
	0.1048	0.1018	0.1018	0.1018	0.1018	0.1018	0.1018	0.1018	0.1018	0.1048

are asymptotically normal and efficient. In order to compute the coefficients of the ordered variables, only tables of $f(x)$, $F(x)$ and t_i are necessary. For normal samples, Teichroew [14] gives t_i to 10 D for $n \leqq 20$, and an extension to $n \leqq 100$ with 24 D is being prepared (Ruben [12]). For logistic samples, explicit formulae have already been given.

5. Numerical tables. Tables 3A and 3B refer to the estimation of θ from the smallest k observations in a sample of size $n = 10$ from a normal distribution. They give the coefficients of $y_1, y_2, \cdots, y_k$ for $k \leqq 10$ in

(i) the best linear unbiased estimate, $\theta^* = \begin{bmatrix} \mu^* \\ \sigma^* \end{bmatrix}$,

(ii) the linearized maximum likelihood estimate, $\theta^0 = \begin{bmatrix} \mu^\circ \\ \sigma^\circ \end{bmatrix}$

Table 4 gives the coefficients of μ and σ in the expectation of θ^0. Suppose in general that

$$\mathcal{E}\theta^0 = B\theta \tag{55}$$

Then $B^{-1}\theta^0$ is an unbiased estimate of θ, and the efficiencies of its elements, relative to μ^* and σ^* respectively, have been calculated from the table of $\mathfrak{D}m$ in Sarhan and Greenberg [13] when, as above, $n = 10$, $u = 1$, and $v = 2, 3, \cdots 10$. These efficiencies never fall below 0.9998, a result which suggests that θ^0, corrected for bias, can be used in place of θ^*, with negligible loss of efficiency, for all sample sizes of practical importance.

TABLE 3B

Coefficients of ordered variables when estimating the standard deviation. θ^* above, θ° below

k	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}
2	−1.8608	1.8608								
	−2.1366	2.0404								
3	−0.9625	−0.4357	1.3981							
	−1.0767	−0.4586	1.4738							
4	−0.6520	−0.3150	−0.1593	1.1263						
	−0.7190	−0.3330	−0.1611	1.1681						
5	−0.4419	−0.2491	−0.1362	−0.0472	0.9243					
	−0.5374	−0.2631	−0.1414	−0.0425	0.9499					
6	−0.3931	−0.2063	−0.1192	−0.0501	0.0111	0.7576				
	−0.4266	−0.2175	−0.1250	−0.0498	0.0180	0.7740				
7	−0.3252	−0.1758	−0.1058	−0.0502	−0.0006	0.0469	0.6107			
	−0.3513	−0.1849	−0.1114	−0.0517	0.0022	0.0545	0.6218			
8	−0.2753	−0.1523	−0.0947	−0.0488	−0.0077	0.0319	0.0722	0.4746		
	−0.2963	−0.1600	−0.0998	−0.0510	−0.0069	0.0358	0.0799	0.4830		
9	−0.2364	−0.1334	−0.0851	−0.0465	−0.0119	0.0215	0.0559	0.0936	0.3423	
	−0.2539	−0.1399	−0.0897	−0.0490	−0.0122	0.0234	0.0602	0.1009	0.3505	
10	−0.2044	−0.1172	−0.0763	−0.0436	−0.0142	0.0142	0.0436	0.0763	0.1172	0.2044
	−0.2196	−0.1231	−0.0807	−0.0462	−0.0151	0.0151	0.0462	0.0807	0.1231	0.2196

TABLE 4

Expectation of θ^0

k	μ^0		σ^0	
	μ	σ	μ	σ
2	0.9007	0.2560	−0.0962	1.2446
3	0.9574	0.1104	−0.0616	1.1492
4	0.9821	0.0539	−0.0450	1.1066
5	0.9962	0.0260	−0.0346	1.0827
6	1.0054	0.0108	−0.0270	1.0678
7	1.0119	0.0022	−0.0208	1.0583
8	1.0166	−0.0023	−0.0153	1.0529
9	1.0204	−0.0038	−0.0097	1.0523
10	1.0243	0.0000	0.0000	1.0668

TABLE 5

$1/n$	h_1	h_2	h_3	h_4	h_5
0.5000	0.4274				
0.3333	0.3013	0.3013			
0.2500	0.2316	0.2326			
0.2000	0.1879	0.1888	0.1897		
0.1667	0.1580	0.1588	0.1597		
0.1429	0.1362	0.1370	0.1378	0.1378	
0.1250	0.1198	0.1204	0.1212	0.1212	
0.1111	0.1068	0.1074	0.1081	0.1082	0.1082
0.1000	0.0964	0.0970	0.0976	0.0976	0.0976

Table 5 also refers to normal samples. Used in conjunction with the relation

$$h_i = h_{n+1-i}, \tag{56}$$

it gives the values of h_i for $n = 2, 3, \cdots 10$ and $1 \leqq i \leqq n$. That there is close agreement between θ^+ and θ^0 can be inferred from Table 5 in particular and (10) in general.

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