which is therefore a confidence statement with a confidence coefficient greater than or equal to the confidence coefficient of (2.9). Thus, if (2.3) has a probability $1 - \alpha$, (2.9) has a probability $1 - \beta \geqq 1 - \alpha$, and if (2.9) has a probability $1 - \beta$, then (2.11) has a probability $1 - \gamma \geqq 1 - \beta$. The bounds in (2.11) are the ones obtained in [2] in a different way.

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A NOTE ON THE NORMAL DISTRIBUTION

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1. It is well known that a necessary and sufficient condition for the independence of the sample mean and variance is that the parent population be normal. This was first shown by R. C. Geary [2], and later Lukacs [3] gave a somewhat simpler proof using characteristics functions.

By using the method of Lukacs one can derive a similar theorem concerning the sample mean and the mean square successive difference.

2. Let $x_1, \cdots, x_n$ be independent and identically distributed with density $f(x)$ and mean μ and variance σ^2.

Let

$$\bar{x} = n^{-1} \sum_{j=1}^{n} x_j,$$

$$\delta_k^2 = 2^{-1}(n - k)^{-1} \sum_{j=1}^{n-k} (x_{j+k} - x_j)^2 \qquad k = 1, 2, \cdots, n - 1.$$

The following theorem can be proved:

Theorem: *A necessary and sufficient condition that $f(x)$ be the normal density is that δ_k^2 and $\bar{x}$ be independent.*

Proof: If δ_k^2 and $\bar{x}$ are independent, then we follow Lukacs [3] step for step, replacing

$$s^2 = n^{-2}[(n - 1) \sum x_\alpha^2 - 2 \sum \sum x_\alpha x_{\beta+1}]$$

Received on June 14, 1955.

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by

$$\delta_k^2 = 2^{-1}(n - k)^{-1}\left[\sum_{j=1}^{n-k} x_{j+k}^2 + \sum_{j=1}^{n-k} x_j^2 - 2\sum_{j=1}^{n-k} x_{j+k}x_j\right],$$

so that

$$\varphi(t_1, t_2) = \int \cdots \int e^{it_1\bar{x}+it_2\delta_k^2} f(x_1) \cdots f(x_n)\, dx_1 \cdots dx_n = \varphi_1(t_1)\varphi_2(t_2),$$

or

$$\left.\frac{\partial\varphi(t_1, t_2)}{\partial t_2}\right|_{t_2=0} = \varphi_1(t_1) \left.\frac{\partial\varphi_2(t_2)}{\partial t_2}\right|_{t_2=0}.$$

It is easy to show that

$$\varphi_1(t_1) = [\psi(t_1/n)]^n,$$

where

$$\psi(t) = \int e^{itx} f(x)\, dx,$$

and

$$\left.\frac{\partial\varphi(t_1, t_2)}{\partial t_2}\right|_{t_2=0} = i\left\{[\psi(t_1/n)]^{n-1}\int x^2 e^{it_1x/n} f(x)\, dx - [\psi(t_1/n)]^{n-2}\left[\int xe^{it_1x/n} f(x)\, dx\right]^2\right\},$$

$$\left.\frac{\partial\varphi_2(t_2)}{\partial t_2}\right|_{t_2=0} = i\sigma^2.$$

This leads to the same differential equation

$$-\psi(t)\frac{d^2\psi}{dt^2} + \left(\frac{dx}{dt}\right)^2 = \sigma^2[\psi(t)]^2$$

obtained by Lukacs, and the solution of which is the characteristic function of the normal distribution.

The converse is a special case of a lemma by Daly [1], which says that $\bar{x}$ and $g(x_1, \cdots, x_n)$ are independent in the normal case if $g(x_1, \cdots, x_n) = g(x_1 + a, \cdots, x_n + a)$. Since δ_k^2 is invariant under a translation, the theorem is proved.

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