ON INFINITELY DIVISIBLE RANDOM VECTORS[1]

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1. Summary. A normally distributed random vector X is well known to be representable by $A \cdot Y$ (in the sense of having identical distributions), where A is a matrix of constants and Y is a random vector whose component random variables are independent. A necessary and sufficient condition for any infinitely divisible random vector to be so representable is given. The limiting case is discussed as are connections with the multivariate Poisson distribution and stochastic processes.

2. Notation and preliminaries. Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a probability space; that is, Ω is an abstract point set, $\mathcal{B}$ a Borel Field of subsets of Ω, and $\mathcal{P}$ a probability measure defined on $\mathcal{B}$. If $m \geqq 1$ is an integer and $X, Y, Z \cdots$ a set of m-dimensional vectors defined on Ω, we write $X \sim Y$ to signify that the associated probability measures (or distribution functions) of X and Y are identical. Since the relationship indicated by $\sim$ is reflexive, symmetric and transitive, the use of this symbol is in the best of taste and tradition.

We abbreviate the terms cumulative distribution function, characteristic function, random vector, and infinitely divisible by c.d.f., c.f., r.v., and i.d., respectively, and occasionally string some of these together. A bar over a set signifies complementation and the notation R^m is used for m-dimensional Euclidean space.

3. Infinitely divisible vectors. Recall that an r.v. X and likewise its c.d.f., say $F(x_1, x_2, \cdots, x_m)$, and its c.f., say $\varphi(t_1, t_2, \cdots, t_m)$, are called i.d.[2] if for every positive integer n, $X \sim$ sum of n independent (identically distributed) r.v.'s. P. Lévy ([4], p. 220) has given a necessary and sufficient condition (NSC) that X be i.d., viz.,

$$\begin{aligned}\varphi(t_1, \cdots, t_m) = \exp\Bigg\{ & i \sum_{j=1}^{m} \gamma_j t_j - \frac{1}{2} \sum_{j,k=1}^{m} \sigma_{jk} t_j t_k \\ & + \int_{R^m} \left[e^{i(t_1 u_1 + \cdots + t_m u_m)} - 1 - \frac{i(t_1 u_1 + \cdots + t_m u_m)}{1 + |u|^2} \right] dN(u_1, \cdots, u_m) \Bigg\},\end{aligned} \tag{1}$$

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[2] In many works X is defined to be i.d. if for every positive integer n, $X_n = X_{n1} + X_{n2} + \cdots + X_{nn}$, where $X_{n1}, X_{n2}, \cdots, X_{nn}$ are mutually independent. Such a definition places demands on the basic space Ω. A discussion of this point occurs in Appendix 2 of "Limit Distributions of Sums of Independent Random Variables" by Gnedenko-Kolmogoroff, Addison Wesley. The current definition obviates such questions.

where γ_j are real coefficients $\Sigma = \{\sigma_{ij}\}$ is a positive semidefinite matrix, $| u |$ is the Euclidean length of the row vector $u' = (u_1, u_2, \cdots, u_m)$ and $\mu_N(A) = \int_A dN(u_1, \cdots, u_m)$ is a nonnegative additive set function (not necessarily finite) defined on the Borel sets A of R^m and such that

$$(1') \qquad \int_{S_\epsilon} | u^2 | \, dN(u_1, \cdots, u_m) < \infty, \qquad \int_{\bar{S}_\epsilon} dN(u_1, \cdots, u_m) < \infty,$$

(with S_ϵ an m-dimensional sphere of radius $\epsilon > 0$ centered at the origin and $\bar{S}_\epsilon$ its complement in R^m) for arbitrary ϵ.

Let $\varphi(t)$, $N(u)$, and $G(u)$ abbreviate $\varphi(t_1, \cdots, t_m)$, $N(u_1, \cdots, u_m)$ and $G(u_1, \cdots, u_m)$, respectively. Analogous to the one-dimensional case, an alternative and frequently more convenient form of (1) is given by

$$(2) \qquad \varphi(t) = \exp\{i\gamma' t - \tfrac{1}{2} t' \Sigma t\} \exp\left\{\int_{R^m} \left(e^{it'u} - 1 - \frac{it'u}{1 + | u |^2}\right)\left(\frac{1 + | u |^2}{| u |^2}\right) dG(u)\right\},$$

where t, u, γ are column vectors and

$$\mu_G(A) = \int_A dG(u) = \int_A \frac{| u |^2}{1 + | u |^2} \, dN(u)$$

is, in view of (1′), a finite Lebesgue-Stieltjes measure on the Borel sets of R^m which may be taken to vanish for $A = \{u\colon |u| = 0\}$.[3] Thus, any i.d.c.f. may be characterized by a triplet $[\gamma, \Sigma, G]$.[4]

The first factor in (2) is obviously the c.f. of a multivariate normal distribution, while the second is generated by the Poisson distribution in a sense which will be made more precise later. Thus, every i.d. vector $X \sim X^{(1)} + X^{(2)}$ where $X^{(1)}$ is multinormal and independent of $X^{(2)}$ which will be said to be "Poisson type." It will be convenient to refer to this as the canonical representation of X. If $X^{(2)} = 0$, X will be called "purely normal" while if $X^{(1)} = 0$, X will be dubbed "purely Poisson type."

If an i.d. vector $X = \begin{pmatrix} U \\ V \end{pmatrix}$ is partitionable into subvectors, one of which (say U) is purely normal and the other V purely Poisson type, then U and V must be independent. For $\begin{pmatrix} U \\ V \end{pmatrix} \sim \begin{pmatrix} U^{(1)} \\ V^{(1)} \end{pmatrix} + \begin{pmatrix} U^{(2)} \\ V^{(2)} \end{pmatrix}$ with $\begin{pmatrix} U^{(1)} \\ V^{(1)} \end{pmatrix}$ purely normal and independent of $\begin{pmatrix} U^{(2)} \\ V^{(2)} \end{pmatrix}$ which is purely Poisson type. But U purely normal implies $U^{(2)} = 0$ and V purely Poisson requires $V^{(1)} = 0$. The assertion follows. This observation may be utilized to construct a non-i.d. vector, all of whose marginal random variables are i.d.

[3] For the most frequently encountered case $m = 1$, this $G(u)$ is not in general identical with that used by the authors of the book mentioned in Footnote 2.

[4] It does not appear to have been remarked (even for the case $m = 1$) that a bounded r.v. is i.d. if and only if it is constant with probability one. This may be argued directly from the definition without resorting to (2).

The fact that for an arbitrary nonzero constant vector

$$c = (c_1, c_2, \cdots, c_m) \quad \text{and c.f.} \quad \varphi_{X_1, X_2, \cdots, X_m}(t_1, t_2, \cdots, t_m),$$
$$\varphi(c_1t, c_2t, \cdots, c_mt) = E[\exp\{i(c_1X_1 + \cdots + c_mX_m)t\}] = \varphi_{c'X}(t)$$

shows immediately that if X is an i.d. vector, every linear combination $c'X$ of its component random variables is i.d. The converse, however, is untrue. That is, it is possible for every linear combination $c'X$ to be i.d. without the vector X being i.d.[5] The Wishart distribution provides an example.

For the c.f. of a so-called Γ-variable is well known to be $[1 - (it/\alpha)]^{-\lambda}$, $\lambda, \alpha > 0$ and is manifestly i.d. Hence, the c.f. $[(1 - it_1)(1 - it_2)]^{-\lambda}$, $\lambda > 0$, of the sum of two independent Γ-variables is clearly i.d., whence by the remarks at the beginning of the preceding paragraph, $[1 - i(c_1 + c_2)t - c_1c_2t^2]^{-\lambda}$ is i.d. for arbitrary constants c_1, c_2. But if $Z_j' = (Z_{1j}, Z_{2j})$ are independent normally distributed vectors with mean vector zero and common covariance matrix Σ, then $X' = (X_1, X_2, X_3) = (\sum_1^n Z_{1j}, \sum_1^n Z_{2j}, \sum_1^n Z_{1j}Z_{2j})$ has the Wishart distribution with c.f.

$$| I - \Sigma T |^{-n/2} = [1 - 2i(\sigma_{11}t_1 + \sigma_{22}t_2 + 2\sigma_{12}t_3) + 4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)(t_3^2 - t_1t_2)]^{-n/2}.$$

Thus, *every* linear combination $b'X$ has the c.f.

$$[1 - 2i(b_1\sigma_{11} + b_2\sigma_{22} + 2b_3\sigma_{12})t + 4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)(b_3^2 - b_1b_2)t^2]^{-n/2},$$

which is i.d. by the preceding remarks. On the other hand, P. Lévy has shown [5] that the Wishart distribution is not itself i.d. and that for $n = 1$, it is even indecomposable.

If Y is a k-dimensional r.v. whose component random variables are independent and i.d. and A is an arbitrary $m \times k$ matrix of real constants, $X = AY$ is an i.d. r.v. In what sense is the converse true? That is, if X is an i.d. vector when does there exist a constant matrix A and a finite set of *independent* i.d. random variables $Y_1, \cdots, Y_k$ such that $X \sim AY$?

If X is purely normal it is well known that such a representation is always possible. Thus, it suffices to investigate $X^{(2)}$, the Poisson type r.v. in the canonical representation of X. For if $X^{(2)} \sim A_2Y^{(2)}$ with the k_2 components of $Y^{(2)}$ mutually independent, since $X^{(1)} \sim A_1Y^{(1)}$ with the k_1 components of $Y^{(1)}$ independent, we will have

$$X \sim X^{(1)} + X^{(2)} \sim (A_1, A_2)\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} = AY,$$

with the $k = k_1 + k_2$ components of Y independent random variables.

An answer to the question posed is given by

THEOREM 1. *A NSC that a Poisson-type r.v.* $X \sim AY$ *where the components* $Y_1, \cdots, Y_k$ *of* Y *are independent non-degenerate i.d. random variables and* A *is an* $m \times k$ *matrix of constants, no column of which consists entirely of zeros, is that in* (2), μ_G *vanish identically except on* k *different rays through the origin. Then* k *is*

[5] It is presumed that the assigned distributions of all linear combinations are compatible with the existence of a joint distribution.

the minimum number of random variables for which a representation $X \sim AY$, with the components $Y_1, \cdots, Y_k$ independent Poisson-type random variables, is possible.

Sufficiency. From the hypothesis of Theorem 1, the c.f. of the i.d. vector X may be supposed characterized by $[\gamma, \Sigma, G]$, with $\gamma' = (0, \cdots, 0)$, $\Sigma = \{0\}$, i.e., $X^{(1)} = (0, \cdots, 0)$. By hypothesis, μ_G assigns positive mass only along k rays, say R_j, whose direction cosine vectors are $r_j' = (r_{1j}, r_{2j}, \cdots, r_{mj})$. Let $G_j(s) = \mu_G(A_j^s)$, where $A_j^s = \{u : u \, \varepsilon \, R^m, u' = \rho r_j', -\infty < \rho < s\}$ and $h(t'u, |u|)$ denotes the integrand of (2). Then

$$\int_{R^m} h(t'u, |u|) \, dG(u) = \sum_{j=1}^{k} \int_{R^1} h(\rho t' r_j, \rho) \, dG_j(\rho)$$

and

$$\begin{aligned} \varphi_X(t) &= \prod_{j=1}^{k} \exp\left\{\int_{R^1} \left(e^{i(t'r_j)\rho} - 1 - \frac{i(t'r_j)\rho}{1+\rho^2}\right)\left(\frac{1+\rho^2}{\rho^2}\right) dG_j(\rho)\right\} \\ &= \prod_{j=1}^{k} \varphi_j(t'r_j), \end{aligned} \tag{3}$$

where $\varphi_j(t)$ is a univariate i.d. c.f. characterized by $[0, 0, G_j]$.

Let $Y_1, \cdots, Y_k$ be independent i.d. random variables with Y_j having c.f. $\varphi_j(t)$ as defined in (3) and take A to be the $m \times k$ matrix whose jth column is r_j. Then if $Z = AY$,

$$\begin{aligned} \varphi_{Z_1, \cdots, Z_m}(t_1, \cdots, t_m) &= E\,[\exp\{it'Z\}] = E\,[\exp\{i(t'A)Y\}] \\ &= \prod_{j=1}^{k} E\,[\exp\{i(t'r_j)Y_j\}] = \prod_{j=1}^{k} \varphi_j(t'r_j). \end{aligned} \tag{4}$$

Thus, $X \sim Z = AY$. This representation in terms of the distributions of k independent i.d. random variables (k being the number of rays with positive mass) is unique to within a relabelling of the variables and multiplication of each variable by a nonzero constant. The columns of A must then be adjusted accordingly.

Necessity. Since $X \sim AY$ with the components of Y independent and nondegenerate, the first equality of (3) holds with r_j equal to the jth column of the given matrix $A = \{a_{ij}\}$ divided by the scalar norming factor $(\sum_{i=1}^{m} a_{ij}^2)^{1/2}$. Comparing this with (2), it follows from the uniqueness of the i.d. representation that G and μ_G are as stated in the theorem.

Note that if $k < m$, the mass of the distribution of $X^{(2)}$ is concentrated in a space of lower dimensionality than R^m (i.e., the distribution of $X^{(2)}$ is singular).

A family of distributions $\mathfrak{F} = \{F\}$ has been defined in [7] to be factor-closed if $F = G_1 * G_2$, $F \, \varepsilon \, \mathfrak{F}$ implies $G_1, G_2 \, \varepsilon \, \mathfrak{F}$. Then we have as a

COROLLARY. If some component X_i of $X^{(2)}$ has a distribution belonging to a factor-closed family $\mathfrak{F}$, the distributions of $r_{ij}Y_j$ belong to $\mathfrak{F}$, for $j = 1, 2, \cdots, k$.

To avoid trivialities, let all components X_i of X be nondegenerate. Then no row of A is a zero vector. If $X^{(1)} = 0$ and $m = k$, then the components of X (i.e., $X^{(2)}$) are independent if and only if the rows of A may be permuted so as to form a diagonal matrix. This is palpably sufficient; on the other hand, if A cannot be so juggled, some Y_i has nonzero coefficients in two (necessarily independent) linear forms in the independent random variables Y_j. But this implies [2] that Y_i is normally distributed. The proof of the theorem, however, shows that when $X^{(1)} = 0$, the Y_j are all purely Poisson type, producing a contradiction.

Let α_j' be a k-tuple with 1 in the jth position and zeros elsewhere, $j = 1, \cdots, k$. Then if $k \leqq m$ and all α_j belong to the m-manifold spanned by the m rows of B, no representation of an i.d. vector X in the form BZ with the k components of Z independent but not *all* i.d. random variables is possible. For in such a case $C_j'B = \alpha_j$ has a nonzero solution C_j for all $j = 1, 2, \cdots, k$ whence $C_j'X \sim C_j'BZ = \alpha_j'Z$. But $C_j'X$ and therefore Z_j is i.d. If, e.g., $k > m$ such a representation is not summarily precluded.

It is, in general, untrue that an m-dimensional random vector $Y \sim AX$ where the components of X are independent random variables and the matrix A is $m \times k$. This may be seen with the familiar multinomial distribution.

Example. In r independent repetitions of an experiment, let $Y_1, \cdots, Y_{m+1}$ be the number of occurrences, respectively, of the mutually exclusive and exhaustive outcomes $A_1, \cdots, A_{m+1}$ with (single) trial probabilities $p_1, \cdots, p_{m+1}$, $\left(\sum_1^{m+1} p_i = 1\right)$; take $Y' = (Y_1, Y_2, \cdots, Y_m)$ and suppose there exists an integer $k \geqq 1$ and constant vectors $a_j' = (a_{j1}, \cdots, a_{jk})$ such that $Y \sim AX$ with the components of X independent random variables. Then

$$\prod_{j=1}^{k} \varphi_j\left(\sum_{\nu=1}^{m} a_{j\nu} t_\nu\right) = \left[\sum_{\nu=1}^{m} p_\nu(e^{it_\nu} - 1) + 1\right]^r \tag{5}$$

Setting $t_\mu = t$, $t_\nu = 0$ for $\nu \neq \mu$,

$$\prod_{j=1}^{k} \varphi_j(a_{j\mu} t) = [p_\mu(e^{it} - 1) + 1]^r.$$

Since the classical binomial family is factor-closed [7],

$$\varphi_j(a_{j\mu}t) = e^{itc_j}[p_\mu(e^{it} - 1) + 1]^{r_j} \text{ with } 0 < r_j \leqq r,$$

$$\sum r_j = r, \sum c_j = 0,$$

or

$$\varphi_j(t) = e^{itc_j/a_{j\mu}}[p_\mu(e^{it/a_{j\mu}} - 1) + 1]^{r_j}, \qquad j = 1, 2, \cdots, k.$$

Since the left-hand side is independent of μ, so is the right-hand side, whence $p_\mu = p$, $a_{j\mu} = a_j$, $\mu = 1, 2, \cdots, m$. Thus, if the multinomial probabilities are not identical, (5) cannot hold. However, even if $p_i \equiv p$, (5) would imply

$$\prod_{j=1}^{k} \varphi_j\left(\sum_{\nu=1}^{m} a_{j\nu} t_\nu\right) = \prod_{j=1}^{k} \left[p\left(\exp\left[i \sum_{\nu=1}^{m} t_\nu\right] - 1\right) + 1\right]^{r_j}$$

$$\equiv \left[\sum_{\nu=1}^{m} p(e^{it_\nu} - 1) + 1\right]^r,$$

which is impossible since the middle expression is a function of $\sum_{i=1}^{m} t_i$ only (and hence a degenerate multivariate distribution) whereas the right-hand side is not.

The following theorem covers the case that the measure μ_G is not necessarily wholly concentrated on a finite number of rays through the origin.

THEOREM 2. *If X is an i.d. random vector, then there exists a sequence of vectors $\{Y_n\}$ consisting of independent i.d. components, and a sequence of finite matrices $\{A_n\}$ such that the distribution of $A_n Y_n$ converges to the distribution of X as $n \to \infty$. The components of Y_n can each be taken to be of the form $(Y - b)$, where Y is a Poisson random variable if X is purely Poisson type.*

PROOF. As earlier, we may suppose $X^{(1)}$ is zero. Let

$$h(u) = \left(e^{it'u} - 1 - \frac{it'u}{1 + |u|^2}\right)\left(\frac{1 + |u|^2}{|u|^2}\right).$$

There is a double sequence of positive constants

$$\lambda_{n,1}, \cdots, \lambda_{n,k(n)}, \qquad n = 1, 2, \cdots$$

and a double sequence of m-tuples,

$$u_{n,1}, \cdots, u_{n,k(n)}, \qquad n = 1, 2, \cdots$$

$$u'_{n,i} = (u_{n,i}^{(1)}, \cdots, u_{n,i}^{(m)}), \qquad i = 1, 2, \cdots, k(n)$$

such that

$$\sum_{i=1}^{k(n)} \lambda_{n,i} h(u_{n,i}) \to \int_{R^m} h(u)\, dG(u), \tag{6}$$

as $n \to \infty$. Now $\lambda_{n,i} h(u_{n,i})$ is the log of the c.f. of a random vector

$$([Y_{ni} - b_{ni}]u_{ni}^{(1)}, \cdots, [Y_{ni} - b_{ni}]u_{ni}^{(m)}), \tag{7}$$

where the b_{ni} are appropriately chosen constants, and Y_{ni} is a Poisson random variable with parameter λ_{ni}. Hence, the left-hand side of (6) is the log of the c.f. of a sum of $k(n)$ vectors of the form (7), where $Y_{n1}, \cdots, Y_{nk(n)}$ are mutually independent. In other words, the left-hand side of (6) is the log of the c.f. of the vector

$$\begin{pmatrix} u_{n1}^{(1)} & & u_{n,k(n)}^{(1)} \\ \cdot & & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & & \cdot \\ u_{n1}^{(m)} & & u_{n,k(n)}^{(m)} \end{pmatrix} \cdot \begin{pmatrix} \bar{Y}_{n1} \\ \cdot \\ \cdot \\ \cdot \\ \bar{Y}_{nk(n)} \end{pmatrix},$$

where $\bar{Y}_{ni} = [Y_{ni} - b_{ni}]$, $i = 1, \cdots, k(n)$ are mutually independent i.d. random variables. This completes the proof.

4. Multivariate Poisson distribution. Let V denote the set of $2^m - 1$ vertices (excluding the origin) of the unit cube in the first orthant of R^m and lying along the m-axes; let V_j signify the vertex with one as the jth coordinate and zeros for the others, $j = 1, 2, \cdots, m$; let V_{ij}, $i < j$, represent the vertex with one's for the ith and jth coordinates but zeros for the remaining; $\cdots$; finally, let $V_{1,2,\cdots,m}$ denote the vertex $(1, 1, \cdots, 1)$.

In (2), define $G(u)$ by $\mu_G(V_j) = a_j \geqq 0, j = 1, \cdots, m, \mu_G(V_{ij}) = a_{ij} \geqq 0$, $i < j; \cdots, \mu_G(V_{1,2,\cdots,m}) = a_{1,2,\cdots,m} \geqq 0$, and for any Borel set B of R^m, $\mu_G(B) = \mu_G(VB)$ where the measure of the empty set is zero. Then if $z_j = e^{it_j}$, (2) becomes

$$\varphi(t) = \exp\left\{\sum_{j=1}^{m} a_j z_j + \sum_{i<j} a_{ij} z_i z_j + \cdots + a_{1,2,\cdots,m} \prod_{j=1}^{m} z_j - A_m\right\}, \tag{8}$$

where A_m is a constant such that $\varphi(0, 0, \cdots, 0) = 1$. The c.f. in (8) is that of the multivariate Poisson discussed in [6]. Since $G(u)$ is of the form prescribed by Theorem 1, with $k = 2^m - 1$, it follows from this theorem (supposing the constants a_j, a_{ij}, $\cdots$, $a_{12,\cdots,m}$ strictly positive) that there are $2^m - 1$ random variables Y_j and a constant matrix A such that $X \sim AY$. The matrix A may be chosen so that its $2^m - 1$ columns are the vectors (vertices) of V. By the corollary to Theorem 1, the Y_j are also Poisson distributed with parameters $a_1, a_2, \cdots, a_m$; $a_{12}, a_{13,\cdots}, a_{m-1,m}$; $\cdots$; $a_{1,2,\cdots,m}$. Since the classical Poisson distribution is not invariant under scale change, the matrix A is uniquely determined to within a permutation of its columns by the stipulation that the Y_j be independent Poisson random variables.

Furthermore, the multivariate Poisson distributions specified in (8) are the only i.d. distributions which are marginally Poisson. For, under this last proviso, $G(u)$ in (2) must be such that the projection of μ_G on the jth coordinate axis concentrates all mass at the point $(0, 0, \cdots, 0, 1, 0, \cdots, 0)$. This, in turn, requires that μ_G be as defined in the previous paragraph.

More generally, let $\mathfrak{F} = \{F(x; b_1, \cdots, b_r; c_1, \cdots, c_r\}$ be a family of univariate i.d. distributions whose c.f.'s are characterized by $[0, 0, G]$ with μ_G a discrete measure assigning mass $c_h > 0$ to $u = b_h \neq 0, h = 1, 2, \cdots, r$. Let X be an i.d. vector with the prescribed marginal distribution $F_{X_i}(x) = F(x; b_1, \cdots, b_r; c_1^i, \cdots, c_r^i)$, $i = 1, 2, \cdots, m$. Then, as earlier, there is a unique family of i.d. distributions for X having the stated marginals. Its c.f.'s are characterized by $[\gamma, \Sigma, G_d]$ where $\gamma' = (0, 0, \cdots, 0)$, $\Sigma = \{0\}$, and μ_{G_d} is a discrete measure assigning mass $d_j \geqq 0$ to the $(r + 1)^m - 1$ points $(u_1, u_2, \cdots, u_m)$ where $u_i = b_i$ or 0 (but u_i not all zero). Here the independent random variables may be taken to have the classical Poisson distribution and

$$X \sim DY = \begin{pmatrix} \sum_{j=1}^{k} d_{1j} Y_j \\ \cdot \\ \cdot \\ \cdot \\ \sum_{1}^{k} d_{mj} Y_j \end{pmatrix} = \sum_{j=1}^{k} \begin{pmatrix} d_{1j} Y_j \\ \cdot \\ \cdot \\ \cdot \\ d_{mj} Y_j \end{pmatrix}.$$

It is degenerate vectors of this form based on a single Poisson random variable Y_j rather than nondegenerate vectors having the most general multivariate Poisson distribution that spawn i.d. vectors.

It was pointed out above that if X is a multivariate i.d. vector, all of whose components have Poisson distributions marginally, then X must have a multivariate distribution specified by (8) and $X \sim AY$ when A is a finite matrix and Y is a vector of independent Poisson random variables. The purpose of the next remarks is to indicate that in general a comparable situation does not prevail. For example, suppose that U_1, U_2, U_3 are independent gamma variables whose c.f.'s are all $(1 - it)^{-\lambda}$, $(\lambda > 0)$. Then

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 101 \\ 011 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

is an i.d. vector with c.f. $\{(1 - it_1)(1 - it_2)[1 - i(t_1 + t_2)]\}^{-\lambda}$ whose marginals X_1, X_2 are gamma variables. On the other hand, in [3] it is shown that if $|\rho| < 1$. then

$$[(1 - it_1)(1 - it_2) + \rho^2 t_1 t_2]^{-2\lambda} \tag{9}$$

is a c.f. for all $\lambda > 0$ (and hence i.d.) and its marginals clearly have the same distribution as do X_1 and X_2. Thus there is no unique i.d. family having gamma marginals. Suppose $\rho \neq 0$ to avoid the trivial case of independence; then it is easy to verify that (9) cannot be the c.f. of a finite linear combination of independent gamma variables.

5. Connection with stochastic processes. It is a familiar fact that in the one-dimensional case the theory of i.d. random variables has a close connection with the theory of stochastic processes with independent increments. The analogue for multivariate i.d. vectors should be apparent, but it may be worth making some of the facts explicit.

Let U be a random vector whose values are the vectors of the set V defined at the beginning of Section 4. Denote these values by $u_1, \cdots, u_k$ and let their corresponding probabilities be $p_1, \cdots, p_k$, where $k = 2^m - 1$. Let $U_1, U_2, \cdots,$ be an infinite sequence of independent random vectors, each distributed as U. Let $X'(t)$, $(t \geqq 0, X'(0) = 0)$, be a Poisson process with stationary, independent increments. It is well known that waiting times for "jumps" in $X'(t)$ are independent, identically distributed exponential random variables. That is, an equivalent way of defining this process is in terms of an infinite sequence of independent, identically distributed random variables $W_1, W_2, \cdots,$ such that $P(W_i < w) = \lambda \int_0^w e^{-\lambda y}\, dy$ for $w > 0$ and zero otherwise $(\lambda > 0)$ as follows:

$$X'(0) = 0, 0 \leqq t \leqq W_1,$$

$$X'(t) = 1, W_1 < t \leqq W_1 + W_2,$$

$$X'(t) = 2, W_1 + W_2 < t \leqq W_1 + W_2 + W_3,$$

etc. Analogously, we can now define a *multivariate* Poisson process $X(t)$ as follows:

$$X(0) = 0, \text{ (zero } m\text{-vector)}, 0 \leqq t \leqq W_1,$$

$$X(t) = U_1, W_1 < t \leqq W_1 + W_2,$$

$$X(t) = U_1 + U_2, W_1 + W_2 < t \leqq W_1 + W_2 + W_3,$$

etc. Making use of the well known fact that the conditional distribution of $W_1, \cdots, W_r$ given that $X'(t) = r$ is that of the ordered values of r independent random variables, each uniformly distributed in $(0, t)$, it is easy to compute that the c.f. of $X(t)$ is

$$\sum_{j=0}^{\infty} [C(\theta)]^j \frac{(\lambda t)^j e^{-\lambda t}}{j!} = \exp\left\{\lambda t \sum_{j=1}^{k} (e^{i\theta' u_j} - 1) p_j\right\},$$

where

$$C(\theta) = C(\theta_1, \cdots, \theta_m) = \sum_{j=1}^{k} e^{i\theta' u_j} p_j$$

is the c.f. of the random vector U and $u_1, \cdots, u_k$ is the set of the k possible values of U. Making use of the material in Section 4, we see that we can choose the p_j's and λ so that $X(t)$ has any prescribed i.d. multivariate Poisson distribution. We remark also that $X(t)$ has independent, stationary increments for exactly the same reasons that $X'(t)$ does.

Consider now the somewhat more general case in which $U_1, U_2, \cdots$ are independent, identically distributed random vectors (m-tuples) having an *arbitrary* distribution with c.f.

$$C(\theta) = \int_{R_m} e^{i\theta' u}\, dF(u),$$

where F is the distribution function of U_1. If we define $X(t)$ as above but in terms of these more general U_i's, then the c.f. of $X(t)$ is

$$\exp\left\{\lambda t \int_{R_m} (e^{i\theta' u} - 1)\, dF(u)\right\}$$

We recognize this to be a multivariate i.d. c.f. either from the Lévy form or from the fact that $X(t)$ has independent increments. We cannot obtain the most general multivariate i.d. c.f. in this way. On the other hand, we can find a sequence of constant vectors $a_1, a_2, \cdots$ and scalars $b_1, b_2, \cdots$ and distribution functions $F_1, F_2, \cdots$ such that if $X_n(t)$ is determined by F_n as above, then as $n \to \infty$,

$$(X_n(t) - a_n)/b_n$$

converges in law to an arbitrary Poisson i.d. vector. Thus, the most general Poisson-type i.d. vector can be approximately obtained in terms of a Poisson-like stochastic process with independent exponential waiting times between "jumps" and whose "jumps" are random vectors.

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